CHAPTER 5

GRAPHS AND EMPIRICAL EQUATIONS

The equation of a curve being given, any number of points on the curve can be constructed by assigning values to either coördinate, calculating the corresponding values of the other coördinate and plotting the resulting points. When enough points have been located a smooth curve drawn through them may be taken as an approximation to the required curve. We desire as quickly as possible to obtain a satisfactory approximation. To some extent this is accomplished by plotting points rather sparsely where the curve is nearly straight and more closely where it bends rapidly. The following are some of the things it may be helpful to note:

(1) Points where the curve crosses the axes and the side of an axis on which it lies between two consecutive crossings (Art. 36).

(2) Values of either coördinate for which the other coördinate is real and values for which it is imaginary (Art. 37).

(3) Symmetry (Art. 38).

(4) Infinite values of the coördinates. Asymptotes (Art. 39).

(5) Direction of the curve near a point (Art. 40).

Art. 36. Intersections with the Axes

The points where a curve meets the x-axis are found by letting $y = 0$ in the equation and solving for x. Similarly, points on the y-axis are found by letting $x = 0$ and solving for y. The x-coördinates of the points on the x-axis and the y-coördinates of the points on the y-axis are called the *intercepts* of the curve on the coördinate axes.

Example 1. $y = x(x-1)(x+2)$. The curve crosses the x-axis where $y = 0$, that is, where $x = 0, 1, -2$. These points divide the x-axis and the curve into four parts; namely, the part on the

left of $x = -2$, that between $x = -2$ and $x = 0$, that between $x = 0$ and $x = 1$ and the part on the right of $x = 1$. Construct these parts separately.

On the left of $x = -2$, each factor of $x(x-1)(x+2)$ is negative and consequently the whole product is negative. Hence on the left of $x = -2$, y is negative and the curve lies below the x-axis. Between $x = -2$ and $x = 0$, x and $x - 1$ are negative and $x + 2$ is positive. The product is then positive and so the curve lies above the x-axis. Similarly between $x = 0$ and $x = 1$ the curve is below the x-axis and on the right of $x = 1$ it is above. Using these facts and plotting a few points on each part of the curve the graph of Fig. 36*a* is obtained.

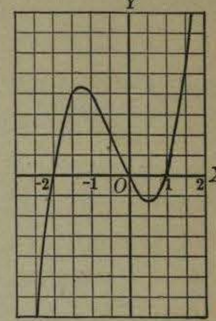

Fig. 36*a*.

Ex. 2. $y - 2 = x^2(x+2)^2$. The curve meets the line $y = 2$ at $x = 0, -2$. Since $x^2(x+2)^2$ is never negative, y can never be less than 2. Hence the curve touches but does not cross the line $y = 2$ at $x = 0$ and $x = -2$. (Fig. 36*b*.)

Ex. 3. $x = y^4 + 2y^3 + 3y^2 + 4y + 2$. The curve meets the y-axis where $x = 0$, that is, where $y^4 + 2y^3 + 3y^2 + 4y + 2 = 0$. Proceeding as

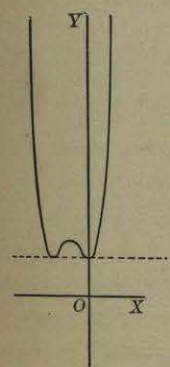

Fig. 36*b*.

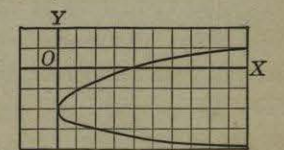

Fig. 36*c*.

in Art. 4 it is found that -1 is a root of this equation. Hence $y + 1$ is a factor of the polynomial. Division gives

$$x = (y+1)(y^3 + y^2 + 2y + 2).$$

In the same way it is found that

$$y^3 + y^2 + 2y + 2 = (y + 1)(y^2 + 2).$$

Consequently,

$$x = (y + 1)^2 (y^2 + 2).$$

Since $y^2 + 2 = 0$ has no real roots, the curve meets the y-axis only at $y = -1$. The factor $(y + 1)^2$ is positive both above and below $y = -1$. Hence x is always positive and the curve does not cross the y-axis. After plotting a few points the curve in Fig. 36*c* is obtained.

Art. 37. Real and Imaginary Coördinates

When the equation of the curve is of even degree in one of the coördinates, that coördinate may be real for certain values of the other coördinate and imaginary for certain others. The plane is then divided into strips (Fig. 37*a*) containing a part of the curve and strips not containing a part. These strips being determined the part of the curve in each strip is plotted separately.

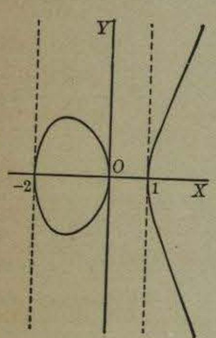

FIG. 37*a*.

Example 1. $y^2 = x(x - 1)(x + 2)$. The curve crosses the x-axis at $x = -2$, $x = 0$ and $x = 1$. The lines $x = -2$, $x = 0$ and $x = 1$ divide the plane into four parts. On the left of $x = -2$, the product $x(x - 1)(x + 2)$ is negative and y is imaginary. Between $x = -2$ and $x = 0$ the product is positive and y is real. Similarly, between $x = 0$ and $x = 1$, y is imaginary and, on the right of $x = 1$, y is again real. The curve therefore consists of two pieces, one between $x = -2$ and $x = 0$ and the other on the right of $x = 1$. The equation can be written

$$y = \pm\sqrt{x(x - 1)(x + 2)}.$$

To each value of x correspond two values of y differing only in sign. The curve therefore consists of points at equal distances above and below the x-axis (Fig. 37*a*).

Ex. 2. $x^2 - 4xy + 4y^2 - y - 1 = 0$. Solving for x,

$$x = 2y \pm \sqrt{y + 1}.$$

The value of x is real if $y > -1$, imaginary if $y < -1$. The curve therefore lies above the line $y = -1$. It consists of pairs of points at equal distances right and left of the line $x = 2y$ (Fig. 37*b*).

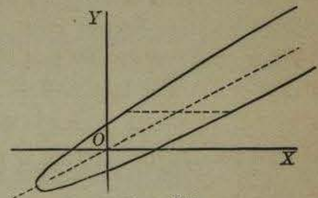

FIG. 37*b*.

Art. 38. Symmetry

Two points P, P' are said to be *symmetrical with respect to a line* if the segment PP' is bisected perpendicularly by that line. In Fig. 38*a*, P and P' are symmetrical with respect to the x-axis, P and P'', Q and R with respect to the y-axis.

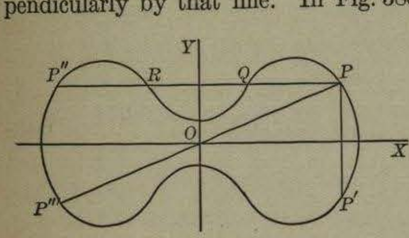

FIG. 38*a*.

Two points P and P''' are called *symmetrical with respect to a point* O if the segment PP''' is bisected by O.

A curve is called *symmetrical with respect to an axis* if all chords perpendicular to the axis meet the curve in pairs of points symmetrical with respect to the axis. The curve in Fig. 38*a* is symmetrical with respect to both coördinate axes.

A curve is called *symmetrical with respect to a center* if all chords through the center meet the curve in pairs of points symmetrical with respect to the center. The curve in Fig. 38*a* is symmetrical with respect to the origin.

A curve $f(x, y) = 0$ is symmetrical with respect to the x-axis if

$$f(x, y) = f(x, -y).$$

For then any point $P(x_1, y_1)$ being on the curve the point $P'(x_1, -y_1)$ is also on the curve. Hence any line $x = x_1$ perpendicular to the x-axis and meeting the curve in a point P will meet it in two points P, P' symmetrical with respect to the x-axis. In particular, *a curve is symmetrical with respect to the x-axis if its equation contains* *only even powers of y.* Similarly, the curve $f(x, y) = 0$ is symmetrical with respect to the y-axis if

$$f(x, y) = f(-x, y),$$

and, in particular, *a curve is symmetrical with respect to the y-axis if its equation contains only even powers of x.*

The curve $f(x, y) = 0$ is symmetrical with respect to the origin if

$$f(x, y) = \pm f(-x, -y).$$

For then if $P(x_1, y_1)$ is on the curve $P'''(-x_1, -y_1)$ is also on the curve, and so any line through the origin meeting the curve in a point P will meet it in two points P, P''' symmetrical with respect to the origin. In particular, *a curve is symmetrical with respect to the origin if all the terms in its equation are of even degree or if all are of odd degree.*

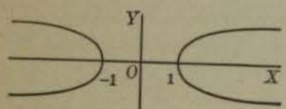

FIG. 38b.

Example 1. $y^2 = \frac{x^2 - 1}{x^2 + 1}$. The curve crosses the x-axis at $x = \pm 1$. The value of y is real only when x is in absolute value equal to or greater than 1. There are consequently two parts of the curve, one on the right of $x = 1$, the other on the left of $x = -1$. Since the equation contains only even powers of x and y, the curve is symmetrical with respect to both coördinate axes and with respect to the origin. Since $x^2 - 1 < x^2 + 1$, y is always less than 1. When x is very large, however, y is nearly 1.

Ex. 2. $y = x^3 - 3x^2 + 3x + 1$. This equation can be written

$$y - 2 = (x - 1)^3.$$

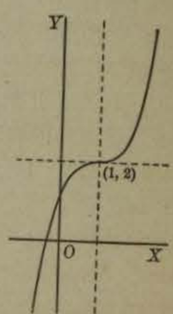

FIG. 38c.

The expressions $y - 2$ and $x - 1$ are the coördinates of a point $P(x, y)$ relative to the lines $y = 2$, $x = 1$ used as axes. Since the equation contains only odd powers of $y - 2$ and $x - 1$ the curve is symmetrical with respect to the point $(1, 2)$.

Art. 39. Infinite Values

In some cases a variable increases in absolute value beyond any assignable bound. Such a variable is said to become infinite and a fictitious value represented by the symbol ∞ is attached to it. But the symbol and the name infinity are used only to express that a variable goes beyond all bounds.

Zero and infinity have the following relations:

(1) $\frac{0}{a} = 0, \quad \frac{a}{0} = \infty, \quad a \cdot \infty = \infty \cdot a = \infty$, if a is not zero.

(2) $\frac{\infty}{a} = \infty, \quad \frac{a}{\infty} = 0, \quad a \cdot 0 = 0 \cdot a = 0$, if a is not infinite.

No definite value can be assigned to the symbols $0/0$, ∞/∞, $0 \cdot \infty$ and $\infty - \infty$.

These relations become evident when 0 and ∞ are interpreted as less than any assignable quantity and greater than any assignable quantity respectively.

A branch of a curve extending to an infinite distance can only be traced until it runs off the diagram. In such cases the curve usually consists of two or more pieces not connected together. Sometimes two pieces are related as at A, B, Fig. 39*a*, the one going off in a certain direction, the other returning from that direction. Sometimes they are related as at C, D, the one going off one side of the paper, the other returning from the other side. Sometimes there is no return branch.

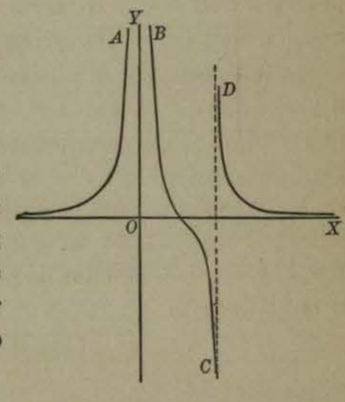

FIG. 39*a*.

If a branch of a curve when indefinitely prolonged approaches a straight line in such a way that the distance between the two approaches zero, the straight line is called an *asymptote* of the curve. Both coördinate axes and the line CD are asymptotes of the curve, Fig. 39a. If when indefinitely prolonged the distance between a branch of one curve and a branch of another approaches zero, the two curves are called *asymptotic*.

Example 1. $y = \dfrac{x-1}{x^2(x-2)}$. The graph is shown in Fig. 39a. It crosses the x-axis at $x = 1$. The value of y is infinite when $x = 0$ or 2. When x is a little less than zero, $x - 1$ and $x - 2$ are negative and so y is positive. When x is a little greater than zero y is again positive. In both cases y is very large. The curve thus goes up one side of the y-axis and comes down the other. When x is a little less than 2, y is negative but when x is a little greater than 2, y is positive. The curve then goes down the left side of CD and reappears at the top. As x increases indefinitely, y approaches zero. Consequently, at a great distance from the origin the curve comes closer and closer to the x-axis. The two axes and the line CD are asymptotes of the curve.

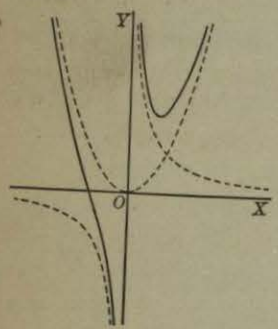

FIG. 39b.

Ex. 2. $y = x^2 + \dfrac{1}{x}$. When x is very small, x^2 is very small and y is approximately $1/x$. Near the y-axis the curve is then asymptotic to the hyperbola $y = 1/x$. The y-axis is an asymptote to both curves. When x is very large, $1/x$ is very small and y is approximately equal to x^2. As x increases indefinitely, the curve therefore approaches the parabola $y = x^2$ to which it is consequently asymptotic.

Art. 40. Direction of the Curve

To determine the shape of a curve near a particular point it is often useful to find the direction along which the curve or a branch of the curve approaches that point. In Fig. 40a, for example, are shown three ways that a branch of a curve can approach a horizontal line. In (1) the curve and line are tangent, in (2) the curve and line intersect at an acute angle and in (3) they are perpendicular

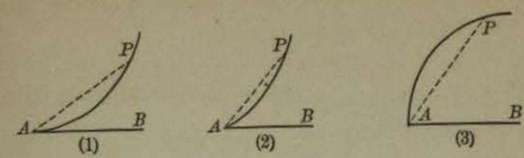

FIG. 40a.

to each other. Let P be a variable point on the curve. As P approaches A the slope of the line AP will approach zero in (1), a finite value not zero in (2) and an infinite value in (3). By finding this slope and determining its limit the direction of the curve near A can be determined.

Example 1. $y^2 = x^3$. The curve passes through the origin (Fig. 40b). If x is negative, y is imaginary. The curve therefore reaches the y-axis at the origin but does not cross it. Since the equation contains only even powers of y, the curve is symmetrical with respect to the x-axis. The slope of OP is

$$\tan\phi = y/x.$$

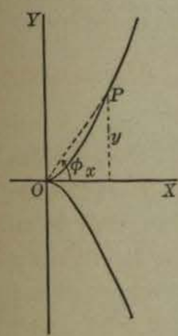

FIG. 40b.

From the equation of the curve, $y = \pm x^{\frac{3}{2}}$. Hence

$$\tan\phi = \pm x^{\frac{3}{2}}/x = \pm x^{\frac{1}{2}}.$$

As P approaches the origin, x approaches zero and consequently $\tan\phi$ approaches zero. The branches of the curve above and below the x-axis are thus tangent to the x-axis and to each other at the origin.

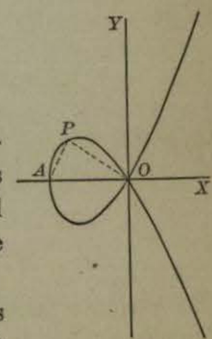

FIG. 40c.

Ex. 2. $y^2 = x^2(x + 2)$. The curve crosses the x-axis at the origin and at A $(-2, 0)$ (Fig. 40c). There is no point of the curve to the left of $x = -2$. Let P (x, y) be any point on the curve. The slope of AP is

$$\frac{y-0}{x+2} = \frac{\pm x\sqrt{x+2}}{x+2} = \frac{\pm x}{\sqrt{x+2}}.$$

As P approaches A, x approaches -2 and the slope of AP increases indefinitely. The curve is therefore perpendicular to the x-axis at A. The slope of OP is

$$\frac{y}{x} = \frac{\pm x\sqrt{x+2}}{x} = \pm\sqrt{x+2}.$$

As P approaches the origin, x approaches 0 and the slope of OP approaches $\pm\sqrt{2}$. At the origin the curve therefore makes with the x-axis the angles $\tan^{-1}(\pm\sqrt{2})$.

Exercises

Make graphs of the following equations:

1. $y = x(x+1)$.
2. $x = y^2(y-2)$.
3. $y = (x-1)(x+2)(x-3)$.
4. $y + 1 = x^4 + 2x$.
5. $y = x(x+1)(2x-3)$.
6. $y - 3 = x^2(x+1)(2x-3)$.
7. $y + 2 = x^3(x+1)^2(2x-3)$.
8. $y = x^2(x+1)^2(2x-3)^2$.
9. $y = x^3 - 1$.
10. $x - 2 = (y+1)^4 - 1$.
11. $x = y^4 + y^2 + 1$.
12. $x = y^4 + y^3 - 4y^2 - 4y$.
13. $y^2 = x(x+1)$.
14. $y^2 = (x^2-1)(x^2-4)$.
15. $(y+1)^2 = (x^2-1)(4-x^2)$.
16. $x^2 = (y-1)^2(y-2)$.
17. $x^2 = (y-1)^2(2-y)$.
18. $(y^2+x)(y^2-x) = 0$.
19. $y^2 + 2y = x^4 + 2x - 1$.
20. $(x+y)^4 - y^4 = 0$.
21. $(y+x)^2 = x^2(x-1)$.
22. $x^4 + y^4 = 1$.
23. $y^4 - 2xy^2 + x^2 - y^2 + 4 = 0$.
24. $y^4 + (2x^2+1)y^2 + x^4 - x^2 = 0$.
25. $x^3 + xy^2 = 1$.
26. $x^2 - 4xy + 8y^2 - y^4 = 0$.
27. $y = \frac{16}{1-x}$.
28. $y = \frac{x}{x+1}$.
29. $x = \frac{2y^2+3y-2}{y-3}$.
30. $y = \frac{(x+1)(x-2)}{x(x-3)}$.
31. $y = x + \frac{1}{x}$.
32. $y = x^2 + \frac{1}{x^2}$.
33. $x^2 = \frac{y^2-1}{y^2-4}$.
34. $y = \frac{1}{x-1} - \frac{1}{x+3}$.
35. $y = x^3 - \frac{1}{x(x-2)}$.
36. $y = \frac{1}{(x-1)^2} - \frac{1}{(x+3)^2}$.
37. $y^2 = \frac{1}{x(x-1)} - \frac{1}{x+3}$.

38. $x^2y^2 + 36 = 4y^2$.
39. $y^2 = \frac{x^2(a+x)}{a-x}$.
40. $x^3y + a^3y - x^4 + a^4 = 0$.
41. $y^3 = x^4$.
42. $x^4 = y^6$.
43. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.
44. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
45. $x^2 + y^{\frac{2}{3}} = 1$.
46. $x^{13} + y^{23} = 1$.
47. $x^{12} + y^{22} = 1$.
48. $y^3 = x^2(x+2)$.
49. $(y+2)^3 = (x-1)(x^2-4)$.
50. $(x+y)^2 = y^2(y+1)$.
51. $(x+y-1)(2x-4y-2) = 4$.
52. $y^2 = x(x-y)(x+y)$.
53. $(x+y-2)^2 = (2x-y-1)^3$.
54. $x(x^2+y^2-1) = 1$.

Art. 41. Sine Curves

When a, b, c are constants, the graph of the equation

$$y = a\sin(bx+c)$$

is called a *sine curve*. As shown in Fig. 41*b*, the graph consists of a series of waves all having the same length and height. The sine of an angle is never greater than 1 and so y is never greater than a. The constant a therefore measures the height of the waves. If x is increased by $2\pi/b$, the angle $bx+c$ is increased by 2π and y is not changed. This is the smallest constant that added to x will leave y unchanged. Hence $2\pi/b$ is the distance from any point of a wave to the corresponding point of the next wave. It is called the *wave length*.

The equation $y = a\cos(bx+c)$ also represents a sine curve; for

$$a\cos(bx+c) = a\sin\left(\frac{\pi}{2} - bx - c\right) = a\sin(b'x+c'),$$

if $b' = -b$, $c' = \frac{1}{2}\pi - c$. Thus a cosine curve is a sine curve with different constants. Also

$$y = A\sin(mx) + B\cos(mx)$$

can be reduced to the form $y = a\sin(mx+b)$ and so represents a sine curve.

In plotting sine curves angles should be expressed in circular measure. A circle being drawn with the vertex of an angle as center, the circular measure of the angle is defined as the ratio of the intercepted arc to the radius (Fig. 41a), that is,

$$\text{Circular measure of angle} = \frac{\text{intercepted arc}}{\text{radius of circle}}.$$

An angle of 180° has a circular measure equal to $\pi = 3.14159 \ldots$, and other angles have proportional measures. For instance, the circular measure of 360° is 2π; 90°, $\frac{1}{2}\pi$; 60°, $\frac{1}{3}\pi$; 45°, $\frac{1}{4}\pi$; 30°, $\frac{1}{6}\pi$. An angle whose circular measure is 1 intercepts an arc equal to the radius. This angle, called a *radian*, is

$$\frac{180°}{\pi} = 57°.3- .$$

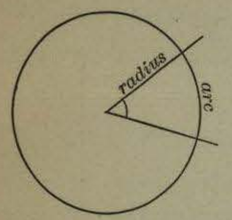

FIG. 41a.

If x is a real number, $\sin x$ means sine of the angle whose circular measure is x. Thus $\sin 2 = \sin (114°.6-)$.

Example 1. Plot the curve $y = 2 \sin (3x)$. The sine of an angle is never greater than 1 nor less than -1. Consequently, on this curve, y cannot be greater than 2 nor less than -2. The curve thus lies between the lines $y = -2$ and $y = 2$. The most important points on a sine curve are those where the sine is a maximum, minimum or zero. Between $3x = 0$ and $3x = 2\pi$, the important values are shown in the following table:

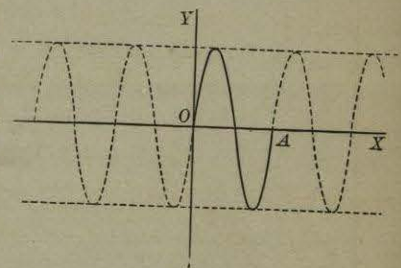

FIG. 41b.

$3x = 0$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π
$x = 0$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$
$y = 0$	2	0	-2	0

This part of the curve extends from O to A (Fig. 41b). When x is increased by $\frac{2}{3}\pi$, $3x$ is increased by 2π and y is not changed. Beyond A a new wave thus begins like that between O and A. In the same way it is seen that the wave on the left of O is like that from A to O. The whole curve thus consists of an infinite number of waves each of wave length $\frac{2}{3}\pi$.

Ex. 2. $x = \cos (2y - 3)$. Since the cosine is never greater than 1 nor less than -1, the curve lies between the lines $x = 1$, $x = -1$. At the point A (Fig. 41c) where $2y - 3 = 0$, x has the maximum value 1. Between this point and B, where $2y - 3 = 2\pi$, the most important values are shown in the following table:

$2y - 3 = 0$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π
$y = \frac{3}{2}$	$\frac{3}{2} + \frac{\pi}{4}$	$\frac{3}{2} + \frac{\pi}{2}$	$\frac{3}{2} + \frac{3}{4}\pi$	$\frac{3}{2} + \pi$
$x = 1$	0	-1	0	1

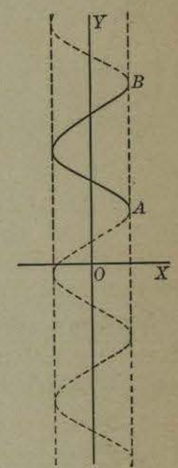

FIG. 41c.

When y is increased or decreased by π, the change in the angle $2y - 3$ is 2π and x is not changed. The curve then consists of a series of waves like that from A to B placed end to end.

Ex. 3. $y + 1 = \sin^{-1} (x - 1)$. This equation is equivalent to

$$x - 1 = \sin (y + 1).$$

The graph is a sine curve with axis $x - 1 = 0$. The curve passes through the point $y = -1$, $x = 1$, extending above and below that point in a series of waves each of vertical length 2π. (Fig. 41d.)

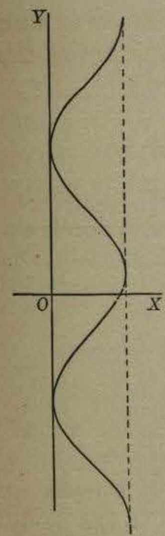

FIG. 41d.

Ex. 4. $y = 2 \sin (\frac{1}{2}x) + 6 \cos (\frac{1}{4}x)$. To construct this curve, draw the curves

$$y_1 = 2 \sin (\tfrac{1}{2}x), \quad y_2 = 6 \cos (\tfrac{1}{4}x)$$

and on each vertical line determine the point P such that $y = y_1 + y_2$, that is,

$$MP = MP_1 + MP_2.$$

To leave $\sin(\frac{1}{2}x)$ and $\cos(\frac{1}{4}x)$ both unchanged, x must be increased by 8π or a multiple of 8π. Hence a section AB from $x = 0$ to

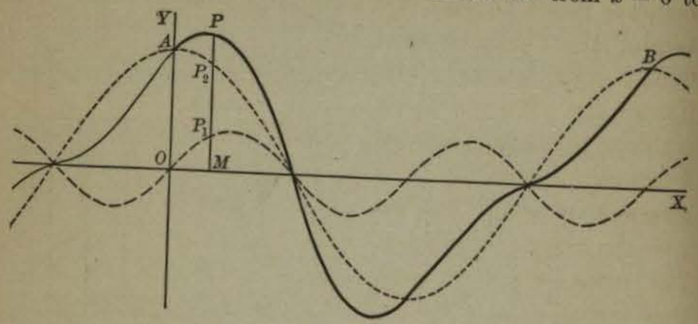

FIG. 41e.

$x = 8\pi$ must be plotted. The whole curve is a series of such sections placed end to end (Fig. 41e).

Art. 42. Periodic Functions

The equations considered in the previous article have the peculiarity that when a certain constant is added to one of the variables the other variable is not changed. Let the equation of a curve be $f(x, y) = 0$. If

$$f(x+k, y) = f(x, y), \quad \text{or} \quad f(x, y+k) = f(x, y),$$

the curve consists of a series of pieces (extending from x to $x + k$ or from y to $y + k$) each obtained by moving the preceding one a distance k in the direction of a coördinate axis. In this case the function $f(x, y)$ is called *periodic* and k is its *period*. The part of the curve from x to $x + k$ or from y to $y + k$ is called a *cycle*. For example, in Ex. 4 of the previous article a cycle extends from any point (x, y) of the curve to the point $(x + 8\pi, y)$. To plot a curve whose equation is periodic it is necessary to plot one cycle and sketch the others from periodicity.

Example 1. Plot the curve $y = \tan(2x)$. If the angle $2x$ is increased by π (x increased by $\frac{1}{2}\pi$) the tangent is not changed. The curve therefore consists of a series of parts each obtained by moving the preceding one a distance $\frac{1}{2}\pi$ to the right. One of these branches passes through the origin. As x increases from 0 to $2x = \frac{1}{2}\pi$, y increases from 0 to infinity. The line $x = \frac{1}{4}\pi$ is therefore an asymptote. As x decreases from 0 to $-\frac{1}{4}\pi$, y decreases to $-\infty$, the line $x = -\frac{1}{4}\pi$ being an asymptote. The branch through the origin thus extends from $x = -\frac{1}{4}\pi$, $y = -\infty$ to $x = \frac{1}{4}\pi$, $y = \infty$.

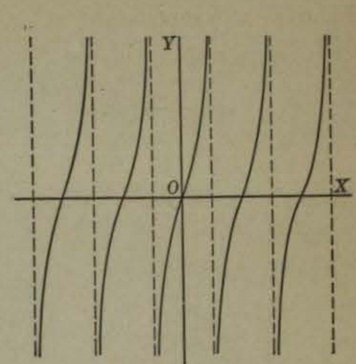

FIG. 42a.

The whole curve consists of a series of such branches at horizontal distances $\frac{1}{2}\pi$ apart (Fig. 42a).

Ex. 2. $y = \sec x$. The secant of an angle is in absolute value never less than 1. Consequently the curve lies outside the lines $y = \pm 1$. Since $\sec(-x) = \sec x$, the curve is symmetrical with respect to the y-axis. Since $\sec(x + 2\pi) = \sec x$, a complete cycle of the curve is contained between $x = 0$ and $x = 2\pi$. The most important points on this part of the curve are given in the following table:

$x = 0$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π
$y = 1$	$\infty, -\infty$	-1	$-\infty, \infty$	1

the expression $\infty, -\infty$ meaning that on the left of this point y is indefinitely large and positive but on the right indefinitely large and negative. The curve is a series of U-shaped branches alternately above $y = 1$ and below $y = -1$ (Fig. 42b).

Ex. 3. $x = \sec^2 y$. In this case x is never less than 1. The curve consists of a series of U-shaped branches on the right of $x = 1$ with asymptotes $y = \pm\frac{\pi}{2}$, $y = \pm\frac{3}{2}\pi$, etc. (Fig 42c).

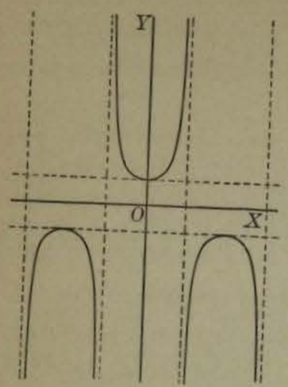

FIG. 42*b*.

Ex. 4. $y = \sin\left(\frac{1}{x}\right)$. The curve crosses the x-axis where

$$\frac{1}{x} = 0, \quad \pm\pi, \quad \pm 2\pi, \quad \pm 3\pi, \quad \ldots,$$

that is, where

$$x = \infty, \quad \pm\frac{1}{\pi}, \quad \pm\frac{1}{2\pi}, \quad \pm\frac{1}{3\pi}, \quad \ldots.$$

Between each pair of consecutive crossings it reaches one of the lines $y = \pm 1$. The curve has an infinite number of waves whose horizontal lengths approach zero near the y-axis (Fig. 42*d*).

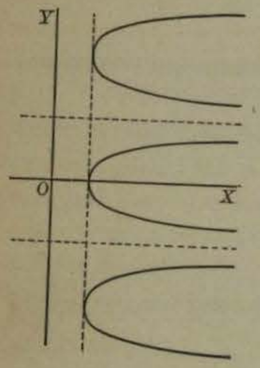

FIG. 42*c*.

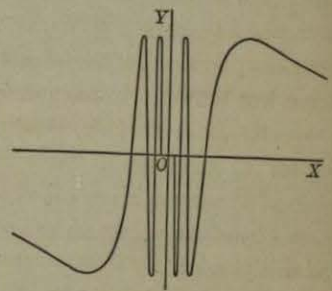

FIG. 42*d*.

Art. 43. Exponential and Logarithmic Curves

If a is a positive constant the function a^x is called an *exponential function*. It is understood that if x is a fraction a^x is the positive root.

If $y = a^x$ then, by definition, x is the logarithm of y to base a. Thus the equations

$$y = a^x, \quad x = \log_a y$$

are equivalent and both represent the same curve.

A particular number of great importance is

$$e = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots = 2.7182 \ldots.$$

Logarithms with e as base are called natural logarithms. The functions e^x and $\log_e x$ are much used in higher mathematics.

In working with exponentials and logarithms the following facts are often useful:

(1) $a^0 = 1$, $\log_a 1 = 0$, if a is not zero or infinite,

(2) $a^{\infty} = \infty$, $a^{-\infty} = 0$, $\log_a \infty = \infty$, $\log_a 0 = -\infty$, if $a > 1$,

(3) $a^{\infty} = 0$, $a^{-\infty} = \infty$, $\log_a \infty = -\infty$, $\log_a 0 = \infty$, if $a < 1$.

Example 1. Plot the curve $y = 2^x$, or $x = \log_2 y$. The curve crosses the y-axis at (0, 1). Since y is always positive the curve lies entirely above the x-axis. As x decreases to $-\infty$, y approaches zero and the curve approaches the x-axis which is an asymptote. As x increases, y increases. The increase in y for a given increase in x is greater the larger x; for, if x changes to $x + h$, the change in y is

$$2^{x+h} - 2^x = 2^x(2^h - 1),$$

and this is larger for larger values of x. If then x is increased by equal amounts h, the changes in y will form a series of steps of increasing height. The curve is thus concave upward and becomes more and more inclined to the x-axis as x increases (Fig. 43*a*).

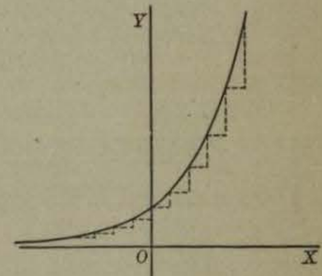

FIG. 43*a*.

Ex. 2. $y = \log_{10}\left(\frac{x-1}{x+1}\right)$ or $x = \frac{1 + 10^y}{1 - 10^y}$. The logarithm of a negative number is imaginary. Hence y is real only when $x > 1$ or $x < -1$. When $x = 1$,

$$y = \log_{10} 0 = -\infty.$$

When $x > 1$, $(x - 1)/(x + 1)$ is less than 1 but approaches 1 as x increases indefinitely. Consequently y is negative but approaches 0 as x increases indefinitely. This part of the curve is thus below the x-axis and has the x-axis and the line $x = 1$ as asymptotes. When $x = -1$, $y = \log_{10}\infty = \infty$. If $x < -1$, $(x - 1)/(x + 1)$ is greater than 1 but approaches 1 as x decreases indefinitely. This part of the curve is therefore above the x-axis and has the x-axis and the line $x = -1$ as asymptotes (Fig. 43*b*).

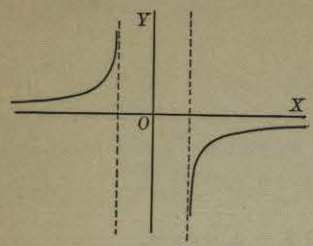

FIG. 43*b*.

Ex. 3. $y = \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$. When x is negative

$$e^{\frac{1}{x}} = \frac{1}{e^{-\frac{1}{x}}} < 1.$$

Consequently y is then negative and the curve is below the x-axis. When x is positive, y is positive and the curve is above the x-axis.

As x approaches 0 from the negative side, $e^{\frac{1}{x}}$ approaches $e^{-\infty} = 0$ and y approaches -1. As x approaches 0 from the positive side, $e^{\frac{1}{x}}$ approaches infinity and y, being the ratio of two very large numbers whose difference is 2, approaches 1. Hence, as x passes through zero from the negative to the positive side, the point (x, y) jumps from $(0, -1)$ to $(0, +1)$. The curve is *discontinuous* at $x = 0$. When x becomes very large, whether it is positive or negative, y approaches zero. The x-axis is therefore an asymptote (Fig. 43*c*.)

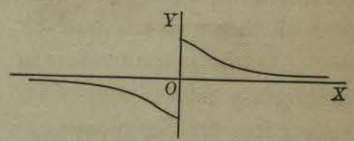

FIG. 43*c*.

Exercises

Plot the graphs of the following equations:

1. $y = \sin(2x)$.
2. $x = 2\sin(\frac{1}{2}y)$.
3. $y = 4\cos\left(x - \frac{\pi}{4}\right)$.
4. $y = 2\cos x + 3\sin x$.
5. $x = \sin y + \sin(2y)$.
6. $y = \cos(x-1) + \sin(x+1)$.
7. $y - 2 = \cos(2x + 1)$.
8. $y = \sin^2 x$.
9. $y + 1 = \cos^{-1}(x - 3)$.
10. $y = 2\tan(3x)$.
11. $x = \tan\left(y + \frac{\pi}{4}\right)$.
12. $y - 1 = \cot(x - 3)$.
13. $x = \cot^{-1}(2y)$.
14. $y = \tan^2 x$.
15. $y^2 = \tan x$.
16. $y = \sec(2x)$.
17. $x = 1 + \csc y$.
18. $y = \sec x + \tan x$.
19. $y = \sec x \csc x$.
20. $y = x \sin x$.
21. $y = \cos\left(\frac{1}{x}\right)$.
22. $y = x\sin\left(\frac{1}{x}\right)$.
23. $y = e^x$.
24. $x = 2^{-y}$.
25. $y = 10^{x^2}$.
26. $y = e^{-x}\sin x$.
27. $x = \frac{1}{2}(3^y - 3^{-y})$.
28. $y = \frac{1}{2}(e^x + e^{-x})$.
29. $y\log_{10} x = 1$.
30. $y = \log_e[x(x - 2)]$.
31. $y = \dfrac{10^{\frac{1}{x}}}{10^{\frac{1}{x}} - 1}$.
32. $y = \frac{1}{x}e^{-\frac{1}{x}}$.

Art. 44. Empirical Equations

Pairs of corresponding values of two variable quantities being given, it is sometimes desirable to find an equation connecting them. Let the pairs of values be plotted and draw a curve through the resulting points. However many points are given, the section of the curve between consecutive points can be arbitrarily drawn. Consequently an infinite number of curves can be drawn through the points. Each curve has an equation. An infinite number of equations are then satisfied by the given pairs of values. From a table of corresponding values it is not then possible to find the exact equation connecting the quantities.

It is usually assumed that if a smooth curve is drawn through or near the points, its equation will represent approximately the relation of the two quantities. Such an approximate equation is called *empirical*. A table of values being given, an infinite number of empirical equations are approximately satisfied by these values. The simplest equation should be chosen that has the required degree of accuracy. The choice of such an equation is largely a matter of judgment.

The following are types of equations it may be well to consider:

(1) $y = mx + b$.
(2) $y = ax^2 + bx + c$.
(3) $y = ax^n$.
(4) $y = ab^x$.

For convenience of plotting different unit lengths are often used along the two axes. This amounts to a uniform contraction in the direction of an axis. Equation (1) then still represents a straight line and (2) a parabola, but the constants in the equations have a different geometrical meaning.

Example 1. From the following values find an approximate equation connecting x and y:

$x = 0$	2.2	5.0	7.0	10	15
$y = 3.3$	4.0	6.0	6.5	8.4	11

These values are plotted in Fig. 44*a*. The points are seen to lie almost on a line. A line is drawn so that the points are about equally distributed on the two sides. This line crosses the y-axis at about (0, 3) and the line $x = 10$ at approximately (10, 8.4). The equation of the line through these points is

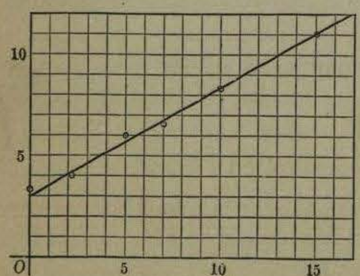

FIG. 44*a*.

$$y = 0.54\,x + 3,$$

which is the empirical equation required.

Ex. 2. Measurements of train resistance are given in the following table, where V = miles per hour, R = resistance in pounds per ton.

$V = 20$	40	60	80	100	120
$R = 5.5$	9.10	14.9	22.8	33.3	46.

The curve (Fig. 44*b*) looks like a parabola with horizontal axis. Let

$$R = A + BV + CV^2.$$

There being three coefficients, A, B, C, in this equation, the curve can be made to pass through only three of the given points. By more advanced methods the parabola could be found which fits closest to all the points. If a parabola is passed through three properly chosen points it will, however, usually be accurate enough. The points chosen should be spread over the whole curve and should not include any that appear to be faulty. In the present case numbers 1, 3 and 5 will be used. Substituting the coördinates of these points in the equation of the parabola,

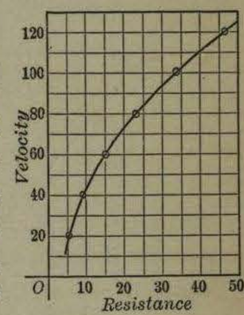

FIG. 44*b*.

$$5.5 = A + 20B + 400C,$$
$$14.9 = A + 60B + 3{,}600C,$$
$$33.3 = A + 100B + 10{,}000C.$$

The solution of these equations is

$$A = 4.18, \quad B = 0.01, \quad C = 0.0028.$$

The empirical equation found is then

$$R = 4.18 + 0.01\,V + 0.0028\,V^2.$$

The curve (Fig. 44*b*) drawn from this equation is seen to pass very close to all the points.

Ex. 3. In the table below are given the loads which cause the failure of long wrought-iron columns with round ends, in which P/a is the load in pounds per square inch and l/r is the ratio of the length of the column to the least radius of gyration of its cross-section.

l/r	P/a	$\log(l/r)$	$\log(P/a)$
140	12,800	2.1461	4.1072
180	7,500	2.2553	3.8751
220	5,000	2.3424	3.6990
260	3,800	2.4150	3.5798
300	2,800	2.4771	3.4472
340	2,100	2.5315	3.3222
380	1,700	2.5798	3.2304
420	1,300	2.6232	3.1139

Try the formula $P/a = C\,(l/r)^n$. Taking logarithms of both sides,

$$\log(P/a) = \log C + n \log(l/r).$$

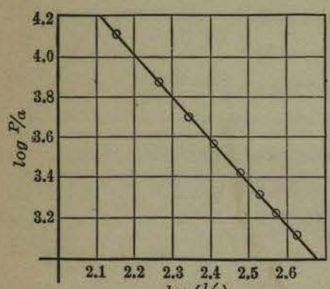

FIG. 44c.

This is a first degree equation connecting $\log(P/a)$ and $\log(l/r)$. If the formula is correct, the logarithms should then be coördinates of points on a line. Values of the logarithms are plotted in Fig. 44c. The points are almost on a line joining the first and last. The equation of this line is

$$\log(P/a) = -2.1 \log(l/r) + 8.62.$$

Consequently,

$$P/a = 417{,}000{,}000/(l/r)^{2.1},$$

which is the empirical equation required. This is approximately Euler's formula for the axial unit-load P/a which will cause a long wrought-iron column with round ends to fail.

Ex. 4. The following values were found for the amplitude of vibration of a long pendulum. Here A = amplitude in inches and t = time in minutes since the pendulum was set swinging.

$t =$	0	1	2	3	4	5	6
$A =$	10	4.97	2.47	1.22	0.61	0.30	0.14
$\log A =$	1	0.696	0.393	0.086	−0.215	−0.523	−0.854

Assume an equation of the form $A = ab^t$. Then

$$\log A = \log a + t \log b.$$

Using t and $\log A$ as coördinates the points should lie on a line. Fig. 44d shows this to be the case. The line seems to pass through the points $t = 0$ and $t = 5$. Its equation is then

$$\log A = 1 - 0.305\,t.$$

Consequently,

$$\log a = 1, \quad \log b = -0.305,$$

and so $a = 10$, $b = 0.495$. The equation required is therefore

$$A = 10\,(0.495)^t.$$

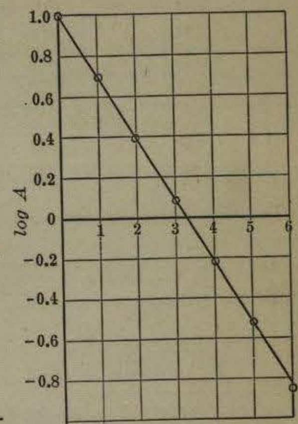

FIG. 44d.

Exercises

From the data in each of the following examples find an empirical equation connecting the quantities measured.

1. Test on square steel wire for winding guns. The stress is measured in pounds per square inch, the elongation in inches per inch.

Stress	Elongation	Stress	Elongation
5,000	0.00000	60,000	0.00216
10,000	0.00019	70,000	0.00256
20,000	0.00057	80,000	0.00297
30,000	0.00094	90,000	0.00343
40,000	0.00134	100,000	0.00390
50,000	0.00173	110,000	0.00444

2. Test on a steel column. The stress is measured in pounds per square inch, the compression in inches per inch.

Stress	Compression	Stress	Compression
3,000	0.00004	15,000	0.00039
6,000	0.00011	18,000	0.00053
9,000	0.00020	21,000	0.00066
12,000	0.00030	24,000	0.00087

3. The melting point θ, in degrees Centigrade, of a lead and zinc alloy containing x per cent lead, is given in the following table.

$x =$	40	50	60	70	80	90
$\theta =$	186	205	226	250	276	304

4. The following table gives the electromotive force E, in microvolts, produced in a lead and cadmium thermo-electric couple when the difference in temperature between the junctions is $\theta°$ C.

$\theta =$	-200	-100	0	100	200	300
$E =$	50	-140	0	475	1300	2425

5. The following table gives the number of grams S of anhydrous ammonium chloride which dissolved in 100 grams of water make a saturated solution at $\theta°$ absolute temperature.

$\theta =$	273	283	288	293	313	333	353	373
$S =$	29.4	33.3	35.2	37.2	45.8	55.2	65.6	77.3

6. The hysteresis losses in soft sheet iron subjected to an alternating magnetic flux are given in the following table, where B is flux density in kilolines per square inch, and P is watts lost per cubic inch for one cycle per second.

$B =$	20	40	60	80	100	120
$P =$	0.0022	0.0067	0.0128	0.0202	0.0289	0.0387

7. The observed temperatures θ of a vessel of cooling water at times t, in minutes, from the beginning of observation are given in the following table:

$t =$	0	1	2	3	5	7	10	15	20
$\theta =$	92°	85.3°	79.5°	74.5°	67°	60.5°	53.5°	45°	39.5°

8. Measurements showing the decay in activity of radium emanation are given in the following table:

Time in hours =	0	20.8	187.6	354.9	521.9	786.9
Relative activity =	100	85.7	24.0	6.9	1.5	0.19

CHAPTER 6

POLAR COÖRDINATES

Art. 45. Definitions

We shall now define another kind of coördinates called *polar*. Let O (Fig. 45a) be a fixed point and OX a fixed line. The point

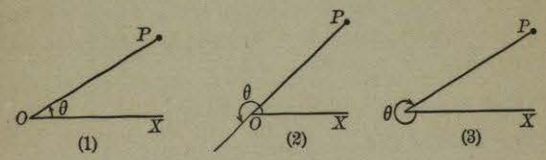

FIG. 45a.

O is called the *pole*, or origin, the line OX is called the *initial line*, or axis. The polar coördinates of a point P are the radius $r = OP$ and the angle θ from OX to OP.

The angle θ is any angle extending from OX to the line OP, the angle being considered positive when measured in the counter-clockwise direction (Fig. 45a, 1 or 2) and negative when measured in the clockwise direction (Fig. 45a, 3).

The radius r is considered positive when OP is the terminal side of θ (Fig. 45a, 1 or 3) and negative when θ terminates on OP produced.

A given point P is seen to have many pairs of polar coördinates, θ being any angle from OX to OP. A given pair of polar coördinates, however, determines a definite point obtained by constructing the angle θ and laying off r forward or backward along the terminal side according as r is positive or negative.

The point whose polar coördinates are r, θ is represented by the symbol (r, θ). To signify that P is the point (r, θ) the notation $P\ (r, \theta)$ is used.