CHAPTER 11

LINES AND CURVES

Art. 73. The Straight Line

A straight line is the intersection of two planes. It is then represented in rectangular coördinates by two first degree equations. The line is best constructed by finding two points on it. The most convenient points for this purpose are usually its intersections with two coördinate planes. If the line passes through the origin a second point can be found by assigning an arbitrary value to one of the coördinates and calculating the values of the other two.

Example 1. Construct the line represented by the equations

$$y + z = 3, \quad 4x + 3y - 3z = 3,$$

and find its direction cosines.

The line cuts the yz-plane where $x = 0$, that is, where

$$y + z = 3, \quad 3y - 3z = 3.$$

The solution of these equations is $y = 2$, $z = 1$. The intersection with the yz-plane is then A (0, 2, 1). In the same way the intersection with the xz-plane is found to be B (3, 0, 3). Draw the line through A and B (Fig. 73a). Its direction cosines are

$$\cos\alpha = \frac{3}{\sqrt{17}}, \quad \cos\beta = \frac{-2}{\sqrt{17}}, \quad \cos\gamma = \frac{2}{\sqrt{17}}.$$

Line through a Point with a Given Direction. — Let the line pass through P_1 (x_1, y_1, z_1) and have direction angles α, β, γ (Fig. 73b). If P (x, y, z) is any point on the line

$$x - x_1 = P_1P\cos\alpha, \quad y - y_1 = P_1P\cos\beta, \quad z - z_1 = P_1P\cos\gamma.$$

Hence

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}. \qquad (73a)$$

These are the equations of the line. Since two quantities equal to a third are equal to each other, they are equivalent to two independent equations.

If the cosines are proportional to A, B, C, equations (73a) are equivalent to

$$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C}. \qquad (73b)$$

These are the equations of a line through (x_1, y_1, z_1) with direction cosines proportional to A, B, C.

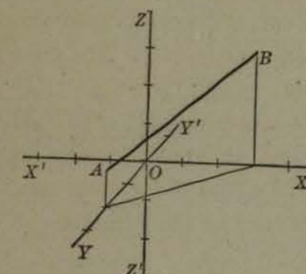

FIG. 73a.

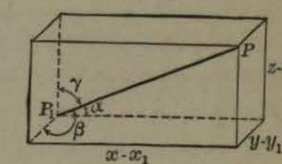

FIG. 73b.

Ex. 2. Find the equations of the line through (1, 0, 1) and (−2, 1, 0).

The direction cosines of the line are proportional to 3, −1, 1. Since the line passes through (1, 0, 1) its equations are then

$$\frac{x-1}{3} = \frac{y-0}{-1} = \frac{z-1}{1}.$$

These are equivalent to the two equations

$$\frac{x-1}{3} = \frac{y-0}{-1}, \quad \frac{y-0}{-1} = \frac{z-1}{1},$$

and consequently to $x + 3y = 1$, $y + z = 1$.

Exercises

Construct the lines represented by the following equations and find their direction cosines:

1. $x + y = 1, \quad y = 2z.$
2. $x - y = z, \quad x + y = 0.$
3. $x + y + z = 1, \quad 2x - 3y + 4z = 5.$
4. $x = 2, \quad y = 3.$

5. Find the equations of the line through the points (2, 3, −1) and (3, 4, 2).

6. Find the equations of the line through (0, 1, 2) parallel to the vector [3, 1, 5].

7. Find the equation of the line through (1, 1, 1) perpendicular to the plane $x + 2y - z = 3$.

8. Find the angle between the lines $x + y - z = 0$, $x + z = 0$ and $x - y = 1$, $x - 3y + z = 0$.

9. Find the angle between the line $x - y + z = 1$, $x = 2$ and the plane $z = x - 3y$.

10. Show that the lines $x + y + z = 1$, $2x - y + 3z = 2$ and $3y - z = 2$, $3x + 4z = 1$ are parallel.

11. Show that the lines $x + 2y = 1$, $2y - z = 1$ and $x - y = 1$, $x - 2z = 3$ meet in a point and are perpendicular.

Art. 74. Curves

A curve is the intersection of two surfaces. It is then represented by two simultaneous equations. Since an indefinite number of surfaces can be passed through a curve, it can be represented in an indefinite number of ways by a pair of equations.

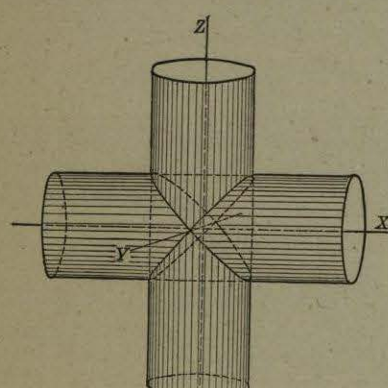

FIG. 74a.

Example 1. Show that the equations

$$x^2 + y^2 + z^2 = 1,$$
$$x + y + z = 1$$

represent a circle.

The first equation represents a sphere, the second a plane. The two equations represent the circle in which the sphere and plane intersect.

Ex. 2. Determine the locus represented by the equations

$$x^2 + y^2 = a^2, \quad y^2 + z^2 = a^2.$$

These equations represent circular cylinders of radius a (Fig. 74a). The locus required is the intersection of the cylinders. Subtraction of the equations gives $x^2 - z^2 = 0$. Therefore $x = \pm z$. All points of the intersection thus lie in the two planes $x = z$ and $x = -z$. The locus is two ellipses in which the planes cut the cylinders.

Projecting Cylinders. — If a rectangular coördinate is eliminated from the equations of a curve, the resulting equation usually represents the cylindrical surface with the curve as directrix and generators parallel to the axis of the eliminated coördinate. The intersection of this cylinder with the plane of the other two coördinates is then the projection of the curve on that coördinate plane. These statements are illustrated in the examples solved below.

Ex. 3. Find the cylinders with generators parallel to the coordinate axes and cutting the curve

$$z = x^2 + y^2, \quad z = x.$$

Eliminating z we get $x^2 + y^2 = x$. Since this equation contains only x and y, it represents a cylinder with generators parallel to the z-axis. Since the equation of the cylinder is a consequence of the equations of the curve, all points on the curve lie on the cylinder. Furthermore, if x and y satisfy the equation of the cylinder, a value of z can be found such that x, y, z satisfy the equations of the curve. That is, each generator of the cylinder cuts the curve. Therefore $x^2 + y^2 = x$ is the equation of the cylinder generated by lines parallel to the z-axis and cutting the curve. In the same way the equation of

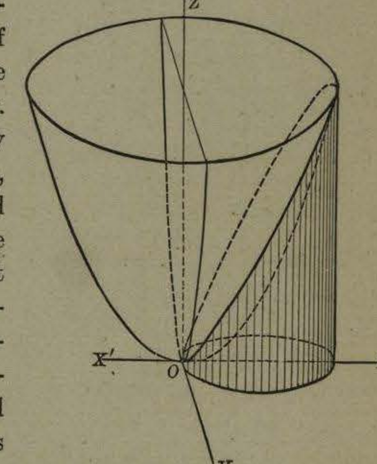

FIG. 74b.

the cylinder parallel to the x-axis is found to be $y^2 + z^2 = z$. Lines parallel to the y-axis and cutting the curve lie in the plane $z = x$. They however generate only the strip of this plane in which the curve lies. This shows that the elimination of a coördinate may give more than the surface generated by lines cutting the curve and parallel to the axis of that coördinate.

Ex. 4. Find the equations of the projections of the line

$$2x + y - z = 0, \quad x - y + 2z = 3$$

on the coördinate planes.

Eliminating z, we get

$$5x + y = 3.$$

This is the equation of a plane through the line parallel to the z-axis. The equation of the projection on the xy-plane is then

$$z = 0, \quad 5x + y = 3.$$

In the same way the projections on the xz- and yz-planes are found to be $y = 0$, $3x + z = 3$ and $x = 0$, $3y - 5z + 6 = 0$.

Exercises

Draw the following curves and find their projections on the coördinate planes:

1. $x+y+z=1, \quad x^2+y^2+z^2=1.$
2. $x^2 + y^2 = a^2, \quad x^2 + y^2 = z^2.$
3. $z^2 = x^2 + y^2, \quad x + y = 1.$
4. $z = xy, \quad z = 2x.$
5. $z = x^2 + y^2, \quad z = y^2 + z^2.$
6. $x^2+y^2+z^2=a^2, \quad x^2+z^2=ax.$
7. $r = a, \quad \theta = z.$
8. $\phi = \frac{1}{3}\pi, \quad \rho = \theta.$

9. Show that the equations $x^2 + y^2 - z^2 = 1$, $y - z = 1 - x$ represent a pair of lines.

10. Show that the circle $x^2 + y^2 + z^2 = 6$, $y + 2x = 1$ and the line $y + z = 1$, $x + y + z = 2$ intersect.

11. Find the intersection of the circle $x^2 + y^2 + z^2 + 6x = 0$, $x + y = 0$ and the plane $x + z = 1$.

Art. 75. Parametric Equations

The locus of a point whose coördinates are given functions of a parameter is usually a curve. For, if one of the equations is solved for the parameter and the value substituted in the other two, two equations between the coördinates are obtained. Thus

$$x = t, \quad y = t^2, \quad z = t^3$$

are parametric equations of a curve. To plot the curve, we can assign values to the parameter, calculate the corresponding values of the coördinates and plot the resulting points. By eliminating

the parameter the equation of a surface through the curve is obtained. If the parameter is eliminated between two of the parametric equations, the resulting equation considered as an equation in a coördinate plane represents the projection of the curve on that plane. For example the projections of the curve given above have the equations

$$y = x^2, \quad z = x^3, \quad z^2 = y^3.$$

Example. — The **helix** is a curve traced on the surface of a right circular cylinder by a point that advances in the direction of the axis of the cylinder while it rotates around the axis of the cylinder, the amount of advance being proportional to the angle of rotation.

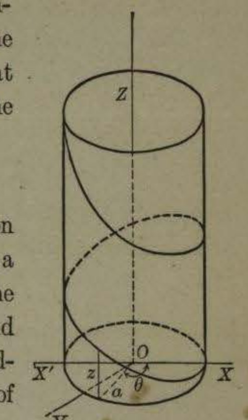

FIG. 75.

To find the equations of the helix, let the axis of z be the axis of the cylinder, a the radius of the cylinder, and let the x-axis pass through a point of the helix. If θ is the angle of rotation,

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = k\theta,$$

k being the ratio of the advance in the direction of the axis to the angle of rotation.

Exercises

Construct the following curves and find their projections on the coördinate planes:

1. $x = 1 + t,\ y = 2 - t,\ z = 3t.$
2. $x = \cos\theta,\ y = \sin\theta,\ z = 2\theta.$
3. $x = t\sin t,\ y = t\cos t,\ z = t.$
4. $x = \sin t,\ y = \cos t,\ z = \tan t.$

5. A conical helix is described by a point moving on the surface of a right circular cone, the distance of the point from the vertex of the cone being proportional to the angle of rotation about the axis. Find parametric equations for the curve.

6. Find the equation of the twisted surface generated by perpendiculars from points of a helix to the axis of the cylinder on which it lies.

7. Neglecting friction the position of a bullet starting from the origin with velocity $[a, b, c]$ after t seconds is given by the equations

$$x = at, \quad y = bt, \quad z = ct - 16.1\,t^2.$$

Construct the curve and find its projections on the coördinate planes if it starts with a velocity of 1000 ft. per second in the direction $\alpha = 60°$, $\beta = 45°$, $\gamma = 60°$.

8. The wheel of a gyroscope rotates with constant speed around its axis while the axis turns with constant speed about a fixed point of itself. Find equations of the curve described by a fixed point on the periphery of the wheel.

Natural Values of Trigonometric Functions

Deg.	Rad.	Sin.	Cos.	Tan.	Deg.	Rad.	Sin.	Cos.	Tan.
0	.0000	.0000	1.0000	.0000	**45**	.7854	.7071	.7071	1.0000
1	.0175	.0175	.9998	.0175	46	.8029	.7193	.6947	1.0355
2	.0349	.0349	.9994	.0349	47	.8203	.7314	.6820	1.0724
3	.0524	.0523	.9986	.0524	48	.8378	.7431	.6691	1.1106
4	.0698	.0698	.9976	.0699	49	.8552	.7547	.6561	1.1504
5	.0873	.0872	.9962	.0875	**50**	.8727	.7660	.6428	1.1918
6	.1047	.1045	.9945	.1051	51	.8901	.7771	.6293	1.2349
7	.1222	.1219	.9925	.1228	52	.9076	.7880	.6157	1.2799
8	.1396	.1392	.9903	.1405	53	.9250	.7986	.6018	1.3270
9	.1571	.1564	.9877	.1584	54	.9425	.8090	.5878	1.3764
10	.1745	.1736	.9848	.1763	**55**	.9599	.8192	.5736	1.4281
11	.1920	.1908	.9816	.1944	56	.9774	.8290	.5592	1.4826
12	.2094	.2079	.9781	.2126	57	.9948	.8387	.5446	1.5399
13	.2269	.2250	.9744	.2309	58	1.0123	.8480	.5299	1.6003
14	.2443	.2419	.9703	.2493	59	1.0297	.8572	.5150	1.6643
15	.2618	.2588	.9659	.2679	**60**	1.0472	.8660	.5000	1.7321
16	.2793	.2756	.9613	.2867	61	1.0647	.8746	.4848	[illegible]
17	.2967	.2924	.9563	.3057	62	1.0821	.8829	.4695	[illegible]
18	.3142	.3090	.9511	.3249	63	1.0996	.8910	.4540	[illegible]
19	.3316	.3256	.9455	.3443	64	1.1170	.8988	[illegible]	[illegible]
20	.3491	.3420	.9397	.3640	**65**	1.1345	.9063	[illegible]	[illegible]
21	.3665	.3584	.9336	.3839	66	1.1519	[illegible]	[illegible]	2.2460
22	.3840	.3746	.9272	.4040	67	1.1694	[illegible]	[illegible]	2.3559
23	.4014	.3907	.9205	.4245	68	1.1868	[illegible]	[illegible]	2.4751
24	.4189	.4067	.9135	.4452	69	[illegible]	[illegible]	.3584	2.6051
25	.4363	.4226	.9063	.4663	**70**	[illegible]	[illegible]	.3420	2.7475
26	.4538	.4384	.8988	.4877	71	[illegible]	[illegible]	.3256	2.9042
27	.4712	.4540	.8910	.5095	72	[illegible]	.9511	.3090	3.0777
28	.4887	.4695	.8829	.5317	[illegible]	[illegible]	.9563	.2924	3.2709
29	.5061	.4848	.8746	.5543	[illegible]	[illegible]	.9613	.2756	3.4874
30	.5236	.5000	.8660	[illegible]	[illegible]	[illegible]	.9659	.2588	3.7321
31	.5411	.5150	.8572	[illegible]	[illegible]	[illegible]	.9703	.2419	4.0108
32	.5585	.5299	.8480	[illegible]	[illegible]	1.3439	.9744	.2250	4.3315
33	.5760	.5446	[illegible]	[illegible]	78	1.3614	.9781	.2079	4.7046
34	.5934	.5592	[illegible]	[illegible]	79	1.3788	.9816	.1908	5.1446
35	.6109	.5736	[illegible]	[illegible]	**80**	1.3963	.9848	.1736	5.6713
36	.6283	.5878	[illegible]	.7265	81	1.4137	.9877	.1564	6.3138
37	.6458	.6018	[illegible]	.7536	82	1.4312	.9903	.1392	7.1154
38	.6632	[illegible]	[illegible]	.7813	83	1.4486	.9925	.1219	8.1443
39	.6807	[illegible]	[illegible]	.8098	84	1.4661	.9945	.1045	9.5144
40	.6981	[illegible]	[illegible]	.8391	**85**	1.4835	.9962	.0872	11.4301
41	.7156	[illegible]	[illegible]	.8693	86	1.5010	.9976	.0698	14.3007
42	.7330	[illegible]	[illegible]	.9004	87	1.5184	.9986	.0523	19.0811
43	.7505	[illegible]	[illegible]	.9325	88	1.5359	.9994	.0349	28.6363
44	[illegible]	[illegible]	[illegible]	.9657	89	1.5533	.9998	.0175	57.2900
45	[illegible]	[illegible]	[illegible]	1.0000	**90**	1.5708	1.0000	.0000	∞