CHAPTER 9

COÖRDINATES OF A POINT IN SPACE

Art. 60. Rectangular Coördinates

Let $X'X$, $Y'Y$ and $Z'Z$ be three scales with a common zero point O called the *origin*. The lines $X'X$, $Y'Y$ and $Z'Z$ are called *coördinate axes* and are referred to as the x-axis, y-axis and z-axis respectively. They determine three *coördinate planes* XOY, YOZ and ZOX called the xy-, yz- and zx-planes. These planes divide space into eight portions called *octants*. The portion $O\text{-}XYZ$ is sometimes called the first octant. The other octants are not usually numbered.

Through any point P pass planes perpendicular to the coördinate axes meeting them in M, R and T. The numbers at these points in the scales are called the coördinates x, y, z of P. This point is represented by the symbol (x, y, z). To indicate the coördinates of P the notation $P\ (x, y, z)$ is used.

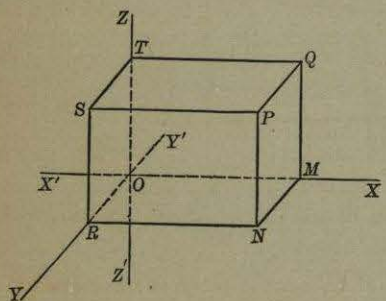

FIG. 60.

Usually the x-axis is drawn to the right, the y-axis forward and the z-axis upward. The x-coördinate of P is then the segment SP considered positive when drawn to the right, the y-coördinate is QP considered positive when drawn forward and the z-coördinate is NP considered positive when drawn upward.

To plot the point P having coördinates x, y, z draw $OMNP$ making $OM = x$, $MN = y$ and $NP = z$. The result is a plane figure.

By shading and dotting lines it can however be given the appearance of a space construction.

The points in a coördinate plane have one coördinate equal to zero. With respect to the other two coördinates these points can be treated as in plane geometry. For example, the points in the yz-plane whose coördinates y and z satisfy a first degree equation lie on a line.

Many results of solid geometry are similar to those already found in the plane. A formula in two coördinates x and y is often extended to space by adding a similar term containing z. With a little attention to these relations many formulas of space geometry can be inferred from analogy with those in the plane.

Art. 61. Projection

The projection of a segment AB upon a line RS is the segment of that line intercepted between planes through A and B perpen-

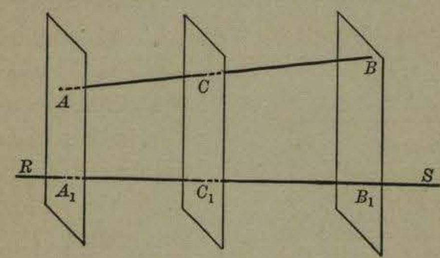

FIG. 61a.

dicular to RS. The projection of AB upon a plane is the segment between the feet of perpendiculars from A and B to the plane. Since parallel planes or lines intercept two fixed lines proportionally, it follows, as in plane geometry, that segments of a line have the same ratios as their projections on a line or plane. Thus, in Figs. 61a or 61b,

$$AB : BC = A_1B_1 : B_1C_1.$$

In using this proportion AB and BC must have algebraic signs if their projections have algebraic signs and conversely.

Projection on a Directed Line. — By the angle between two lines that do not intersect is meant the angle between intersecting lines parallel to them. Two lines determine two angles less than 180°, one acute and one obtuse. If the lines are directed they however determine a definite angle, the one less than 180° with the arrows on its sides pointing away from its vertex.

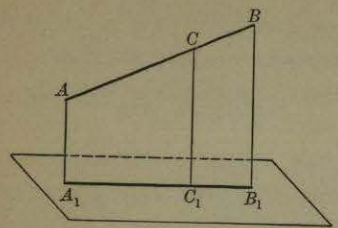

FIG. 61*b*.

Let θ be the angle between a directed segment AB and a directed line RS (Fig. 61*c*). Through A draw AB' parallel to RS to meet the plane through B perpendicular to RS. If the projection A_1B_1 is considered positive when it is drawn in the direction RS

$$A_1B_1 = AB' = AB\cos\theta,$$

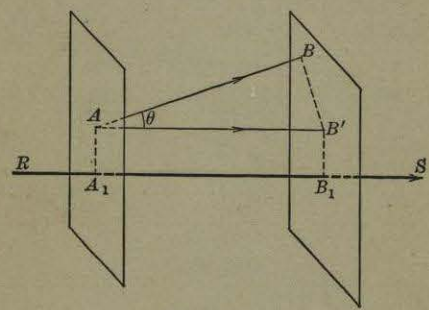

FIG. 61*c*.

that is, *the projection of a segment AB on a directed line RS is equal to the product of the length of the segment by the cosine of the angle between the segment and line.*

Projection of a Broken Line. — Let $ABCD$ be a broken line joining A and D. Project on RS. If segments of RS are considered opposite in sign when drawn in opposite directions

$$A_1B_1 + B_1C_1 + C_1D_1 = A_1D_1.$$

This is true whether A, B, C, D are in a plane or not. Therefore, *the algebraic sum of the projections of the parts of a broken line joining two points is equal to the projection of the segment from the first to the last of those points.*

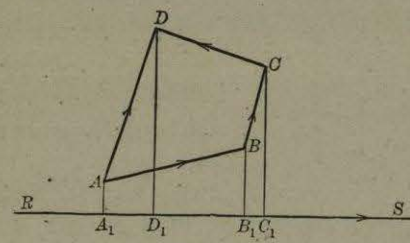

FIG. 61*d*.

Projections on the Axes. — Let P_1P_2 be the segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ (Fig. 61*e*). Through P_1 and P_2 pass planes perpendicular to the x-axis meeting it in M_1 and M_2. P_1M_1 and P_2M_2 are then perpendicular to OX. If segments are considered positive when drawn in the positive direction along the x-axis, the projection of P_1P_2 is

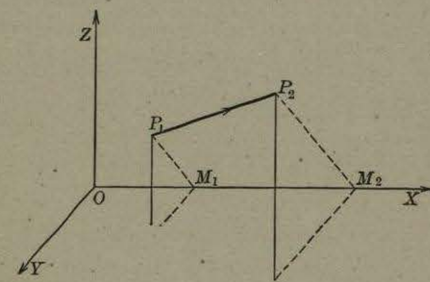

FIG. 61*e*.

$$M_1M_2 = OM_2 - OM_1 = x_2 - x_1.$$

Similarly the projections on the y- and z-axes are $y_2 - y_1$ and $z_2 - z_1$. Therefore *in length and sign the projection of a segment on a coördinate axis is equal to the difference obtained by subtracting the coördinate of the beginning from that of the end of the segment.*

Exercises

1. Plot the points (0, 0, 1), (1, 1, 0), (1, 1, 1), (−1, 2, 3), (−1, −2, −3).

2. What is the x-coördinate of a point in the yz-plane? What are the x- and y-coördinates of a point on the z-axis?

3. Where are all the points for which $z = -1$? What is the locus of points for which $x = 1$ and $y = 2$?

4. Determine the distance of (x, y, z) from each coördinate axis. What is its distance from the origin?

5. Find the feet of the perpendiculars from (1, 2, 3) to the coördinate planes and to the coördinate axes.

6. Given P (1, 0, 1), Q (2, 1, 5), R (3, −1, 2), find the projections of PQ, QR and RP on the coördinate axes. Show that the sum of the projections on each axis is zero.

7. In exercise 6 find the angles between PQ and the coördinate axes.

8. The projections of AB on OX, OY and OZ are 3, −1 and 2 respectively. Those of BC are 2, −3 and 1. Find the projections of AC.

9. Find the coördinates of the middle point of the segment joining (3, 2, −1) and (2, −3, 4).

10. Given A (2, 1, −2), B (1, 3, 1), C (−1, 7, 7), show that the projections of A, B, C on the xy-plane are three points on a line. Also show that the projections on the yz-plane are points of a line. Hence show that A, B, C lie on a line.

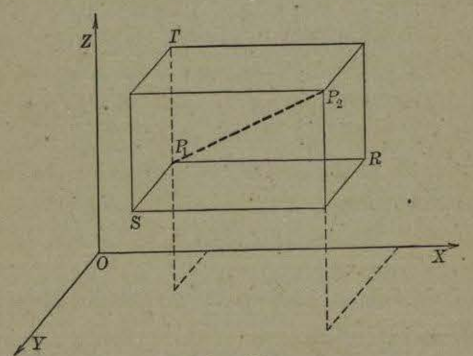

Fig. 62.

Art. 62. Distance between Two Points

Let P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) be the points. On P_1P_2 as diagonal construct a box with edges parallel to the coördinate axes (Fig. 62). Since projections on parallel lines are equal, the edges

of the box are equal to the projections of P_1P_2 on the coördinate axes. Consequently

$$P_1R = x_2 - x_1, \quad P_1S = y_2 - y_1, \quad P_1T = z_2 - z_1.$$

Since the square on the diagonal of a rectangular parallelopiped is equal to the sum of the squares of its three edges,

$$P_1P_2^2 = P_1R^2 + P_1S^2 + P_1T^2,$$

whence

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{62}$$

Art. 63. Vectors

A vector is a segment with given length and direction. The components of a vector in space are its projections on the coördinate axes. In this book the vector with components a, b, c will be represented by the symbol $[a, b, c]$. The vector P_1P_2 from P_1 (x_1, y_1, z_1) to P_2 (x_2, y_2, z_2) has components equal to $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$. This is expressed by the equation

$$P_1P_2 = [x_2 - x_1, y_2 - y_1, z_2 - z_1]. \tag{63}$$

As in plane geometry, the sum of two vectors is obtained by placing the second on the end of the first and drawing the vector from the beginning of the first to the end of the second. If v_1 and v_2 are vectors beginning at the same point, $v_2 - v_1$ is the vector from the end of v_1 to the end of v_2. If r is a number, rv is the vector r times as long as v and having the same direction if r is positive but the opposite direction if r is negative.

The components of the sum of two vectors are obtained by adding corresponding components, those of the difference by subtracting corresponding components and those of the product rv by multiplying the components of v by r.

Example. Find the point P (x, y, z) on the line through A (−1, 2, 3) and B (3, −4, 2) such that $AP = 3\,BP$.

Using the given coördinates,

$$AP = [x + 1, y - 2, z - 3], \quad BP = [x - 3, y + 4, z - 2].$$

If then $AP = 3\,BP$,

$$x + 1 = 3\,(x - 3), \quad y - 2 = 3\,(y + 4), \quad z - 3 = 3\,(z - 2).$$

Consequently $x = 5$, $y = -7$, $z = \frac{3}{2}$.

Exercises

1. Show that the triangle formed by the points (1, 0, 2), (0, −2, 3) and (2, −3, 0) is isosceles.

2. By showing that the sum of the distances AB and BC is equal to AC show that the points A (1, 2, −1), B (0, 5, 1), C (−2, 11, 5) lie on a line.

3. Find the center of the sphere through the points (0, 0, 0), (2, 0, 0), (0, 4, 0) and (0, 0, 6).

4. Given A (1, −1, 2), B (3, 2, −1), find the point on AB produced which is four times as far from A as from B. Also find the point one-fourth of the way from A to B.

5. Given A (1, 3, 1), B (3, 5, 4), find the vectors which are the projections of AB on the coördinate planes. Show that the sum of the squares of the projections is twice the square of AB.

6. Show that the points A (1, 0, −1), B (−2, 1, 3), C (−1, 3, 6), D (2, 2, 2) are the vertices of a parallelogram.

7. AE is the diagonal of a parallelopiped with edges AB, AC and AD. Given A (1, 1, 0), B (2, 3, 0), C (3, 0, 1), D (2, 1, 4), find the coördinates of E.

8. Given A (2, 4, 5), B (1, 3, 0), C (3, 0, 2), D (6, 1, 9), find the middle points of AB and CD. Then find the middle point P of the two middle points. Show that P satisfies the vector equation

$$PA + PB + PC + PD = 0.$$

9. If r and s are numbers the vector $rAB + sAC$ lies in the plane ABC. Consequently, if

$$AD = rAB + sAC,$$

the points A, B, C, D lie in a plane. By finding values of r and s satisfying this equation show that A (0, 0, 1), B (1, −1, 4), C (−2, 1, −3), D (3, 2, 0) lie in a plane.

Art. 64. Direction of a Line

The angles between a directed line and OX, OY and OZ are represented by the letters α, β and γ. These are sometimes called the *direction angles* of the line. The cosines of α, β and γ are called the *direction cosines* of the line.

Let P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) be two points on the line. Construct a box with P_1P_2 as diagonal and edges parallel to the coördinate axes. If the positive direction along each edge is taken as that of the parallel axis, the direction angles for P_1P_2 are those between P_1P_2 and the edges of the box. Consequently

$$\left.\begin{aligned} \cos\alpha &= \frac{P_1A}{P_1P_2} = \frac{x_2 - x_1}{P_1P_2}, \\ \cos\beta &= \frac{P_1B}{P_1P_2} = \frac{y_2 - y_1}{P_1P_2}, \\ \cos\gamma &= \frac{P_1C}{P_1P_2} = \frac{z_2 - z_1}{P_1P_2}. \end{aligned}\right\} \qquad (64a)$$

The direction cosines of the lines are therefore the components of P_1P_2 divided by its length.

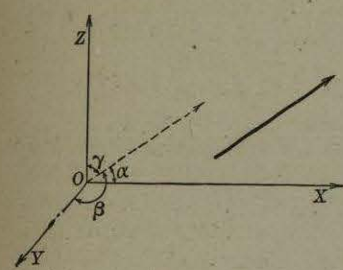

FIG. 64a.

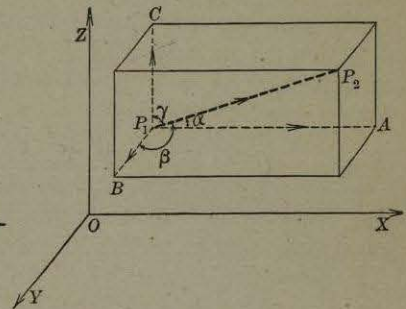

FIG. 64b.

Since $P_1P_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$, it is seen that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \qquad (64b)$$

that is, *the sum of the squares of the direction cosines of a line is equal to unity.*

If it is known that the direction cosines of a line are proportional to three numbers a, b, c, since the sum of the squares of the cosines is equal to unity, the exact cosines are obtained by dividing these proportional numbers by the square root of the sum of their squares. The two square roots give two sets of direction cosines corresponding to the two directions along the line.

Example 1. Find the direction cosines of the vector [2, −3, 5].

The cosines are the ratios of the components to the length of the vector. Consequently,

$$\cos\alpha = \frac{2}{\sqrt{38}}, \quad \cos\beta = \frac{-3}{\sqrt{38}}, \quad \cos\gamma = \frac{5}{\sqrt{38}}.$$

Ex. 2. The direction cosines of a line are proportional to -1, 2 and 1. Find their values and construct a line having the given cosines.

The segment from the origin to the point $(-1, 2, 1)$ has direction cosines proportional to its components -1, 2 and 1. Therefore the line through the origin and the point $(-1, 2, 1)$ has the direction required. Its direction cosines are

$$\cos\alpha = \frac{-1}{\pm\sqrt{6}}, \quad \cos\beta = \frac{2}{\pm\sqrt{6}}, \quad \cos\gamma = \frac{1}{\pm\sqrt{6}},$$

the two signs in the denominators corresponding to the two directions along the line.

Art. 65. The Angle between Two Directed Lines

Let the lines have direction angles $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$. Let OP_1 and OP_2 be lines through the origin with the same directions. Projecting on OP_2,

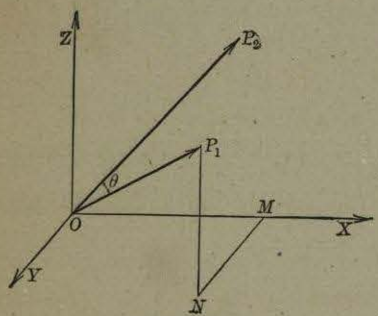

FIG. 65.

$$\text{proj. } OP_1 = \text{proj. } OM + \text{proj. } MN + \text{proj. } NP_1.$$

Consequently,

$$OP_1 \cos\theta = OM \cos\alpha_2 + MN\cos\beta_2 + NP_1\cos\gamma_2.$$

But $OM = OP_1 \cos\alpha_1$, $MN = OP_1 \cos\beta_1$, $NP_1 = OP_1 \cos\gamma_1$. Substituting these values and cancelling OP_1,

$$\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2. \qquad (65a)$$

That is, *the cosine of the angle between two directed lines is equal to the sum of the products of corresponding direction cosines.*

If the lines are perpendicular the angle θ is 90° and its cosine is zero. In that case

$$\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0. \qquad (65b)$$

Therefore, *two lines are perpendicular when the sum of the products of corresponding direction cosines is zero.*

In equation (65*b*) the direction cosines of either line can be replaced by any numbers proportional to them. In particular, *two vectors are perpendicular when the sum of the products of corresponding components is zero.*

Example 1. Find the angle between the vectors $v_1[-2, 0, 1]$ and $v_2[1, 2, 0]$. The direction cosines of the vectors are

$$\cos\alpha_1 = \frac{-2}{\sqrt{5}}, \quad \cos\beta_1 = 0, \quad \cos\gamma_1 = \frac{1}{\sqrt{5}},$$
$$\cos\alpha_2 = \frac{1}{\sqrt{5}}, \quad \cos\beta_2 = \frac{2}{\sqrt{5}}, \quad \cos\gamma_2 = 0.$$

If θ is the angle between the vectors,

$$\cos\theta = -\tfrac{2}{5} + 0 + 0 = -\tfrac{2}{5}.$$

The negative sign shows that the angle is obtuse.

Ex. 2. Show that the points $A\ (4, 2, -1)$, $B\ (3, -2, 3)$, $C\ (1, 1, 0)$ are vertices of a right triangle.

In this case

$$CA = [3, 1, -1], \quad CB = [2, -3, 3].$$

The sum of the products of corresponding components is

$$6 - 3 - 3 = 0.$$

The sides CA and CB are then perpendicular and the triangle is a right triangle.

Exercises

1. Determine the direction cosines of the coördinate axes.
2. A straight line in the xy-plane has a slope equal to $\sqrt{3}$. Find its direction angles and direction cosines.
3. Determine the direction cosines of the segment from the origin to the point (2, 3, 6).
4. How many lines through the origin make angles of 60° with the x- and y-axes? What angles do they make with the z-axis?
5. A line makes angles of 45° with the yz- and zx-planes. Find its direction cosines.
6. A line makes equal angles with the coördinate axes. Find those angles.
7. Construct a line with direction cosines proportional to 1, 4 and 8. Find its direction cosines.

8. Find the angles between the vector [1, 2, 2] and the coördinate axes.

9. Determine the angles of the triangle with vertices (1, 0, 0), (0, 1, 0) and (0, 0, 2).

10. Show that the line joining the origin to the point (1, 1, 1) is perpendicular to the line through (1, 1, 0) and (0, 0, 2).

11. A line L_1 in the xz-plane makes an angle of 60° with the line L_2 in the xy-plane with equation $x + 2y = 4$. Find the angle between L_1 and the x-axis.

12. Find the angles between the segments from (1, 1, 1) to the points (2, 0, 3), (0, 3, −2) and (3, 1, 3). Show that one of these angles is equal to the sum of the other two. What do you conclude about the four points?

13. Show that the line through the origin and the point (1, −1, −1) is perpendicular to the line joining any pair of the four points in Ex. 12.

Art. 66. Cylindrical and Spherical Coördinates

Let O be the origin and OX the initial line of a system of polar coördinates in the xy-plane. Let the projection of a point P on the xy-plane have coördinates r and θ. The *cylindrical coördinates* of P are r, θ and z. The angle θ is considered positive when measured from OX toward OY and r is positive when N lies on the terminal side of the angle.

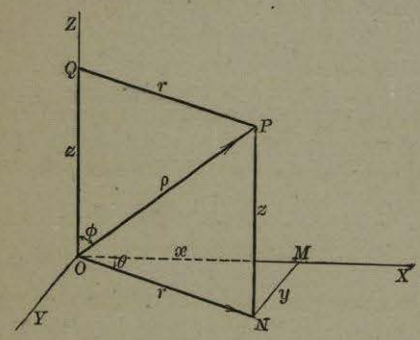

FIG. 66.

In the plane ONP let ρ and ϕ be the polar coördinates of P, the z-axis being initial line and O the origin. The *spherical coördinates* of P are ρ, θ and ϕ. In this case ϕ is considered positive when measured from OZ toward the terminal side of θ and ρ is positive when P lies on the terminal side of ϕ. In some books these coördinates are called polar.

The relations of the rectangular, cylindrical and spherical coördinates of a point can easily be determined from the right triangles

OMN and OPQ in Fig. 66. The most important equations connecting the coördinates are

$$\left.\begin{array}{ll} x = r\cos\theta, & y = r\sin\theta, \\ r = \rho\sin\phi, & z = \rho\cos\phi, \\ r^2 = x^2 + y^2, & \rho^2 = x^2 + y^2 + z^2. \end{array}\right\} \tag{66}$$

Example. Determine the cylindrical and spherical coördinates of the point (1, 2, 3).

From Fig. 66 it is seen that

$$\tan\theta = \frac{y}{x} = 2, \qquad r = \sqrt{x^2 + y^2} = \sqrt{5},$$

$$\tan\phi = \frac{r}{z} = \frac{1}{3}\sqrt{5}, \quad \rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{14}.$$

The cylindrical coördinates of the point are then

$$r = \sqrt{5}, \quad \theta = \tan^{-1} 2, \quad z = 3$$

and its spherical coördinates are

$$\rho = \sqrt{14}, \quad \theta = \tan^{-1} 2, \quad \phi = \tan^{-1}(\tfrac{1}{3}\sqrt{5}).$$

Exercises

1. Using cylindrical coördinates (r, θ, z) construct the points $\left(2, \frac{\pi}{2}, 1\right)$, $\left(-3, \frac{\pi}{4}, 0\right)$, $\left(2, -\frac{\pi}{6}, -1\right)$ and find their rectangular and spherical coördinates.

2. Using spherical coördinates (ρ, θ, ϕ) construct the points $\left(1, \frac{\pi}{4}, \frac{\pi}{3}\right)$, $\left(-1, \frac{\pi}{6}, 0\right)$, $\left(2, \pi, \frac{\pi}{2}\right)$ and find their rectangular and cylindrical coördinates.

3. What is the locus of all points for which $r = 1$? For which $\theta = \frac{\pi}{6}$? What is the locus of points for which $r = 1$ and $\theta = \frac{\pi}{6}$?

4. Determine the locus of points in each of the following cases:

(a) $\rho = 2$, (b) $\theta = \frac{2}{3}\pi$, (c) $\phi = \frac{\pi}{6}$, (d) $\phi = \frac{\pi}{2}$, (e) $\phi = \pi$.

5. Determine the locus of points in each of the following cases:

(a) $r = 2$, $z = 3$, (b) $\theta = 60°$, $z = -2$, (c) $r = -1$, $\theta = \pi$.

6. Determine the locus of points in each of the following cases:

(a) $\rho = 2,\ \theta = \pi$, (b) $\rho = -2,\ \phi = \frac{\pi}{6}$, (c) $\theta = \frac{\pi}{3},\ \phi = \frac{\pi}{4}$.

7. Determine the distances of a point from the coördinate axes in spherical and cylindrical coördinates.

8. The spherical coördinates of P are $\rho = 2$, $\theta = 30°$, $\phi = 45°$. Find the angles between OP and the coördinate axes.

CHAPTER 10

SURFACES

Art. 67. Loci

An equation represents a locus if every point on the locus has coördinates satisfying the equation and every point with coördinates satisfying the equation lies on the locus.

One equation between the coördinates of a point in space usually represents a surface. Thus, the equation $z = 0$ represents the xy-plane, for any point in the xy-plane has a z-coördinate equal to zero and every point with z-coördinate equal to zero lies in the xy-plane. Similarly the equation $x^2 + y^2 + z^2 = 1$ represents a sphere with radius 1 and center at the origin. In particular cases one equation may represent a straight line or curve. Thus, the only real points for which $x^2 + y^2 = 0$ are the points $x = 0, y = 0$ on the z-axis.

Two simultaneous equations usually represent a curve or straight line; for each equation represents a surface and the two equations represent the intersection of two surfaces, that is, a straight line or curve. Thus, the equations

$$x^2 + y^2 + z^2 = 3, \quad z = 1$$

represent the circle in which a sphere and plane intersect.

Three simultaneous equations are usually satisfied by the coördinates of a definite number of points. These points are found by solving the equations simultaneously. In particular cases the equations may have no solution or may be satisfied by the coördinates of all points on a curve or surface.

Art. 68. Equation of a Plane

A line perpendicular to a curve or surface is called a *normal* to that curve or surface.