CHAPTER 7

PARAMETRIC REPRESENTATION

Art. 52. Definition of Parameter

In some cases it is more convenient to express the coördinates of a point on a curve in terms of a third variable than in terms of each other. Such a third variable is often called a *parameter* and the equations connecting the coördinates with the parameter are called *parametric equations*.

Example 1. A rod 5 feet long moves with its ends in the coördinate axes. Find the locus of the point P, 2 feet from the end in the x-axis.

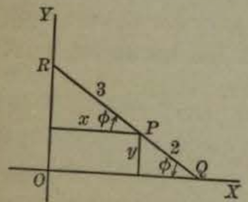

FIG. 52*a*.

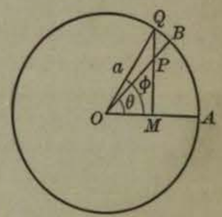

FIG. 52*b*.

Let QR be the rod and $\phi = \angle OQR$ (Fig. 52*a*). By hypothesis $PQ = 2$ and $PR = 3$. Let x, y be the coördinates of P. From Fig. 52*a* it is seen that

$$x = 3\cos\phi, \quad y = 2\sin\phi.$$

These are the equations expressing the coördinates of P in terms of the parameter. To find the equation connecting x and y we eliminate the parameter. The result is

$$\frac{x^2}{9} + \frac{y^2}{4} = \sin^2\phi + \cos^2\phi = 1.$$

The locus of P is therefore an ellipse.

Ex. 2. Find the locus of the point P, intersection of OB and MQ, (Fig. 52*b*) given that $MQ = \text{arc } AB$.

Let the radius of the circle be a and $\angle AOQ = \phi$. Let r, θ be the polar coördinates of P. Then

$$\text{arc } AB = a\theta, \quad MQ = a\sin\phi.$$

Consequently, $\theta = \sin\phi$. Also

$$r = \frac{OM}{\cos\theta} = \frac{a\cos\phi}{\cos(\sin\phi)}.$$

Polar parametric equations of the locus are therefore

$$\theta = \sin\phi, \quad r = a\cos\phi\sec(\sin\phi).$$

Elimination of ϕ gives the equation connecting r and θ in the form

$$r = a\sqrt{1-\theta^2}\sec\theta.$$

Almost any quantity that varies from point to point of a curve can be used as a parameter. The equations connecting the coördinates with the parameter naturally depend on the parameter. There are then an infinite number of parametric equations of the same curve. The coördinate axes and the parameter must be fixed before the parametric equations have a definite form.

Exercises

1. A circle of radius a has its center at the origin. Express the coördinates x, y of any point P on the circle in terms of the angle ϕ between the x-axis and OP and so obtain parametric equations for the circle.

2. $A\ (0, a)$ is a fixed point on the y-axis and M a movable point on the x-axis. MP is perpendicular to AM and equal in length to it. Express the coördinates of P in terms of a and the angle ϕ from the right end of the x-axis to MP. By eliminating ϕ find the rectangular equation of the locus of P.

3. O is the center of a fixed circle tangent to AB at A. Through any point Q of AB passes another line tangent to the circle at R. On RQ determine the point P such that $RQ = QP$. Taking OA as x-axis and O as origin express the coördinates of P in terms of $\phi = \angle AOQ$. By eliminating the parameter ϕ determine the coördinate equation of the locus of P.

4. A segment AB through the point $C\ (2, 1)$ has its ends A and B in the x- and y-axes respectively. If $P\,(x, y)$ is the middle point of AB

express x and y in terms of the angle $\phi = OAB$. By eliminating ϕ determine the rectangular equation of the curve described by P as AB turns about C.

5. A string, held taut, is unwound from a circle. Taking the origin at the center of the circle and the initial line through the point where the string begins to unwind, express the polar coördinates of the point P at the end of the string in terms of the radius of the circle and the angle at the center subtended by the arc unwound. Find the polar equation of the locus of P.

Art. 53. Locus of Parametric Equations

Suppose the coördinates x and y are given functions of a parameter ϕ,

$$x = f_1(\phi), \qquad y = f_2(\phi). \tag{1}$$

If a value is assigned to ϕ the resulting values of x and y are coördinates of a certain point. The totality of such points will usually be found to form a curve. The above equations represent the curve in the sense that if any value be assigned to ϕ the resulting point lies on the curve and if any point (x, y) be taken on the curve there is a value of ϕ such that $x = f_1(\phi)$, $y = f_2(\phi)$.

To plot the curve we assign values to ϕ, considered as independent variable, calculate x and y and plot the resulting points.

To find the coördinate equation eliminate ϕ between the parametric equations. Let the result be

$$f(x, y) = 0. \tag{2}$$

This equation, being a consequence of the parametric equations, is satisfied by the coördinates of any point on the curve. If, conversely, for every pair of values x, y satisfying (2) a value of ϕ can be found such that $x = f_1(\phi)$, $y = f_2(\phi)$, then (2) is the coördinate equation of the curve.

Example 1. Plot the curve whose parametric equations are $x = t$, $y = t^3$ and find its rectangular equation.

t	x	y
$\pm\infty$	$\pm\infty$	$\pm\infty$
± 2	± 2	± 8
$\pm\frac{3}{2}$	$\pm\frac{3}{2}$	$\pm\frac{27}{8}$
± 1	± 1	± 1
$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	$\pm\frac{1}{8}$
0	0	0

In the table values of x and y are placed after the corresponding values of the parameter t. The points represented by these pairs of coordinates are plotted (Fig. 53a) and a smooth curve drawn through

the resulting points. By eliminating t the rectangular equation of the curve is found to be $y = x^3$.

Ex. 2. $x = 3\cos\phi$, $y = 2\sin\phi$. While ϕ increases from 0 to $\frac{\pi}{2}$,

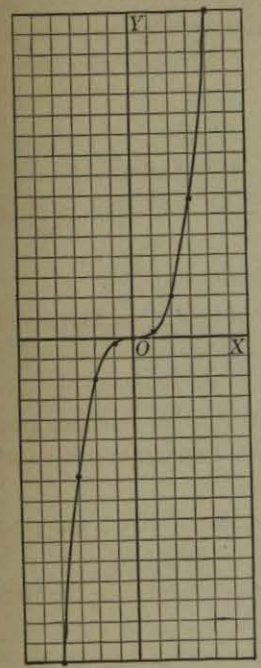

FIG. 53a.

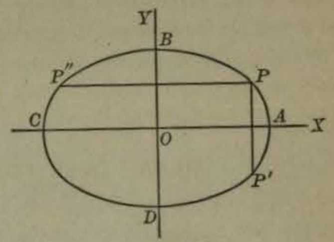

FIG. 53b.

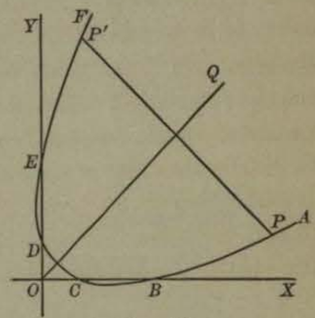

FIG. 53c.

x decreases from 3 to 0 and y increases from 0 to 2 (AB, Fig. 53b). As ϕ continues to π, y decreases to 0 while x decreases to -3 (BC, Fig. 53b). Then y decreases to -2 at $\phi = \frac{3}{2}\pi$ while x increases to 0. Finally, x increases to 3 and y to 0 at $\phi = 2\pi$. The symmetry with respect to the x-axis could have been foreseen since a change in the sign of ϕ makes a change in the sign of y but leaves x unchanged (P to P'). The symmetry with respect to the y-axis is indicated by the fact that a change from ϕ to $\pi - \phi$ changes the sign of x and makes no change in y (P to P'').

Ex. 3. $x = t(t-1)$, $y = (t+1)(t+2)$. The curve crosses the x-axis at $t = -1$ and $t = -2$ (C and B, Fig. 53c). It crosses the y-axis at $t = 0$ and $t = 1$ (D and E). When t is a large negative number x and y are both positive. As t increases to -2, x and y both decrease (A to B). As t goes from -2 to -1, y is negative, x positive and decreasing (B to C). Between $t = -1$ and $t = 0$, y is positive and increasing, x positive and decreasing (C to D). Between $t = 0$ and $t = 1$, x is negative, y positive and increasing (D to E). When $t > 1$, x and y are both positive and increase with t (E to F). If t is replaced by $-t-1$, the x and y coördinates interchange values (P to P'). Hence the curve is symmetrical with respect to the line OQ bisecting the angle between the axes. By subtraction

$$(1) \qquad y - x = 4t + 2, \quad \text{or} \quad t = \tfrac{1}{4}(y - x - 2).$$

Substituting t in the equation $x = t(t-1)$ and simplifying

$$(2) \qquad (y - x)^2 - 8(y + x) + 12 = 0.$$

Conversely, if (1) and (2) are solved for x and y the parametric equations are obtained. Therefore (2) is the rectangular equation of the curve. It is a parabola.

Ex. 4. $x = \cos(2\phi)$, $y = \sin\phi$. When $\phi = 0$ we get the point $A\,(1, 0)$. As ϕ increases, x decreases and y increases until $\phi = \dfrac{\pi}{2}$ at $B\,(-1, 1)$. As ϕ continues to increase the point turns back, retraces the arc BA and continues to $C\,(-1, -1)$ where $\phi = \frac{3}{2}\pi$. As ϕ continues to increase the point oscillates back and forth along the path CAB (Fig. 53*d*).

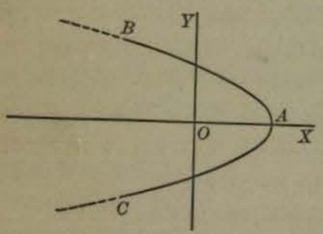

FIG. 53*d*.

Since $x = \cos(2\phi) = 1 - 2\sin^2\phi$ and $y = \sin\phi$,

$$x = 1 - 2y^2.$$

This equation is not however equivalent to the parametric equations. It represents a parabola extending to an infinite distance on the left, whereas the parametric equations (since $\sin\phi$ and $\cos 2\phi$ are never greater than 1) represent only the piece CAB.

Ex. 5. Find the intersections of the curves $x = a\cos\theta$, $y = b\cos(2\theta)$ and $x = a\sin\phi$, $y = b\sin(2\phi)$.

At a point of intersection both pairs of parametric equations are satisfied. Hence

$$\sin\phi = \cos\theta, \quad \sin(2\phi) = \cos(2\theta).$$

The general solution of the first of these equations is

$$\phi = \frac{\pi}{2} + 2n\pi - \theta$$

where n is any positive or negative integer. This value substituted in the second equation gives

$$\sin(2\theta) = \cos(2\theta).$$

Consequently $\tan(2\theta) = 1$ and $\theta = \dfrac{n\pi}{2} + \dfrac{\pi}{8}$. Hence

$$x = \pm a\cos\left(\frac{\pi}{8}\right), \quad \pm a\sin\left(\frac{\pi}{8}\right),$$

$$y = \quad b\cos\left(\frac{\pi}{4}\right), \quad -b\cos\left(\frac{\pi}{4}\right).$$

Art. 54. Parametric from Coördinate Equations

When the parametric equations of a curve are given its coördinate equation is obtained by eliminating the parameter. The converse problem is, given the coördinate equation and the definition of a parameter to find the parametric equations. To do this we use the definition of the parameter to obtain at least one equation connecting the coördinates and the parameter. This and the coördinate equation give two equations in three unknowns (two coördinates and a parameter). By solving these equations for the coördinates we obtain the parametric equations.

Example 1. Find parametric equations of the curve

$$x^3 + y^3 = xy,$$

using the ratio y/x as parameter.

Call the parameter t. Then by hypothesis

$$(1) \qquad \frac{y}{x} = t.$$

This and the equation of the curve, solved simultaneously, give

(2) $$x = \frac{t}{1+t^3}, \quad y = \frac{t^2}{1+t^3}.$$

If we eliminate t from these equations we get the coördinate equation. Therefore the equations (2) are parametric equations of the curve; for if x, y are the coördinates of a point on the curve a value $t = y/x$ can be found such that x, y, t satisfy the parametric equations and conversely if x, y, t are any numbers satisfying the parametric equations then x, y are the coördinates of a point on the curve.

Ex. 2. Find the parametric equations of the curve

$$xy = 2x + 2y - 3,$$

using as parameter the slope of the line joining (x, y) and $(1, 1)$.

Call the parameter t. By hypothesis

(3) $$t = \frac{y-1}{x-1}.$$

This and the equation of the curve, solved simultaneously, give

(4) $$x = 2 + \frac{1}{t}, \quad y = t + 2.$$

Conversely, elimination of t from (4) gives the equation of the curve. Consequently, the equations (4) are parametric equations of the given curve.

Ex. 3. Find parametric equations of the parabola

$$x = 1 - y^2,$$

using the parameter ϕ defined by the equation $y = \sin\phi$.

Substituting $y = \sin\phi$ in the equation of the curve we get $x = 1 - \sin^2\phi = \cos^2\phi$. The equations obtained are therefore

$$x = \cos^2\phi, \quad y = \sin\phi.$$

Since x and y given by these equations cannot be greater than 1, these are not parametric equations of the whole parabola but only of a piece of it. This is due to the fact that when y is numerically greater than 1 there is no parameter ϕ defined by the equation $y = \sin\phi$.

Exercises

Plot the curves represented by the following parametric equations and determine the corresponding coördinate equations:

1. $x = 1 + 2t, \quad y = 2 - 3t.$
2. $x = t^2, \quad y = 2t.$
3. $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}.$
4. $x = t^2(t-1), y = t^2(t+1).$
5. $x = \frac{t+1}{t-1}, \quad y = \frac{t}{t-2}.$
6. $x = \sin\phi, \; y = \sin\left(\phi + \frac{\pi}{6}\right).$
7. $x = \sec\phi, \quad y = \tan\phi.$
8. $x = t\sin t, \quad y = t\cos t.$
9. $r = \cos(2\phi), \quad \theta = \sin\phi.$
10. $r = t + \frac{1}{t}, \quad \theta = t - \frac{1}{t}.$
11. $r = \phi\sin\phi, \quad \theta = \phi\cos\phi.$
12. $x = a(\theta - \sin\theta), y = a(1 - \cos\theta).$
13. $r = \tan\phi, \quad \theta = \sin(3\phi).$
14. $x = a(3\cos\phi + \cos 3\phi),$ $y = a(3\sin\phi - \sin 3\phi).$

15. Plot two turns of the curve

$$x = \cos\phi, \quad y = \sin\left(\frac{7\pi\phi}{22}\right)$$

and show that the complete curve passes indefinitely near any point within a unit square.

16. Find the polar parametric equations of the curve

$$x = t\cos t, \quad y = t\sin t,$$

using the same parameter.

17. Find rectangular parametric equations of the curve

$$r = t^2, \quad \theta = 1 + t.$$

18. Show that $x = \sin t$, $y = \cos(2t)$ and $x = \sin(2t)$, $y = \cos(4t)$ are the same curve.

19. Are $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$ and $x = 2^t + 2^{-t}$, $y = 2^t - 2^{-t}$ the same curve?

20. Are $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$ and $x = \cos\theta + \sec\theta$, $y = \cos\theta - \sec\theta$ the same curve?

Find the intersections of the following pairs of curves:

21. $\left.\begin{matrix} x = t^2 \\ y = 2t \end{matrix}\right\}, \; x^3 + y^3 = 8xy.$

22. $\left.\begin{matrix} x = 5\cos\theta \\ y = 5\sin\theta \end{matrix}\right\}, \; \left.\begin{matrix} x = 1 + t \\ y = 2 + t \end{matrix}\right\}.$

23. $\left.\begin{matrix} x = a\cos\theta \\ y = a\sin\theta \end{matrix}\right\}, \; \left.\begin{matrix} x = a\phi\cos\phi \\ y = a\phi\sin\phi \end{matrix}\right\}.$

24. $\left.\begin{matrix} x = 2a\cos^2\phi \\ y = \dfrac{2a\cos^3\phi}{\sin\phi} \end{matrix}\right\}, \; r = 4a\cos\theta.$

25. Find rectangular parametric equations of the circle, radius a, with center on the x-axis and passing through the origin, using as parameter the polar coördinate θ.

26. Find parametric equations for the parabola

$$y^2 = 4x,$$

using the ratio $\dfrac{y}{x}$ as parameter.

27. Find parametric equations of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

using the parameter ϕ defined by the equation $x = a\cos\phi$.

28. Find parametric equations for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

using the parameter ϕ defined by the equation $x = a\sec\phi$.

29. Find parametric equations of the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

using the parameter ϕ defined by the equation $x = a\sin^3\phi$.

30. Find parametric equations of the curve

$$y^2 + 4y = x^3 - 2x^2 + x - 4,$$

using as parameter the slope of the line joining (x, y) and $(1, -2)$.

Art. 55. Locus Problems

The Cycloid. — When a circle rolls along a straight line a point of its circumference describes a curve called a cycloid.

Let a circle of radius a and center C roll along the x-axis (Fig. 55a). Let N be the point of contact with the x-axis and $P\,(x, y)$

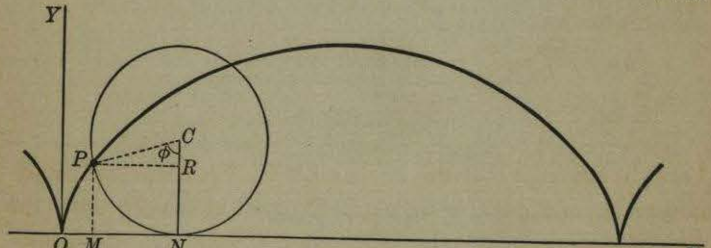

FIG. 55a.

the point tracing the cycloid. Take as origin the point O found by rolling the circle to the left until P meets OX. Take as parameter the angle $NCP = \phi$. Since the point of contact moves the same distance along the circle as along the straight line,

$$ON = \text{arc}\,NP = a\phi.$$

Consequently

$$x = OM = ON - MN = ON - PR = a\phi - a\sin\phi,$$
$$y = MP = NC - RC = a - a\cos\phi.$$

The parametric equations of the cycloid are then

$$x = a\,(\phi - a\sin\phi), \quad y = a\,(1 - \cos\phi).$$

The Epicycloid. — When a circle rolls on the outside of a fixed circle a point on its circumference describes an epicycloid.

Let a circle of radius a and center C roll on the outside of a circle of radius b and center O. Let N be the point of contact and $P\,(x, y)$ the point describing the epicycloid. Let A be the point obtained by rolling the moving circle backward until P meets the fixed circle.

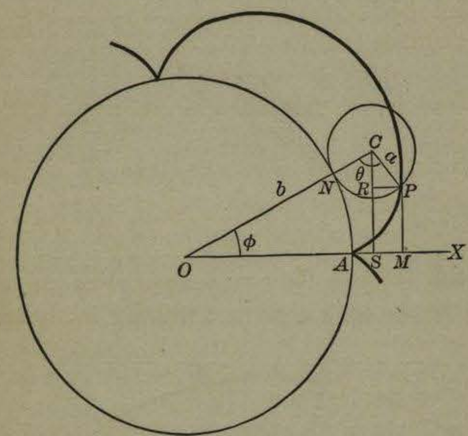

FIG. 55b.

Let O be the origin, OA the x-axis. Let $OCP = \theta$ and take as parameter the angle $AOC = \phi$. Since the point of contact moves the same distance along both circles $\text{arc}\,AN = \text{arc}\,NP$ and consequently $b\phi = a\theta$. Also

$$RCP = OCP - OCR = \theta - \left(\frac{\pi}{2} - \phi\right) = \theta + \phi - \frac{\pi}{2}.$$

Therefore

$$x = OM = OS + RP = OC\cos\phi + CP\sin(RCP)$$
$$= (a+b)\cos\phi - a\cos(\theta+\phi) = (a+b)\cos\phi - a\cos\left(\frac{a+b}{a}\phi\right).$$

$$y = MP = SC - RC = OC \sin\phi - CP \cos(RCP)$$

$$= (a+b)\sin\phi - a\sin(\theta+\phi) = (a+b)\sin\phi - a\sin\left(\frac{a+b}{a}\phi\right).$$

The parametric equations of the epicycloid are then

$$x=(a+b)\cos\phi - a\cos\left(\frac{a+b}{a}\phi\right),\quad y=(a+b)\sin\phi - a\sin\left(\frac{a+b}{a}\phi\right).$$

Exercises

1. A circle of radius a moves with its center in the x-axis and a straight line passes through the center of the circle and a fixed point on the y-axis. Using as parameter the angle between the line and y-axis, find parametric equations for the curves traced by the intersections of moving line and circle.

2. Let AB be a given line, O a given point k units distant from AB. Draw any line through O meeting AB in M and let P be the point on this line such that

$$OM \cdot MP = k^2.$$

Find the parametric equations of the locus of P, using O as the origin, the perpendicular from O to AB as x-axis and the angle between OX and OP as parameter.

3. Let OA be the diameter of a fixed circle and LK the tangent at A. A variable line through O intersects the circle at B and LK at C. Through B draw a line parallel to LK and through C a line perpendicular to LK and call the intersection of these lines P. The locus of P is a curve called the witch. Find its parametric equations using the tangent at O as x-axis and the angle from the x-axis to OC as parameter. Also find the rectangular and polar equations of the curve.

4. Let O be the center of a circle, radius a, A a fixed point and B a moving point on the circle. If the tangent at B meets the tangent at A in C and P is the middle point of BC, find the equations of the locus of P using the angle AOB as parameter. Also find the rectangular equation.

5. $OBCD$ is a rectangle with $OB = a$, $BC = c$. Any line is drawn through C meeting OB in E and the triangle EPO is constructed so that the angles CEP and EPO are right angles. Find the locus of P, using the angle DOP as parameter, OB as x-axis and O as origin. Also find the rectangular equation of the locus.

6. A $(0, -a)$ and B $(0, a)$ are two fixed points on the y-axis. H is a variable point on the x-axis. BK is the perpendicular from B to AH meeting it in K. Through K a line is drawn parallel to the x-axis and through H a line is drawn parallel to the y-axis. These lines meet in P.

Find the equations of the locus of P using the angle BAK as parameter. Also find the rectangular and polar equations of the locus.

7. Let OA be the diameter of a fixed circle and LK the tangent at A. Through O draw any line intersecting the circle in B and LK in C and let P be the middle point of BC. Find the equations of the locus of P, using the angle AOP as parameter, OA as y-axis and O as origin. Find the rectangular and polar equations of the same curve.

8. OA is a diameter of a fixed circle and LK the tangent at A. Through O any line is drawn meeting the circle in B and LK in C. Through B a line is drawn perpendicular to OA meeting it in M. MB is prolonged to P so that $MP = OC$. Find the locus of P.

9. CD is perpendicular to OX and distant a units from O. A is a moving point on CD. AB is drawn perpendicular to OA meeting OX in B. BP is perpendicular to OX meeting OA in P. Find the locus of P.

10. Through a moving point B on a fixed circle lines from the ends of a diameter are extended a distance equal to the radius to form the sides of a square whose diagonal is BP. Find the locus of P.

11. The sides of a right angle are tangent to two fixed circles. Find the locus of the vertex.

12. Through two fixed points lines are drawn to form an isosceles triangle with its base in a fixed line. Find the locus of their point of intersection.

13. The angles of a triangle are A, B, C and the opposite sides are a, b, c. If the vertex A moves along the x-axis and B along the y-axis find the locus of C, using the angle between the side AB and the x-axis as parameter.

14. A string is wound around a circle and the end fastened at the center of the circle. A pencil resting against the string keeps it taut. Find the curve described as the string unwinds from the circle.

15. When a wheel rolls along a straight line any point in a spoke describes a trochoid. Let the wheel roll along the x-axis and use as parameter the angle ϕ in the equation of the cycloid. Find the parametric equations of the trochoid described by the point at distance b from the center of the circle.

16. A hypocycloid is the locus described by a point on the circumference of a circle which rolls internally on the circumference of a fixed circle. Find the parametric equations of the hypocycloid when the radius of the moving circle is $\frac{1}{4}$ that of the fixed circle, using a parameter analogous to that in the equations of the epicycloid.

17. A circle with center at the point $(2, 0)$ intersects a circle with center $(0, 2)$ in a point of the line $x = 3$. Find the locus of the other point of intersection of the two circles.