6. Determine the locus of points in each of the following cases:

(a) $\rho = 2,\ \theta = \pi$, (b) $\rho = -2,\ \phi = \frac{\pi}{6}$, (c) $\theta = \frac{\pi}{3},\ \phi = \frac{\pi}{4}$.

7. Determine the distances of a point from the coördinate axes in spherical and cylindrical coördinates.

8. The spherical coördinates of P are $\rho = 2$, $\theta = 30°$, $\phi = 45°$. Find the angles between OP and the coördinate axes.

CHAPTER 10

SURFACES

Art. 67. Loci

An equation represents a locus if every point on the locus has coördinates satisfying the equation and every point with coördinates satisfying the equation lies on the locus.

One equation between the coördinates of a point in space usually represents a surface. Thus, the equation $z = 0$ represents the xy-plane, for any point in the xy-plane has a z-coördinate equal to zero and every point with z-coördinate equal to zero lies in the xy-plane. Similarly the equation $x^2 + y^2 + z^2 = 1$ represents a sphere with radius 1 and center at the origin. In particular cases one equation may represent a straight line or curve. Thus, the only real points for which $x^2 + y^2 = 0$ are the points $x = 0, y = 0$ on the z-axis.

Two simultaneous equations usually represent a curve or straight line; for each equation represents a surface and the two equations represent the intersection of two surfaces, that is, a straight line or curve. Thus, the equations

$$x^2 + y^2 + z^2 = 3, \quad z = 1$$

represent the circle in which a sphere and plane intersect.

Three simultaneous equations are usually satisfied by the coördinates of a definite number of points. These points are found by solving the equations simultaneously. In particular cases the equations may have no solution or may be satisfied by the coördinates of all points on a curve or surface.

Art. 68. Equation of a Plane

A line perpendicular to a curve or surface is called a *normal* to that curve or surface.

Let a normal DN to a plane (Fig. 68*a*) have direction angles α, β, γ. Let P_1 (x_1, y_1, z_1) be a fixed point and P (x, y, z) a variable point in the plane. The direction cosines of DN are $\cos\alpha$, $\cos\beta$ and $\cos\gamma$. Those of P_1P are proportional to $x - x_1$, $y - y_1$, $z - z_1$. Since DN is perpendicular to the plane it is perpendicular to P_1P. Therefore, by Art. 65,

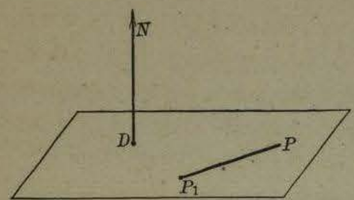

FIG. 68*a*.

$$(x - x_1)\cos\alpha + (y - y_1)\cos\beta + (z - z_1)\cos\gamma = 0. \qquad (68a)$$

This is the equation of the plane through (x_1, y_1, z_1) whose normal makes angles α, β, γ with the coördinate axes.

Let the direction cosines of the normal be proportional to A, B, C. Then

$$\cos\alpha : \cos\beta : \cos\gamma = A : B : C.$$

Since the cosines in equation (68*a*) can be replaced by any proportional numbers, that equation is equivalent to

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0, \qquad (68b)$$

which is therefore the equation of the plane through (x_1, y_1, z_1) perpendicular to the line with direction cosines proportional to A, B, C.

First Degree Equation. — Equations (68*a*) and (68*b*) are of the first degree in x, y, z. Therefore *any plane has an equation of the first degree in rectangular coördinates.*

Conversely, *any equation of the first degree in rectangular coördinates represents a plane;* for any such equation has the form

$$Ax + By + Cz + D = 0, \qquad (68c)$$

A, B, C, D being constant. Let x_1, y_1, z_1 satisfy this equation. Then

$$Ax_1 + By_1 + Cz_1 + D = 0.$$

Subtraction gives

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0,$$

which is equation (68*b*). Therefore *any equation of the form* (68*c*)

represents a plane whose normal has direction cosines proportional to A, B, C.

Example 1. Construct the plane with equation $2x + 3y + z = 6$.

The plane can be determined by its intersections with the coördinate axes. Where the plane crosses the x-axis y and z are zero and so $x = 3$. Similarly it crosses the y-axis at $y = 2$ and the z-axis at $z = 6$. The *intercepts* on the three axes are 3, 2 and 6.

Ex. 2. Construct the plane represented by the equation $x - 2y + z = 0$.

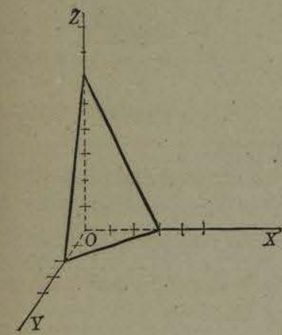

FIG. 68*b*.

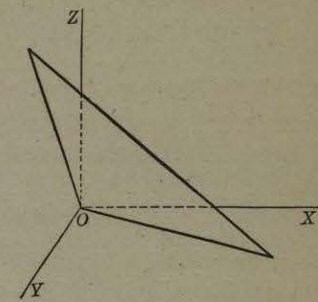

FIG. 68*c*.

The plane passes through the origin and so its intercepts are all zero. It can be determined by its intersections with the coördinate planes. It cuts the xy-plane in the line $z = 0$, $x = 2y$ and the yz-plane in the line $x = 0$, $z = 2y$. In their respective planes these lines are constructed from the equations $x = 2y$ and $z = 2y$ as in plane geometry. The plane through these lines is the one required.

Ex. 3. Find the equation of the plane through $(1, -2, 4)$ perpendicular to the line through A $(2, 1, 0)$ and B $(1, 2, 3)$.

The direction cosines of AB are proportional to $1 - 2$, $2 - 1$, $3 - 0$. Using these values instead of A, B, C in (68*b*) the equation of the plane is found to be

$$-(x - 1) + (y + 2) + 3(z - 4) = 0.$$

Ex. 4. Find the angle between the planes

$$x - y + z = 1, \quad 2x + 3y - z = 2.$$

Between two planes are two angles less than 180°. It is shown in solid geometry that these angles are equal to the angles between lines perpendicular to the two planes. The normals to the two planes have direction cosines proportional to the coefficients 1, −1, 1 and 2, 3, −1. The exact cosines of two normals are then

$$\cos \alpha_1 = \frac{1}{\sqrt{3}}, \quad \cos \beta_1 = \frac{-1}{\sqrt{3}}, \quad \cos \gamma_1 = \frac{1}{\sqrt{3}},$$

$$\cos \alpha_2 = \frac{2}{\sqrt{14}}, \quad \cos \beta_2 = \frac{3}{\sqrt{14}}, \quad \cos \gamma_2 = \frac{-1}{\sqrt{14}}.$$

By equation (65*a*) the angle θ between the two normals satisfies the equation

$$\cos \theta = \frac{-2}{\sqrt{42}}.$$

The negative sign shows that this is the obtuse angle. The acute angle between the two normals or between the two planes is $\cos^{-1}\left(\frac{2}{\sqrt{42}}\right)$.

Exercises

Construct the planes represented by the following equations and find the direction cosines of their normals:

1. $x + 2y + 4z = 4$.
2. $x - y + 3z = 5$.
3. $x + y + z = 0$.
4. $2x - 3y = 0$.
5. $3x + 4z = 0$.
6. $z + 5 = 0$.

7. Find the equation of the plane through the origin perpendicular to the line with direction angles $\alpha = 60°$, $\beta = 45°$, $\gamma = 60°$.

8. Find the equation of the plane through (1, 1, 0) perpendicular to the vector [3, −5, 4].

9. Find the equation of the plane with intercepts on OX, OY and OZ equal to 1, 2 and 3.

10. Show that the planes $x + 2y - z = 1$ and $2x + 4y - 2z = 3$ are parallel.

11. Show that the planes $x + y - z = 0$ and $2x - 3y - z = 0$ are perpendicular.

12. Find the angle between the planes $x + 2y + 2z = 0$ and $x - 4y + 8z = 9$.

13. Show that the angle between a line and a plane is the complement of the angle between the line and the normal to the plane. Find the angle between the plane $x - 2y + z = 0$ and the line through the points (3, 0, 1) and (0, 2, −1).

Art. 69. Equation of a Sphere

A sphere is the locus of points at a constant distance from a fixed point. The fixed point is the center, and the constant distance the radius of the sphere.

Let $C(x_1, y_1, z_1)$ be the center of a sphere with radius r. If $P(x, y, z)$ is any point on the sphere, its equation is

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2. \tag{69a}$$

When expanded the equation of the sphere has the form

$$x^2 + y^2 + z^2 + Ax + By + Cz + D = 0. \tag{69b}$$

Conversely, any equation of this form represents a sphere if it represents a real surface. To show this, complete the squares in x, y and z separately. The result will have the form

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = d.$$

If d is positive this represents a sphere with center (a, b, c) and radius $\sqrt{d}$. If d is zero the locus is the single point (a, b, c). If d is negative there is no real locus. Hence, whenever the equation represents a surface, that surface is a sphere.

Example. Determine the center and radius of the sphere

$$x^2 + y^2 + z^2 - 2x + 3y = 0.$$

Completing the squares,

$$(x - 1)^2 + (y + \tfrac{3}{2})^2 + z^2 = \tfrac{13}{4}.$$

The center is $(1, -\frac{3}{2}, 0)$ and the radius is $\frac{1}{2}\sqrt{13}$.

Art. 70. Equation of a Cylindrical Surface

A *cylindrical surface* is one generated by lines parallel to a fixed line and cutting a fixed curve. The lines are called *generators* and the fixed curve is called a *directrix*.

A cylindrical surface with generators parallel to a coördinate axis is represented in rectangular coördinates by an equation containing *only two coördinates.* To show this let the generators be parallel to the z-axis. Let the surface intersect the xy-plane in the curve $z = 0, f(x, y) = 0$. Any point $P(x, y, z)$ on the surface has the same x and y as its projection $Q(x, y, 0)$ on the xy-plane. Since the coördinates x and y of Q satisfy the equation $f(x, y) = 0$, so do the coördinates x, y, z of P. Also, if the coördinates of any point P satisfy the equation $f(x, y) = 0$, P is in the vertical line through a point Q of the curve and so lies on the surface. Therefore $f(x, y) = 0$ is the equation of the cylindrical surface.

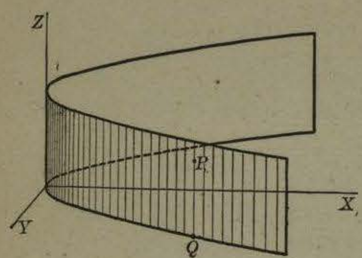

FIG. 70.

Conversely, *any equation in two rectangular coördinates represents a cylindrical surface with generators parallel to the axis of the missing coördinate.* To show this let the equation be $f(x, y) = 0$. Any point P with coördinates satisfying this equation lies in a vertical line through a point Q of the curve $z = 0, f(x, y) = 0$, and any point in such a vertical line has coördinates satisfying the equation. Therefore the equation $f(x, y) = 0$ represents a cylindrical surface whose directrix is the curve $z = 0, f(x, y) = 0$ in the xy-plane.

Example. Find the equation of the cylindrical surface with generators parallel to the x-axis cutting the yz-plane in the circle with radius 2 and center $(0, 1, 1)$.

In the yz-plane the equation of the circle is

$$(y - 1)^2 + (z - 1)^2 = 4.$$

This considered as an equation in space represents the given cylindrical surface.

Art. 71. Surface of Revolution

The surface described by a plane curve revolving about an axis in its plane is called a *surface of revolution.*

In the xz-plane let $f(x, z) = 0$ be the equation of a curve. Let this curve be revolved about the z-axis. The cylindrical coördinates r, z of any point P on the resulting surface are equal to the rectangular coördinates x, z of a point Q on the curve in the xz-plane. Since the coördinates of Q satisfy the equation $f(x, z) = 0$, the cylindrical coördinates of P satisfy the equation

$$f(r, z) = 0.$$

This is then the equation of the surface described by revolving the curve $y = 0, f(x, z) = 0$ about the z-axis.

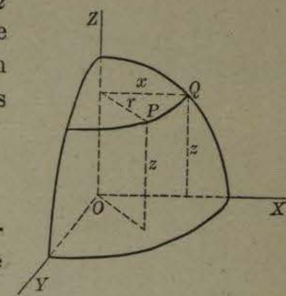

FIG. 71.

Since $r = \sqrt{x^2 + y^2}$, the rectangular equation of the surface is

$$f(\sqrt{x^2 + y^2}, z) = 0.$$

Similar equations are found for the surfaces obtained by revolving about the x- or y-axis. Therefore, *to find the rectangular equation of the surface described by rotating a curve in a coördinate plane about a coördinate axis in that plane, leave the coördinate corresponding to the axis of rotation unchanged in the plane equation of the curve and replace the other coördinate by the square root of the sum of the squares of the other two.*

Example 1. Find the equation of the surface described by rotating the parabola $y^2 = 2x$ about the x-axis.

This means that the parabola in the xy-plane with plane equation $y^2 = 2x$ is to be rotated about the x-axis. The equation of the resulting surface is obtained from $y^2 = 2x$ by leaving x unchanged and replacing y by $\sqrt{y^2 + z^2}$. The required equation is then

$$y^2 + z^2 = 2x.$$

Ex. 2. Show that

$$(x^2 + y^2)^2 + z^2(x^2 + y^2) = 1$$

is the equation of a surface of revolution.

This is indicated by the fact that the equation contains x and y only in the combination $x^2 + y^2$. The surface is generated by revolving about the z-axis the curve with plane equation $x^4 + z^2x^2 = 1$.

Exercises

Describe the surfaces represented by the following equations:

1. $x^2 + y^2 + z^2 = 9$.
2. $x^2 + y^2 + z^2 = 4x$.
3. $x^2 + y^2 + z^2 - 3x + 2y + z = 0$.
4. $2x^2 + 2y^2 + 2z^2 - 5x + 8y - 6z + 4 = 0$.
5. $x^2 + y^2 = 2x$.
6. $x^2 - y^2 = a^2$.
7. $y^2 = az$.
8. $z = \sin x$.
9. $x^2 + y^2 = 2z$.
10. $x^2 + y^2 = z^2$.
11. $x^2 - y^2 = z^2$.
12. $x^2 - z^2 = 0$.
13. $r = a\cos\theta$.
14. $r = az + b$.
15. $\rho = a\cos\phi$.
16. $\rho\sin\phi = a$.

Find the rectangular, cylindrical and spherical equations of the following surfaces:

17. Sphere with radius a and center at the origin.
18. Sphere with center $(0, 0, a)$ passing through the origin.
19. Right circular cylinder with axis OZ and radius a.
20. Right circular cone with axis OZ and vertical angle 90° (between generators in a plane through the axis).

Find the rectangular equations of the following surfaces:

21. Right circular cylinder tangent to the xy-plane along the x-axis.
22. Parabolic cylinder with generators parallel to OY and directrix a parabola in the xz-plane with axis OZ and vertex at the origin.
23. Elliptical cylinder with generators parallel to the z-axis and directrix an ellipse with axes along OX and OY.
24. Prolate spheroid generated by revolving an ellipse about its major axis.
25. Hyperboloid of revolution generated by revolving a hyperbola about one of its axes.
26. Paraboloid of revolution generated by revolving a parabola about its axis.
27. Torus generated by revolving a circle about a line in its plane not cutting the circle.

Art. 72. Graph of an Equation

To construct the graph of a given equation it is customary to draw a series of plane sections and from these sections to determine the appearance of the surface. The sections generally used are those in and parallel to the coördinate planes.

Example 1. **The ellipsoid,**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The sections of this surface in the xz- and yz-planes are the ellipses

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 \quad \text{and} \quad \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

having a common axis on $Z'Z$ (Fig. 72a). The section in a horizontal plane $z = k$ is an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2},$$

with axes parallel to OX and OY. Since the axes of this ellipse are in the xz- and yz-planes and end on the surface, they are chords of the ellipses in those planes. The surface, called an ellipsoid, is thus generated by horizontal ellipses whose axes are chords of the two vertical ellipses.

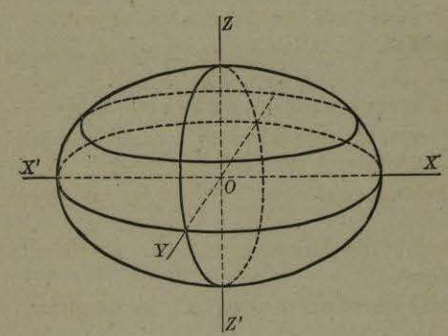

FIG. 72a.

The quantities a, b, c, called the semi-axes, are equal to the intercepts on the coördinate axes. If two of these semi-axes are equal the ellipsoid is called a spheroid. It is then a surface of revolution obtained by revolving an ellipse about one of its axes. If the semi-axes are all equal the ellipsoid is a sphere.

Ex. 2. **The hyperboloid of one sheet,**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

The sections in the xz- and yz-planes are hyperbolas

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1 \quad\text{and}\quad \frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

with a common axis $Z'Z$ (Fig. 72*b*). The section in a horizontal plane $z = k$ is an ellipse.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2}$$

with the ends of its axes on the two hyperbolas.

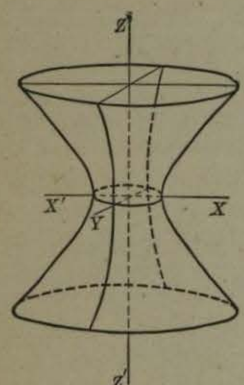

FIG. 72*b*.

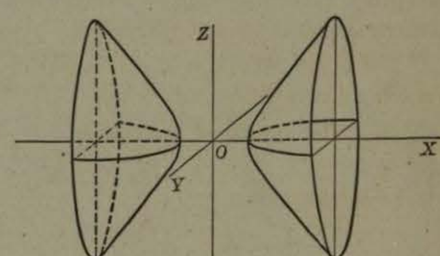

FIG. 72*c*.

If $a = b$ the surface is a hyperboloid of revolution obtained by revolving a hyperbola about its conjugate axis.

Ex. 3. **The hyperboloid of two sheets,**

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

The section in a plane $x = k$ is imaginary if $|x| < a$. The surface therefore consists of two parts, one on the right of $x = a$ and one on the left of $x = -a$. The surface cuts the xy- and xz-planes in hyperbolas and is generated by ellipses parallel to the yz-plane whose axes are chords of these hyperbolas (Fig. 72*c*).

If $b = c$ the surface is a hyperboloid of revolution obtained by revolving a hyperbola about its transverse axis.

Ex. 4. **The elliptic paraboloid,**

$$z = ax^2 + by^2,$$

a and b having the same algebraic sign.

Sections in the xz- and yz-planes are parabolas

$$z = ax^2 \text{ and } z = by^2.$$

The section in a plane $z = k$ is an ellipse

$$ax^2 + by^2 = k$$

with axes parallel to OX and OY if a, b and k have the same sign and imaginary if k is opposite in sign from a and b. The surface is thus generated by horizontal ellipses whose axes are chords of the two parabolas.

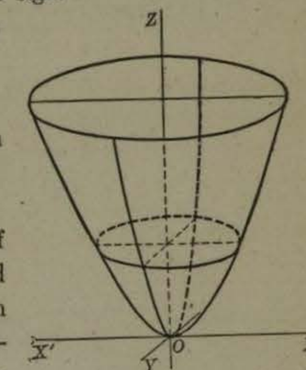

FIG. 72*d*.

If $a = b$ the surface is a paraboloid of revolution obtained by revolving a parabola about its axis.

Ex. 5. **The hyperbolic paraboloid,**

$$z = ax^2 - by^2,$$

a and b being positive.

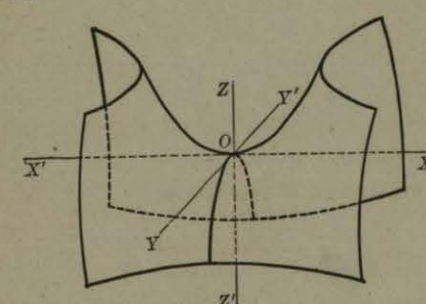

FIG. 72*e*.

The section in the xz-plane is a parabola $z = ax^2$ extending upward. The section in the yz-plane is a parabola $z = -by^2$ extending downward. The section in the xy-plane is a pair of lines $ax^2 - by^2 = 0$. The section in a horizontal plane $z = k$ is a hyperbola $ax^2 - by^2 = k$ whose transverse axis is a chord of the parabola (in the xz- or yz-plane) cut by that plane. The surface has the general shape of a saddle.

Ex. 6. **The hyperbolic paraboloid,**

$$z = kxy.$$

This is a saddle-shaped surface similar to that in Ex. 5. The hyperbolas in horizontal planes are however rectangular with asymptotes parallel to the x- and y-axes.

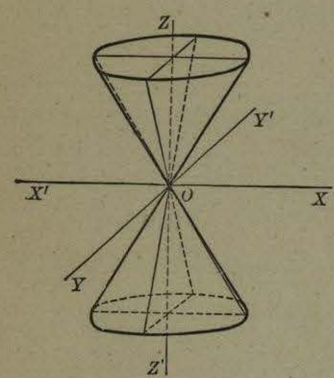

FIG. 72*f*.

Ex. 7. **The cone,**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}.$$

The sections in the xz- and yz-planes are pairs of lines

$$\frac{x}{a} = \pm\frac{z}{c}, \quad \text{and} \quad \frac{y}{b} = \pm\frac{z}{c}.$$

The surface is generated by horizontal ellipses the ends of whose axes are on these lines.

Ex. 8. Describe the surface with spherical equation

$$\rho = a \sin \phi.$$

Since θ does not occur in the equation the surface is one of revolution about the z-axis. The section in the xz-plane is a circle tangent to the z-axis at the origin. The graph is a doughnut-shaped surface generated by revolving this circle about the z-axis.

Exercises

Draw the graphs and describe the surfaces represented by the following equations:

1. $x^2 + 2y^2 + 3z^2 = 6$.
2. $(x-1)^2 + 2(y-2)^2 + 3(z-3)^2 = 6$.
3. $x^2 + 4(y^2 + z^2) = 12$.
4. $x^2 - y^2 + z^2 = 1$.
5. $x^2 - y^2 + z^2 - 2x + 4y = 4$.
6. $x^2 - y^2 - z^2 = 3$.
7. $x^2 + z^2 = y$.
8. $y^2 + 2z^2 - 4x^2 = 12$.
9. $x = z^2 + 2y^2 + 2z - 12y + 19$

10. $x = 2yz$.
11. $x^2 - y^2 = z^2$.
12. $(x+2)^2 + 4(y-1)^2 = (z-3)^2$.
13. $y^2 = z^3$.
14. $z = xy + x + y$.
15. $r = a \cos 2\theta$.
16. $\rho = a \cos \theta$.
17. $\rho \cos \phi = 2$.
18. $r^2 + z^2 = a^2$.
19. $rz = k$.
20. $xyz = a^3$.