CHAPTER 8

TRANSFORMATION OF COÖRDINATES

It is sometimes desirable to move the axes to a new position. The coördinates will then be changed and it is necessary to find the new coördinates. There are two simple cases a combination of which gives any motion. These are translation, in which the origin is moved to a new position without changing the direction of the axes, and rotation, in which the origin is left fixed and the axes turned like a rigid frame about it.

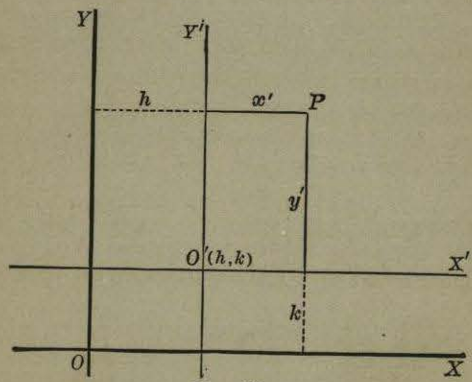

Fig. 56*a*.

Art. 56. Translation of the Axes

Let OX and OY (Fig. 56*a*) be the axes in their first position, $O'X'$ and $O'Y'$ the axes after motion. Let h, k be the coördinates of the new origin with respect to the old axes, x, y and x', y' the old and new coördinates of any point P. From Fig. 56*a* it is seen at once that

$$x = x' + h, \quad y = y' + k. \tag{56}$$

These are the equations connecting the new and old coördinates of any point.

Example 1. The origin is moved to the point $(-1, 2)$ the new axes being parallel to the old. Find the new equation of the curve

$$x^2 + 2x - y + 3 = 0.$$

The equations connecting new and old coördinates are in this case

$$x = x' - 1, \quad y = y' + 2.$$

When these values are substituted for x and y the equation of the curve becomes

$$x'^2 - y' - 0.$$

Ex. 2. Find the equation of the curve $r = a\cos\theta$ referred to a new pole at the point $\left(\frac{a}{\sqrt{2}}, \frac{\pi}{4}\right)$ the new initial line being parallel to the old.

The equations for transformation being given in rectangular coördinates, we change to rectangular coördinates, move the origin and then change back to polar coördinates. The coördinates of the new origin are

$$h = \frac{a}{\sqrt{2}}\cos\left(\frac{\pi}{4}\right) = \frac{a}{2}, \qquad k = \frac{a}{\sqrt{2}}\sin\left(\frac{\pi}{4}\right) = \frac{a}{2}.$$

The rectangular equations for transformation are therefore

$$x = x' + \frac{a}{2}, \qquad y = y' + \frac{a}{2}.$$

The rectangular equation of $r = a\cos\theta$ is

$$x^2 + y^2 = ax,$$

which transforms into

$$x'^2 + y'^2 + ay' = 0.$$

In polar coördinates this is

$$r' + a\sin\theta' = 0,$$

which is the equation required.

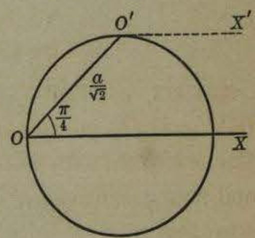

Fig. 56*b*.

Ex. 3. Transform the equation $x^3 + y^3 = 3x - 3y$ to new axes, parallel to the old, so chosen that there are no terms of first degree in the new equation.

Let $x = x' + h$, $y = y' + k$. The equation becomes

$$x'^3 + y'^3 + 3hx'^2 + 3ky'^2 + 3(h-1)x' + 3(k+1)y' + h^3 + k^3 - 3h + 3k = 0.$$

If there are no terms of first degree

$$h - 1 = 0, \quad k + 1 = 0.$$

Consequently, $h = 1$, $k = -1$ and the transformed equation is

$$x'^3 + y'^3 + 3x'^2 - 3y'^2 - 6 = 0.$$

Art. 57. Rotation of the Axes

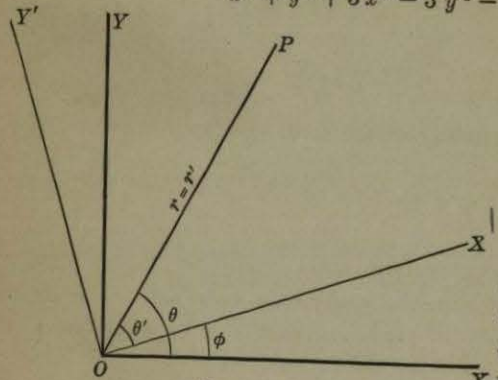

FIG. 57*a*.

The rotation is most easily expressed in polar coördinates. Let the initial line OX be rotated through an angle ϕ to OX' (Fig. 57*a*) the origin remaining fixed. Let the old and new coördinates of any point P be r, θ and r', θ'. From the figure it is seen that

$$r = r', \quad \theta = \theta' + \phi.$$

Using these relations we get

$$x = r\cos\theta = r'\cos(\theta' + \phi) = r'\cos\theta'\cos\phi - r'\sin\theta'\sin\phi,$$
$$y = r\sin\theta = r'\sin(\theta' + \phi) = r'\sin\theta'\cos\phi + r'\cos\theta'\sin\phi.$$

Replacing $r'\cos\theta'$ and $r'\sin\theta'$ by x' and y',

$$\left.\begin{aligned} x &= x'\cos\phi - y'\sin\phi, \\ y &= y'\cos\phi + x'\sin\phi. \end{aligned}\right\} \qquad (57)$$

Example 1. Find the new equation of the curve

$$x^3 + y^3 = x - y$$

after the axes have been rotated through an angle of 45°.

In this case $x = x'\cos(45°) - y'\sin(45°) = \dfrac{x' - y'}{\sqrt{2}}$,

$$y = y'\cos(45°) + x'\sin(45°) = \frac{x' + y'}{\sqrt{2}}.$$

Substituting these values and simplifying

$$x'^3 + 3x'y'^2 + 2y' = 0.$$

Ex. 2. Find the polar equation of the curve

$$x^2 - y^2 - 2xy\sqrt{3} = 2$$

after the origin is moved to the point $\left(-\dfrac{\sqrt{6}}{2}, \dfrac{\sqrt{2}}{2}\right)$ and the axes rotated through an angle of $-30°$.

The equations for moving the origin to $\left(-\dfrac{\sqrt{6}}{2}, \dfrac{\sqrt{2}}{2}\right)$ are

$$x = x' - \frac{\sqrt{6}}{2}, \qquad y = y' + \frac{\sqrt{2}}{2}.$$

If the axes are then rotated through $-30°$

$$x' = \frac{x''\sqrt{3} + y''}{2}, \qquad y' = \frac{y''\sqrt{3} - x''}{2}.$$

Consequently,

$$x = \frac{x''\sqrt{3} + y'' - \sqrt{6}}{2}, \qquad y = \frac{y''\sqrt{3} - x'' + \sqrt{2}}{2}.$$

These values substituted in the original equation give

$$x''^2 - 2\sqrt{2}\,x'' - y''^2 + 1 = 0.$$

Changing to polar coördinates

$$r^2(2\cos^2\theta - 1) - 2\sqrt{2}\,r\cos\theta + 1 = 0,$$

or

$$r = \frac{1}{\sqrt{2}\cos\theta \pm 1}.$$

The curves given by the positive and negative signs in this equation are identical. The equation required is then

$$r = \frac{1}{\sqrt{2}\cos\theta + 1}.$$

Art. 58. Invariants

Quantities associated with a curve are of two kinds, those depending on the position of the axes and those independent of the position of the axes. Quantities of the second kind are called *invariant*. For example, the radius of a circle is invariant (does not depend on the position of the axes) but the coördinates of its center are not invariant (change when the axes change position).

The equation of a curve is not invariant but there are some things connected with it that are invariant. One of the simplest is the degree of a rectangular equation. Wherever the axes are placed this degree is always the same. To see this it is only necessary to note that the equations for change of axes always have the form

$$x = ax' + by' + c, \qquad y = dx' + ey' + f,$$

where a, b, c, d, e, f are constants. A term $x^m y^n$ is equal to a sum of terms none of which are of higher degree than the $(m + n)$th in x' and y'. Hence through change of axes the degree of a polynomial cannot be increased. Neither can it be diminished for, in that case, changing back to the old axes would have to increase it. Therefore the degree of a rectangular equation is invariant.

If a rectangular equation can be factored so can the new equation resulting through a change of axes; for each factor of the old will be changed into a factor of the new. Also since real expressions are replaced by real, if either has real factors the other will have real factors also.

Art. 59. General Equation of the Second Degree

An equation of the second degree in rectangular coördinates has the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{1}$$

Rotating the axes through an angle ϕ the coefficient of $x'y'$ in the resulting equation is

$$\begin{aligned} -2A\cos\phi\sin\phi + B(\cos^2\phi - \sin^2\phi) + 2C\cos\phi\sin\phi \\ = B\cos(2\phi) + (C - A)\sin(2\phi). \end{aligned}$$

This is zero if

$$\tan(2\phi) = \frac{B}{A - C}. \tag{2}$$

If then the axes are turned through an angle ϕ satisfying equation (2), equation (1) takes the form

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0. \tag{3}$$

If A' and C' are both different from zero, completion of the squares gives an equation of the form

$$A'(x - h)^2 + C'(y - k)^2 = R.$$

If R is zero this equation is reducible. If R is not zero the equation can be written

$$\frac{(x' - h)^2}{R/A'} + \frac{(y' - k)^2}{R/C'} = 1.$$

The locus of this equation is an ellipse if the denominators are both positive, a hyperbola if one is positive and one negative, and is imaginary if both are negative.

Let $A' = 0$. Completion of the square in y' terms then gives

$$C'(y' - k)^2 = -D'(x' - h).$$

The locus of this equation is a parabola. Similarly if C' is zero the locus is a parabola. A' and C' cannot both be zero as the equation would then be one of the first degree.

We have thus found that *the second degree equation in rectangular coördinates is either reducible or else its locus is an ellipse, a parabola, a hyperbola, or an imaginary curve.*

Exercises

1. What are the rectangular coördinates of the points (2, 3), (−4, 5) and (5, −7) referred to parallel axes through the point (3, −2)?

2. Find the polar coördinates of the points $\left(3, \frac{\pi}{3}\right)$, $\left(-2, \frac{\pi}{2}\right)$, $\left(4, \frac{2\pi}{3}\right)$ if the origin is moved to the point $\left(2, \frac{\pi}{3}\right)$, the new initial line having the same direction as the old.

3. Find the equation of the curve
$$x^2 + 4y^2 - 2x + 8y + 1 = 0$$
referred to parallel axes through the point (1, −1).

4. Find the equation of the curve
$$y^3 - 6y^2 + 3x^2 + 12y - 18x + 35 = 0$$
referred to parallel axes through (3, 2).

5. Find the equation of the circle
$$r^2 - 5\sqrt{3}\, r\cos\theta - 5r\sin\theta + 21 = 0$$

when the origin is moved to the point $\left(5, \frac{\pi}{6}\right)$, the new initial line being parallel to the old.

6. Find the equation of the parabola

$$r = \frac{2p}{1 - \cos\theta}$$

when the pole is moved to $(-p, 0)$, the direction of the initial line being unchanged.

7. Find the equation of the line

$$2x - 3y + 7 = 0$$

referred to parallel axes through the point $(-5, -1)$.

8. Find the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

when referred to parallel axes through the left-hand vertex.

9. Find the equation of the witch

$$y = \frac{8a^3}{x^2 + 4a^2}$$

referred to parallel axes through the intersection of the y-axis and the curve.

10. Find the equation of the strophoid

$$y^2 = \frac{x^2(a - x)}{a + x}$$

referred to parallel axes through the intersection of the x-axis and the asymptote of the curve.

11. Find the parametric equations of the cycloid

$$x = a(\phi - \sin\phi), \quad y = a(1 - \cos\phi)$$

referred to parallel axes through the highest point of an arch of the curve.

12. Find parametric equations for the cardioid

$$r = a(1 + \cos\theta)$$

referred to parallel axes through the point $x = \frac{1}{2}a$, $y = 0$.

13. Transform the equations

$$x + y - 3 = 0, \quad 2x - 3y + 4 = 0$$

to parallel axes so chosen that the new equations have no constant terms.

14. Transform the equation

$$x^2 + y^2 - 2x + 4y - 6 = 0$$

to parallel axes so chosen that the new equation has no terms of first degree in x and y.

15. Show that by a change to parallel axes the equation

$$y^2 + 4y - 2x + 3 = 0$$

can be reduced to the form

$$y'^2 = 2x'.$$

16. Show that by moving the origin to the point $(p, 0)$ and changing to polar coördinates the equation

$$y^2 = 4px$$

can be reduced to the form

$$r' = \frac{2p}{1 - \cos\theta'}.$$

17. Show that by a change to parallel axes the equation

$$x^2 + 4y^2 - 3x + 5y = 6$$

can be reduced to the form

$$x'^2 + 4y'^2 = b^2,$$

where b is a constant.

18. By a change to parallel axes reduce

$$2x^2 - 6y^2 + 3x - 4y = 12$$

to the form $(x/a)^2 - (y/b)^2 = 1$.

19. What are the new coördinates of the points (0, 1), (1, 0), (1, 1), (−1, −1) after the axes have been rotated through an angle of 60°?

20. Transform the equation

$$x^2 + xy + y^2 = 1$$

to new axes bisecting the angles between the original axes.

21. Find the equation of the curve

$$r^2 \sin\theta \cos\theta = 1$$

referred to the same pole but with the initial line rotated through 45°.

22. By rotating the axes through the proper angle transform the equation $xy = 4$ to a form without xy-term.

23. By a proper change of axes reduce $x^2 + xy = 2$ to the form $(x'/a)^2 - (y'/b)^2 = 1$.

24. By a proper change of axes reduce the equation

$$x + 3y - 4 = 0$$

to the form $x' = 0$.

25. If the curve $y^2 = 4x$ is referred to new axes show that the second degree terms in the new equation form a complete square.

26. If the curve $x^2 + 4y^2 - 5 = 0$ is referred to new axes show that the second degree part of the new equation has imaginary factors.

27. If the curve $x^2 - y^2 - 1 = 0$ is referred to new axes show that the second degree part of the new equation has real factors.