4. The following table gives the electromotive force E, in microvolts, produced in a lead and cadmium thermo-electric couple when the difference in temperature between the junctions is $\theta°$ C.

$\theta =$	−200	−100	0	100	200	300
$E =$	50	−140	0	475	1300	2425

5. The following table gives the number of grams S of anhydrous ammonium chloride which dissolved in 100 grams of water make a saturated solution at $\theta°$ absolute temperature.

$\theta =$	273	283	288	293	313	333	353	373
$S =$	29.4	33.3	35.2	37.2	45.8	55.2	65.6	77.3

6. The hysteresis losses in soft sheet iron subjected to an alternating magnetic flux are given in the following table, where B is flux density in kilolines per square inch, and P is watts lost per cubic inch for one cycle per second.

$B =$	20	40	60	80	100	120
$P =$	0.0022	0.0067	0.0128	0.0202	0.0289	0.0387

7. The observed temperatures θ of a vessel of cooling water at times t, in minutes, from the beginning of observation are given in the following table:

$t =$	0	1	2	3	5	7	10	15	20
$\theta =$	92°	85.3°	79.5°	74.5°	67°	60.5°	53.5°	45°	39.5°

8. Measurements showing the decay in activity of radium emanation are given in the following table:

Time in hours =	0	20.8	187.6	354.9	521.9	786.9
Relative activity =	100	85.7	24.0	6.9	1.5	0.19

CHAPTER 6

POLAR COÖRDINATES

Art. 45. Definitions

We shall now define another kind of coördinates called *polar*. Let O (Fig. 45a) be a fixed point and OX a fixed line. The point

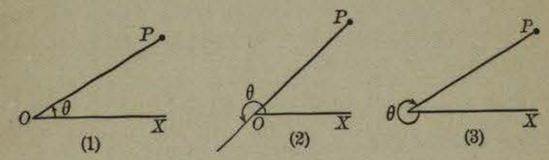

FIG. 45a.

O is called the *pole*, or origin, the line OX is called the *initial line*, or axis. The polar coördinates of a point P are the radius $r = OP$ and the angle θ from OX to OP.

The angle θ is any angle extending from OX to the line OP, the angle being considered positive when measured in the counter-clockwise direction (Fig. 45a, 1 or 2) and negative when measured in the clockwise direction (Fig. 45a, 3).

The radius r is considered positive when OP is the terminal side of θ (Fig. 45a, 1 or 3) and negative when θ terminates on OP produced.

A given point P is seen to have many pairs of polar coördinates, θ being any angle from OX to OP. A given pair of polar coördinates, however, determines a definite point obtained by constructing the angle θ and laying off r forward or backward along the terminal side according as r is positive or negative.

The point whose polar coördinates are r, θ is represented by the symbol (r, θ). To signify that P is the point (r, θ) the notation $P\,(r, \theta)$ is used.

Example 1. Plot the point $P(-1, -\frac{1}{4}\pi)$ and find its other pairs of polar coördinates.

The point is shown in Fig. 45*b*. The angle $-\frac{1}{4}\pi$ terminates on OP produced. The angle XOP is equal to $\frac{3}{4}\pi$. Any other angle terminating on OP or OP produced differs from one of these only by a positive or negative multiple of 2π. Any such angle then has one of the forms $2n\pi - \frac{1}{4}\pi$ or $2n\pi + \frac{3}{4}\pi$, where n is a positive or negative integer. Consequently any pair of polar coördinates of P has one of the forms

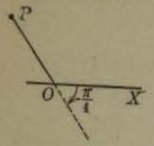

FIG. 45*b*.

$$\left(-1,\ 2n\pi - \frac{\pi}{4}\right), \qquad \left(1,\ 2n\pi + \frac{3}{4}\pi\right).$$

Ex. 2. Plot the point whose polar coördinates are (3, 4).

The coördinates of the point are

$$r = 3, \quad \theta = 4.$$

This means that the circular measure of θ is 4. The angle is approximately 229°. A line is drawn making this angle with OX and on the line the point P is located at a distance 3 from the origin (Fig. 45*c*).

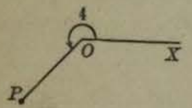

FIG. 45*c*.

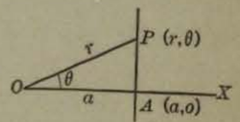

FIG. 45*d*.

Equation of a Locus. — The polar equation of a locus (like its rectangular equation) is satisfied by a pair of coördinates of every point on the locus and not satisfied by the coördinates of any point not on the locus.

Example 1. Find the polar equation of the line through $A\ (a, 0)$ perpendicular to the initial line.

Let $P(r, \theta)$ be any point on the line. From Fig. 45*d* it is seen that

$$r\cos\theta = a.$$

Conversely, any point whose coördinates satisfy this equation lies on the line, for the equation expresses that the projection of OP on OX is a.

Ex. 2. Find the polar equation of the circle whose diameter is the segment from the origin to the point $A\ (a, 0)$.

Let $P\ (r, \theta)$ be any point on the circle (Fig. 45*e*). In the right triangle OAP

$$r = OP = OA\cos\theta = a\cos\theta.$$

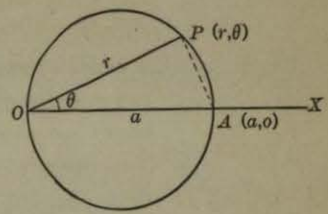

FIG. 45*e*.

Conversely, if $r = a\cos\theta$, the angle at P is a right angle and P lies on the circle. Hence $r = a\cos\theta$ is the equation required.

Exercises

1. Plot the points (0, 30°), (1, 40°), $\left(-2, \frac{\pi}{4}\right)$, $\left(3, -\frac{\pi}{3}\right)$, (−4, −730°).

2. Plot the points whose polar coördinates are (1, 1), (−1, 2), (2, −3), $(\sqrt{2}, \sqrt{3})$.

3. Show graphically that the points $(2\sqrt{2}, 0)$, $\left(2, \frac{\pi}{4}\right)$, $\left(2\sqrt{2}, \frac{\pi}{2}\right)$, $(\infty, \frac{3}{4}\pi)$ lie on a line. On this line what value of r corresponds to $\theta = \frac{5}{4}\pi$?

4. Show graphically that the points (0, 0), $(\frac{3}{2}\sqrt{3}, 60°)$, (3, 90°), $(-\frac{3}{2}, 210°)$ lie on a circle. What is its radius?

5. Show that $\left(1, \frac{\pi}{2}\right)$ and $\left(-1, -\frac{\pi}{2}\right)$ are the same point. Give other pairs of coördinates of this point.

6. Given the point $P(r, \theta)$, find the coördinates of the point Q if P and Q are symmetrical with respect to the initial line; symmetrical with respect to the origin; symmetrical with respect to the line through the origin perpendicular to the initial line.

7. Find the polar equation of the line through the origin making an angle of $\frac{\pi}{6}$ with the initial line. Does $\left(2, \frac{7}{6}\pi\right)$ lie on the line? Do its coördinates satisfy the equation?

8. Find the equation of the circle, radius a, with center at the origin.

9. Find the polar equation of the line parallel to the initial line and passing through the point $\left(a, \frac{\pi}{2}\right)$.

10. Find the polar equation of the circle, radius a, tangent to the initial line at the origin.

Art. 46. Change of Coördinates

The same point can be represented by rectangular or by polar coördinates. It is sometimes desirable to use both systems simultaneously. In this case the x-axis is usually coincident with the initial line and the origin of rectangular coördinates is the pole. A point P (Fig. 46a) then has four coördinates, x, y, r, θ. These are connected by several equations the most important of which are

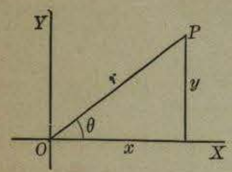

FIG. 46a.

$$\left.\begin{aligned} x &= r\cos\theta, & r &= \pm\sqrt{x^2+y^2}, \\ y &= r\sin\theta, & \tan\theta &= \frac{y}{x}. \end{aligned}\right\} \qquad (46)$$

By the use of these equations (or, better, by the use of Fig. 46a) any expression in rectangular coördinates can be converted into one in polar coördinates and conversely.

Example 1. Find the polar equation of the circle $x^2 + y^2 = x + y$.

From Fig. 46a it is seen that

$$x^2 + y^2 = r^2, \quad x + y = r\cos\theta + r\sin\theta.$$

Hence the polar equation of the circle is

$$r^2 = r\cos\theta + r\sin\theta$$

or

$$r = \cos\theta + \sin\theta.$$

Ex. 2. By changing to rectangular coördinates show that

$$r\,(2\cos\theta + 3\sin\theta) = 4$$

is the polar equation of a straight line.

Since $r\cos\theta = x$, $r\sin\theta = y$, the rectangular equation is

$$2x + 3y = 4,$$

which is the equation of a line whose intercepts are $x = 2$, $y = \frac{4}{3}$.

Art. 47. Straight Line and Circle

Polar Equation of a Straight Line. — Let LK (Fig. 47a) be the line and let OD be the perpendicular upon it from the origin. Let

$$OD = p, \quad XOD = \alpha.$$

If $P\,(r, \theta)$ is any point on the line, $OP = r$, $XOP = \theta$. In the right triangle DOP, $OP\cos(DOP) = OD$, that is,

$$r\cos(\theta - \alpha) = p, \qquad (47a)$$

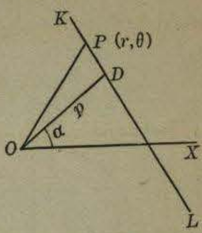

FIG. 47a.

which is the equation required.

Polar Equation of a Circle. — If the circle passes through the origin (Fig. 47b) let its radius be a and its center (a, α). Let A be the end of the diameter through the origin and let $P\,(r, \theta)$ be any point on the circle. In the triangle AOP,

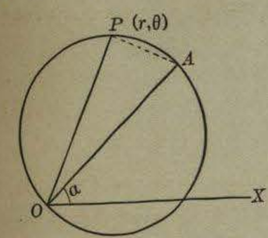

FIG. 47b.

$$OA = 2a, \quad OP = r, \quad AOP = \theta - \alpha.$$

Since $OP = OA\cos(AOP)$, the equation of the circle is then

$$r = 2a\cos(\theta - \alpha). \qquad (47b)$$

If the circle does not pass through the origin (Fig. 47c) let its radius be a and its center $C\,(b, \beta)$. Let $P\,(r, \theta)$ be any point on the circle. In the triangle COP, $CP^2 = OP^2 + OC^2 - 2\,OC \cdot OP\cos(COP)$, that is,

$$a^2 = r^2 + b^2 - 2\,br\cos(\theta - \beta),$$

which is the equation required.

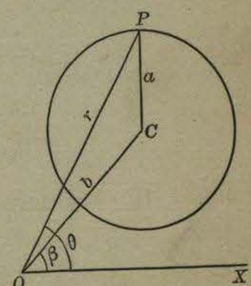

FIG. 47c.

Art. 48. The Conic

The locus of a point moving in such a way that its distance from a fixed point is proportional to its distance from a fixed straight line is called a conic. The fixed point is called a *focus*, the fixed line a *directrix* of the conic. The constant ratio is called the *eccentricity* of the curve. The ellipse, parabola and hyperbola are all conics. The name conic refers to the fact that any section of a cone is a conic.

Let P be any point on a conic whose focus is F and directrix RS (Fig. 48*a*). If e is the eccentricity, the definition of the curve is

$$FP = eNP.$$

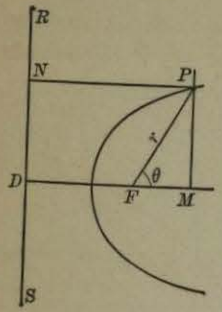

FIG. 48*a*.

Take F as origin and the line through F perpendicular to RS as the x-axis. Let $DF = k$. Then

$$FP = r, \quad NP = DF + FM = k + r\cos\theta.$$

Hence

$$r = e(k + r\cos\theta), \quad \text{or} \quad r = \frac{ke}{1 - e\cos\theta}. \qquad (48a)$$

This is the equation of the conic with focus at the origin and directrix perpendicular to the initial line at the point $(-k, 0)$.

Change to rectangular coördinates, using F as origin and DF as x-axis. Then

$$r = \sqrt{x^2 + y^2}, \qquad r\cos\theta = x.$$

Equation (48*a*) can be written

$$r = ke + re\cos\theta = e(k + x).$$

Consequently,

$$x^2 + y^2 = e^2(k + x)^2. \qquad (48b)$$

The Ellipse. — Suppose $e < 1$. Let a be greater than b and

$$e = \frac{\sqrt{a^2 - b^2}}{a}, \qquad k = \frac{b^2}{\sqrt{a^2 - b^2}}. \quad (48c)$$

Equation (48*b*) is then equivalent to

$$\frac{(x - \sqrt{a^2 - b^2})^2}{a^2} + \frac{y^2}{b^2} = 1.$$

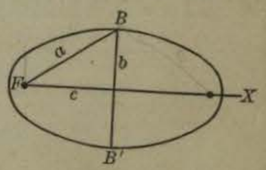

FIG. 48*b*.

This is the equation of an ellipse with semi-axes a and b and major axis horizontal. Since a and b can have any values it follows, conversely, that any ellipse is a conic with eccentricity less than unity.

The center of the ellipse is $(\sqrt{a^2 - b^2}, 0)$. Since the origin is the focus the distance from the center to the focus is then

$$c = \sqrt{a^2 - b^2}. \qquad (48d)$$

This shows that $FB = a$. By symmetry there is another focus at the same distance on the opposite side of the center. Hence an ellipse has two foci on the major axis at a distance from the ends of the minor axis equal to the semi-major axis.

The Parabola. — Suppose $e = 1$. Let $k = \frac{1}{2}a$. Equation (48*b*) is then equivalent to

$$y^2 = a(x + \tfrac{1}{4}a).$$

This is a parabola. Since a can have any value, it follows, conversely, that any parabola is a conic with eccentricity equal to unity. The vertex of the parabola is $(-\frac{1}{4}a, 0)$. The distance from the vertex to the focus is then equal to the absolute value of $\frac{1}{4}a$. Consequently, a parabola $y^2 = ax$ has a focus on its axis at the distance $\frac{1}{4}a$ from the vertex (Fig. 48*c*).

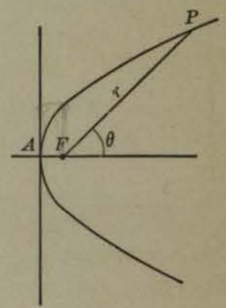

FIG. 48*c*.

The Hyperbola. — Suppose $e > 1$. Let

$$\left.\begin{aligned} e &= \frac{\sqrt{a^2 + b^2}}{a}, \\ k &= \frac{b^2}{\sqrt{a^2 + b^2}}. \end{aligned}\right\} \qquad (48e)$$

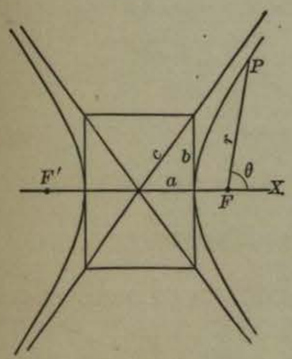

FIG. 48*d*.

Equation (48*b*) is then equivalent to

$$\frac{(x + \sqrt{a^2 + b^2})^2}{a^2} - \frac{y^2}{b^2} = 1.$$

This is the equation of a hyperbola with semi-axes a and b and transverse axis horizontal (Fig. 48*d*). Since a and b can have any values it follows, conversely, that any hyperbola is a conic with eccentricity greater than unity.

The center of the hyperbola is $(-\sqrt{a^2+b^2}, 0)$. Since the origin is the focus the distance from the center to the focus is then

$$c = \sqrt{a^2+b^2}. \tag{48f}$$

By symmetry there is another focus at the same distance on the other side of the center. Therefore, a hyperbola has two foci on its transverse axis at a distance from its center equal to half the diagonal of the rectangle on its axes.

Exercises

Determine the loci represented by the following equations. Construct the graphs. Change to rectangular coördinates.

1. $r\cos\theta = -2.$
2. $r\sin\theta = 3.$
3. $r = \sec\left(\theta + \frac{\pi}{4}\right).$
4. $\theta = -\frac{2}{3}\pi.$
5. $r = 3\cos\theta.$
6. $r = 4\sin\theta.$
7. $r = 2\sin\left(\theta - \frac{\pi}{4}\right).$
8. $r = \cos\theta - \sin\theta.$
9. $r = -2.$
10. $r^2 - 4r\cos\left(\theta - \frac{\pi}{3}\right) + 3 = 0.$
11. $r^2 - 2r(\cos\theta + \sin\theta) + 1 = 0.$
12. $r(2 - \cos\theta) = 2.$
13. $r(2 - 3\cos\theta) = 6.$
14. $r(1 - \cos\theta) = 3.$
15. $r(1 + \sin\theta) = -2.$
16. $r(1 + \sin\theta + \cos\theta) = 4.$
17. $r = 2\sec\theta - 3\csc\theta.$
18. $r\cos(2\theta) = \sin\theta.$

Determine the polar equations of the following loci:

19. $y = 2x - 1.$
20. $y^2 = 4x.$
21. $x^2 + y^2 = 2x.$
22. $xy = 7.$
23. $x^2 - y^2 = 1.$
24. $x^2 + y^2 = 4x + 4y.$

25. Find the polar equations of the circles of radius a tangent to both coördinate axes.

26. Find the polar equation of the parabola with focus at the origin and vertex $\left(-2, \frac{\pi}{6}\right)$.

27. Find the polar equation of the ellipse with focus at the origin, center on the y-axis, eccentricity $\frac{2}{3}$, and passing through the point $\left(1, \frac{\pi}{3}\right)$.

28. Find the eccentricity of the rectangular hyperbola $x^2 - y^2 = a^2$.

Art. 49. Graphing Equations

To plot the graph of a polar equation make a table of pairs of values satisfying the equation, plot the corresponding points and draw a smooth curve through them. It may be useful to look for any of the things mentioned in connection with plotting in rectangular coördinates. In most cases, however, it is sufficient to imagine θ, beginning with some definite value, to gradually increase and determine at each instant merely whether r is increasing or decreasing, draw a curve on which r varies in the proper direction and mark accurately the points where r is a maximum, minimum or

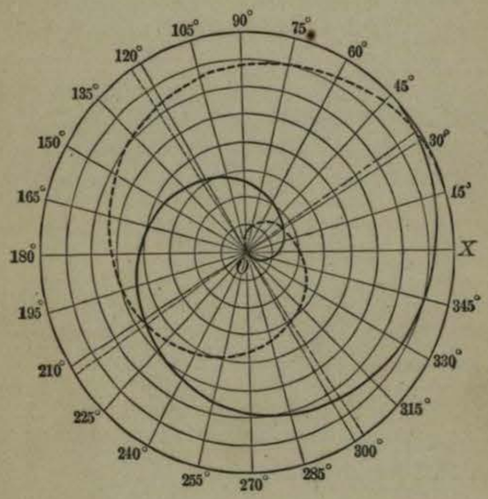

FIG. 49a.

zero. Proceed in the same way with θ decreasing from the initial value.

Example 1. $r = \theta + 1$. The curve passes through the origin at $\theta = -1$. As θ increases from this value, r is positive and steadily increases. While the angle makes a complete turn about the origin, r is increased by 2π. This part of the curve (indicated by the continuous line, Fig. 49a) thus consists of a series of expanding coils. Values of θ less than -1 give negative values of r (indicated by the dotted line). Angles $\theta = -1 \pm k$ give numerically equal values of r. In particular, when k is an odd multiple of $\frac{1}{2}\pi$ the resulting points coincide. Hence the two sets of coils cross on the line perpendicular to $\theta = -1$. A curve like this containing an infinite number of coils is called a *spiral*.

Ex. 2. The cardioid $r = a(1 + \cos\theta)$.

The curve is shown in Fig. 49*b*. As θ increases from 0 to $\frac{1}{2}\pi$, r decreases from $2a$ to a. As θ increases from $\frac{1}{2}\pi$ to π, r decreases to 0. Since $\cos\theta$ cannot be less than -1, r does not become negative. As θ goes from π to 2π, r increases from 0 to $2a$. Since

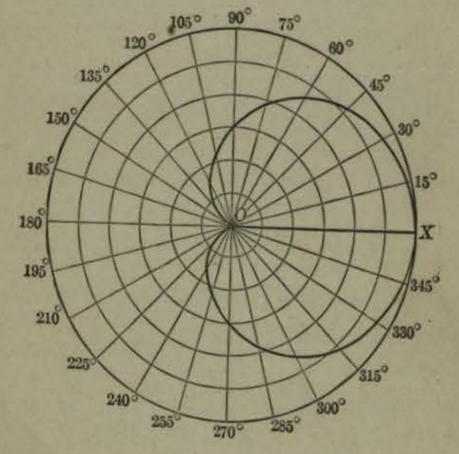

FIG. 49*b*.

$\cos(\theta + 2\pi) = \cos\theta$, values of θ greater than 2π and negative values give points already plotted.

Ex. 3. The lemniscate $r^2 = 2a^2\cos(2\theta)$. To each value of θ correspond two values of r differing only in sign. The curve is therefore symmetrical with respect to the origin. Also angles differing only in sign give the same values of r. Hence the curve is symmetrical with respect to the initial line. As θ increases from 0 to $\frac{1}{4}\pi$, r varies from $\pm a\sqrt{2}$ to 0. When θ is between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$, $\cos(2\theta)$ is negative and and r is imaginary. As θ increases from $\frac{3}{4}\pi$

to π, r goes from 0 to $\pm a\sqrt{2}$. Values of θ greater than π and negative values of θ give points already plotted (Fig. 49*c*).

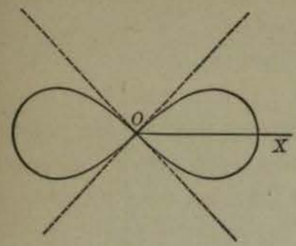

FIG. 49*c*.

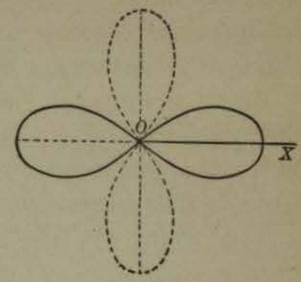

FIG. 49*d*.

Ex. 4. $r = a\cos(2\theta)$. The maximum, minimum and zero values of the cosine occur when the angle is zero or a multiple of $\frac{1}{2}\pi$. The most important points on the curve are then the following:

$2\theta = 0,$	$\frac{\pi}{2},$	$\pi,$	$\frac{3}{2}\pi,$	$2\pi,$	$\frac{5}{2}\pi,$	$3\pi,$	$\frac{7}{2}\pi,$	$4\pi,$
$\theta = 0,$	$\frac{\pi}{4},$	$\frac{\pi}{2},$	$\frac{3}{4}\pi,$	$\pi,$	$\frac{5}{4}\pi,$	$\frac{3}{2}\pi,$	$\frac{7}{4}\pi,$	$2\pi,$
$r = a,$	$0,$	$-a,$	$0,$	$a,$	$0,$	$-a,$	$0,$	$a.$

Values of θ greater than 2π give points already plotted (Fig. 49*d*).

Ex. 5. $r = a\sec(\frac{3}{2}\theta)$. If θ is increased by 4π the value of r is not changed. For

$$a\sec[\tfrac{3}{2}(\theta + 4\pi)] = a\sec(\tfrac{3}{2}\theta + 6\pi) = a\sec(\tfrac{3}{2}\theta).$$

The whole curve is then obtained by plotting points from $\theta = 0$ to $\theta = 4\pi$. The most important points are

$\frac{3}{2}\theta = 0,$	$\frac{\pi}{2},$		$\pi,$	$\frac{3}{2}\pi,$		$\ldots,$	$6\pi,$
$\theta = 0,$	$\frac{\pi}{3},$		$\frac{2}{3}\pi,$	$\pi,$		$\ldots,$	$4\pi,$
$r = a,$	$+\infty,$	$-\infty,$	$-a,$	$-\infty,$	$+\infty,$	$\ldots,$	$a,$

where $+\infty, -\infty$ means that as θ increases to $\frac{1}{3}\pi$, r becomes indefinitely large and positive, while as θ decreases to $\frac{1}{3}\pi$, r becomes indefinitely large and negative. The curve is shown in Fig. 49*e*.

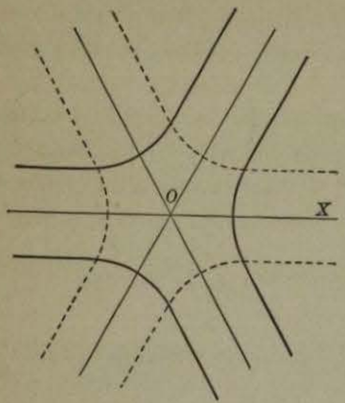

FIG. 49*e*.

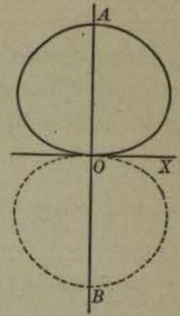

FIG. 49*f*.

Ex. 6. $r^2 = a^2 \sin(\theta)$.

The curve (Fig. 49*f*) is symmetrical with respect to the origin. When θ increases from 0 to $\frac{1}{2}\pi$, the positive value of r increases from 0 to a. As θ increases from $\frac{1}{2}\pi$ to π, r decreases to 0. Between $\theta = \pi$ and $\theta = 2\pi$ the sine is negative and no point is obtained. Negative values of θ and those greater than 2π give points already plotted.

Art. 50. Intersection of Curves

If the polar coördinates of a point satisfy the equation of a curve, the point lies on the curve. A point may, however, lie on a curve although its coördinates (as given) do not satisfy the equation of the curve. This happens because a point has several pairs of polar coördinates. One of these pairs may satisfy a given equation while another does not. Thus the point $B\left(1, -\frac{\pi}{2}\right)$ lies on the curve $r^2 = a^2 \sin(\theta)$ (Fig. 49*f*) but its coördinates do not satisfy the equation. The coördinates $\left(-1, \frac{\pi}{2}\right)$ represent the same point and satisfy the equation.

To find the intersections of two curves we solve their equations simultaneously. The pairs of coördinates thus obtained represent points on both curves. There may, however, be other points of intersection. This happens when some of the pairs of polar coördinates representing a point satisfy one equation, other pairs satisfy the other, but no pair satisfies both. In finding the intersections of curves represented by polar equations the graphs should always be drawn. Any extra intersections will then be seen.

Example 1. Show that the point $\left(a, \frac{\pi}{2}\right)$ lies on the curve $r^2 = a^2 \sin(3\theta)$.

The coördinates given do not satisfy the equation, for $\theta = \frac{\pi}{2}$ gives

$$r^2 = a^2 \sin\left(\tfrac{3}{2}\pi\right) = -a^2.$$

The point $\left(a, \frac{\pi}{2}\right)$ can, however, be written $\left(-a, \frac{3}{2}\pi\right)$ and in this form its coördinates satisfy the equation (Fig. 50*a*).

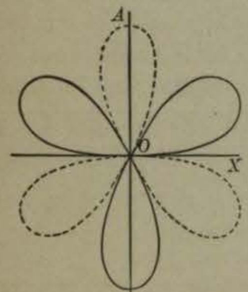

FIG. 50*a*.

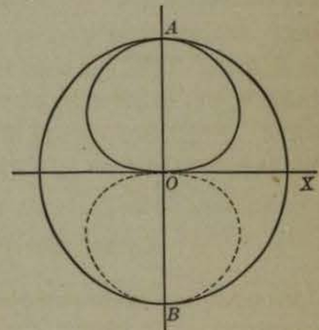

FIG. 50*b*.

Ex. 2. Find the intersections of the curves $r^2 = a^2 \sin\theta$, $r = a$. Solving simultaneously, we get

$$a^2 \sin\theta = a^2.$$

Consequently $\sin\theta = 1$ and $\theta = \frac{\pi}{2}$. One point of intersection is then $A\left(a, \frac{\pi}{2}\right)$. It is seen from the figure that $B\left(-a, \frac{1}{2}\pi\right)$ is also a point of intersection.

Exercises

Plot the following pairs of curves:

1. $r = \sin\theta, \quad y = \sin x.$
2. $r = \cos\theta, \quad y = \cos x.$
3. $r = \tan\theta, \quad y = \tan x.$
4. $r = \sec\theta, \quad y = \sec x.$

Sketch the graphs of the following equations:

5. $r = \theta.$
6. $r = 1 - \theta.$
7. $r = \frac{1}{\theta}.$
8. $r = 2^{\theta}.$
9. $r = a\sin(2\theta).$
10. $r = a\cos(3\theta).$
11. $r = a(1 + \sin\theta).$
12. $r = a(2 + \sin\theta).$
13. $r = a(1 + 2\sin\theta).$
14. $r = a\sin\left(\theta - \frac{\pi}{4}\right).$
15. $r = a\cos(\frac{1}{2}\theta).$
16. $r = a\tan(\frac{3}{2}\theta).$
17. $r = a(1 + \sin 2\theta).$
18. $r = a(1 + 2\cos 3\theta).$
19. $r = 1 + \cos(\frac{3}{2}\theta).$
20. $r = 4 + 5\cos(5\theta).$
21. $r = 1 - 2\cos(\frac{4}{3}\theta).$
22. $r = \sin\theta\cos\theta.$
23. $r = \sin^3\left(\frac{\theta}{3}\right).$
24. $r^2 = 2a^2\sin(2\theta).$
25. $r^2 = 1 + \sin\theta.$
26. $r^2 = \tan(2\theta).$
27. $r = \frac{a}{\cos\theta} + \frac{a}{\sin\theta}.$
28. $r\sin\theta = a\cos(2\theta).$
29. $r = \dfrac{2^{\frac{1}{\theta}} + 1}{2^{\frac{1}{\theta}} - 1}.$

30. Show that the point $(1, \frac{3}{4}\pi)$ lies on the curve $r = \sin(2\theta)$ but that the coördinates given do not satisfy the equation of the curve.

31. Show that the point $\left(1, \frac{3}{2}\pi\right)$ lies on the curve $r = -2\sin\left(\frac{\theta}{3}\right)$ but that the coördinates given do not satisfy the equation of the curve.

32. Show that if a is a constant the equations

$$r = \frac{1}{a\cos\theta + 1}, \quad r = \frac{1}{a\cos\theta - 1}$$

represent the same curve.

33. Show that the equations $r^2 = a^2\cos^2(2\theta)$ and $r = a\cos(2\theta)$ represent the same curve.

34. Why do the equations $r^2 = ar\cos\theta$ and $r = a\cos\theta$ represent the same curve?

Plot the following pairs of curves and find their points of intersection:

35. $r = 2\sec\left(\theta - \frac{\pi}{4}\right), \quad r = 2\sec\left(\theta + \frac{\pi}{3}\right).$

36. $r\sin\theta = a, \quad r\cos\theta = a.$
37. $r^2 = a^2\sin\theta, \quad r^2 = a^2\sin(3\theta).$
38. $r = a\sin(2\theta), \quad r = a(1 - \cos 3\theta).$
39. $r^2 = 2a^2\cos(2\theta), \quad r = a.$

Art. 51. Locus Problems

In finding the equation of the locus of a moving point, either polar or rectangular coördinates can be used. The system should be chosen which seems to fit the problem best, making a change of coördinates in the resulting equation if necessary. If the positions of origin and axes are not specified they should be placed in the most convenient position. This is usually (though not always) the most symmetrical position.

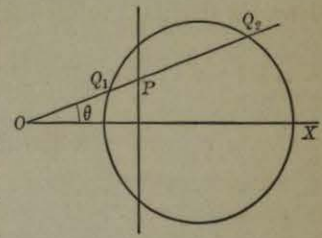

FIG. 51a.

Example 1. Through a fixed point O draw a line intersecting a fixed circle in Q_1 and Q_2 (Fig. 51a). On this line determine the point P such that

$$\frac{2}{OP} = \frac{1}{OQ_1} + \frac{1}{OQ_2}.$$

Find the locus of P as OP turns about O.

Take the origin at O and the initial line through the center $(b, 0)$ of the fixed circle. Let Q_1, Q_2, and P be (r_1, θ), (r_2, θ), and (r, θ) respectively. By hypothesis

$$\frac{2}{r} = \frac{1}{r_1} + \frac{1}{r_2}, \quad \text{or} \quad r = \frac{2r_1r_2}{r_1 + r_2}.$$

The equation of the fixed circle is $r^2 - 2rb\cos\theta + b^2 - a^2 = 0$, that is

$$r = b\cos\theta \pm \sqrt{a^2 - b^2\sin^2\theta}.$$

The two values of r in this equation are r_1 and r_2. Hence

$$r_1 + r_2 = 2b\cos\theta, \quad r_1r_2 = b^2\cos^2\theta - (a^2 - b^2\sin^2\theta) = b^2 - a^2.$$

Therefore

$$r = \frac{b^2 - a^2}{b\cos\theta} = \frac{b^2 - a^2}{b}\sec\theta$$

is the equation of the locus described by P. It is a straight line through the points at which lines through O are tangent to the circle.

Ex. 2. A point moves so that the product of its distances from two fixed points is equal to the square of half the distance between them. Find its locus.

Take the x-axis through the fixed points and the origin midway between them. Let the distance between the fixed points be $2a$. If $P(x, y)$ is any point on the locus (Fig. 51*b*), by hypothesis, $A'P \cdot AP = OA^2$, or

$$\sqrt{(x-a)^2+y^2}\,\sqrt{(x+a)^2+y^2} = a^2.$$

Squaring,

$$(x^2+y^2+a^2)^2 - 4a^2x^2 = a^4.$$

Changing to polar coördinates,

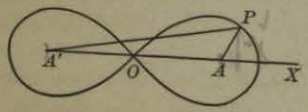

FIG. 51*b*.

$$(r^2+a^2)^2 - 4a^2r^2\cos^2\theta = a^4,$$

or

$$r^2 = 2a^2(2\cos^2\theta - 1) = 2a^2\cos(2\theta),$$

which is the equation of a lemniscate.

Exercises

1. LK is a fixed straight line perpendicular to the initial line at $(a, 0)$. On any line through the origin, intersecting LK in Q, is taken a point P such that

$$OP \cdot OQ = a^2.$$

Find the locus of P as OQ turns about O.

2. LK is a fixed straight line perpendicular to the initial line at $A\,(a, 0)$. Any line through O intersects LK in Q and a point P is taken on this line such that

$$PQ = AQ.$$

Find the locus of P as OQ turns about O.

3. A point P moves in such a way that its distance from a fixed point O multiplied by its distance from a fixed straight line LK is constant. Find the locus of P.

4. A segment of fixed length slides with its ends in the x and y axes. Find the locus of the foot of the perpendicular from the origin to the moving segment.

5. A revolving line passing through the center of a fixed circle intersects the circle in a point P_1 and a fixed straight line in P_2. Find the locus described by the point midway between P_1 and P_2.

6. Let OA be the diameter of a fixed circle and let LK be tangent to the circle at A. Through O draw any line intersecting the circle in D and LK in E. On OE lay off a distance OP equal to DE. Find the locus of P as OE turns about O.

7. From a point O on a fixed circle perpendiculars are dropped upon the tangents of the circle. Taking O as origin and the diameter through O as initial line find the polar equation of the curve generated by the feet of these perpendiculars.

8. A circle rolls along the initial line and a line through the center of the circle turns about the origin. Find the locus of the intersections of the moving line and circle.

9. A circle rolls along the initial line. A line through the origin moves in such a way as to remain tangent to the circle. Find the locus of the point of tangency.

10. Take a fixed point O and a fixed straight line BC. Through O draw any line intersecting BC in D and on this line lay off a constant distance DP measured from D in either direction. Find the locus described by P as the line turns about O.

11. Through a fixed point O on the circumference of a fixed circle draw any line cutting the circle again at D and lay off on this line a constant distance DP measured from D in either direction. Find the locus of P as OP turns about O.

12. A straight line OA of constant length revolves about O. Through A a line is drawn perpendicular to the initial line intersecting it in B. Through B a line is drawn perpendicular to OA intersecting it in P. Find the locus of P.

13. MN is a straight line perpendicular to the initial line at $A\,(a, O)$. From O a line is drawn to any point B of MN. Through B a line is drawn perpendicular to OB intersecting the initial line at C. Through C a line is drawn perpendicular to BC intersecting MN at D. Finally, through D a line is drawn perpendicular to CD intersecting OB at P. Find the locus of P.

14. Two circles whose centers are fixed and whose circumferences touch rotate without slipping. A line through the center of one circle and rotating with it intersects a similar line on the other circle in a point P. Find the locus of P. If the radii are incommensurable show that the locus passes indefinitely near any point of the plane.