CHAPTER 4

SECOND DEGREE EQUATIONS

Art. 29. The Ellipse

If a circle is deformed in such a way that the distances of its points from a fixed diameter are all changed in the same ratio, the resulting curve is called an ellipse.

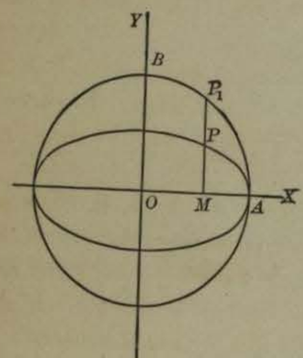

FIG. 29*a*.

For example, in Fig. 29*a*, if each point P_1 of the circle is moved to a point P such that

$$MP/MP_1 = k = \text{constant},$$

the locus of P is an ellipse.

Let the center of the circle be the origin and the fixed diameter the x-axis. Let x_1, y_1 be the coördinates of P_1 and x, y the coördinates of P. Then

$$x_1 = x, \quad y_1 = MP_1 = MP/k = y/k.$$

If a is the radius of the circle, its equation is

$$x_1^2 + y_1^2 = a^2.$$

Replacing x_1 and y_1 by their expressions in terms of x and y,

$$x^2 + \frac{y^2}{k^2} = a^2$$

is found to be the equation of the ellipse. Dividing by a^2 and replacing ka by b, the equation of the ellipse becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad (29)$$

When y is zero, x is $\pm a$, and, when x is zero, y is $\pm b$. Hence a and b are the distances, OA and OB, intercepted on the axes.

Since the equation of an ellipse contains only the squares of x and y, to each value of x correspond two values of y differing only in sign. A line perpendicular to the x-axis then cuts the curve in two points P, P' (Fig. 29*b*) equidistant from the axis. Similarly, a line perpendicular to the y-axis cuts the curve in two points P, P'' equidistant from the y-axis. This is expressed by saying that the curve is symmetrical with respect to both of the coördinate axes. They are called the *axes* of the curve.

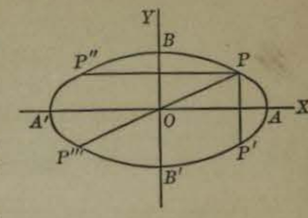

FIG. 29*b*.

Any line through O cuts the curve in two points $P\,(x, y)$, $P'''\,(-x, -y)$ equidistant from O. For this reason the curve is called symmetrical with respect to the origin and the point O is called the *center* of the curve.

The ellipse cuts the axes in four points A', A, B', B. The segments $A'A$ and $B'B$ are sometimes called the axes of the curve. The longer of these is called the *major axis*, the shorter the *minor axis*. The distances OA and OB, equal to a and b, are called the *semi-axes* of the curve. The ends of the major axis are called *vertices*.

Art. 30. The Ellipse in Other Positions

Equation (29) represents an ellipse whose axes are the coördinate axes. The equation can be stated in a form valid in any position. In fact, since $x = NP$, $y = MP$, $a = OA$, $b = OB$, equation (29) is equivalent to

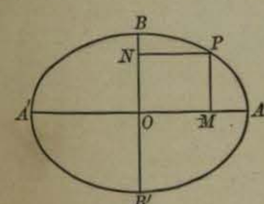

FIG. 30*a*.

$$\frac{NP^2}{OA^2} + \frac{MP^2}{OB^2} = 1, \qquad (30a)$$

that is, *the ratios, obtained by dividing the squares of the distances of any point on the ellipse from the axes by the squares of the parallel semi-axes, have a sum equal to* 1.

For example, if the center of the ellipse is the point (h, k) and the axes of the curve are parallel to the coördinate axes (Fig. 30b),

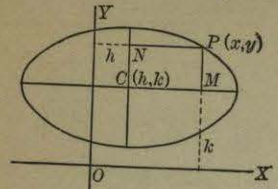

FIG. 30b.

$$NP = x - h, \quad MP = y - k$$

and the equation of the curve is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \qquad (30b)$$

a and b being the semi-axes parallel to OX and OY respectively.

Example 1. Find the equation of the ellipse with vertices $A'(-3, 2)$, $A(5, 2)$ and semi-axes equal to 4 and 1.

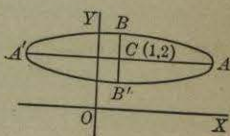

FIG. 30c.

The segment $A'A$ is the major axis (Fig. 30c). Its middle point is the center of the curve. Consequently, the center is $C(1, 2)$. Also the semi-axes are

$$CA = 4, \quad CB = 1.$$

The equation of the ellipse is therefore

$$\frac{(x-1)^2}{16} + \frac{(y-2)^2}{1} = 1.$$

Ex. 2. Show that the equation $9x^2 + 4y^2 + 36x - 24y + 36 = 0$ represents an ellipse. Find its center and axes.

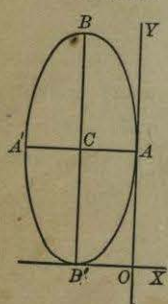

FIG. 30d.

The equation can be written

$$9(x^2 + 4x) + 4(y^2 - 6y) + 36 = 0.$$

Completing the squares in the parentheses,

$$9(x+2)^2 + 4(y-3)^2 = 36,$$

or

$$\frac{(x+2)^2}{4} + \frac{(y-3)^2}{9} = 1.$$

Comparing this with equation (30b) it is seen to represent an ellipse with center $(-2, 3)$. The axes are horizontal and vertical lines through the center. Their equations are $x = -2$, $y = 3$. The

major axis is vertical, the minor axis horizontal. The vertices, at the ends of the major axis, are $B'(-2, 0)$, $B(-2, 6)$.

Ex. 3. Find the equation of the ellipse whose axes are the lines $2x + y = 3$, $x - 2y = 0$, and whose semi-axes along those lines are 1 and 3 respectively.

Let $P(x, y)$ be any point on the curve (Fig. 30e). Then

$$NP = \frac{2x + y - 3}{\pm\sqrt{5}}, \quad MP = \frac{x - 2y}{\pm\sqrt{5}}.$$

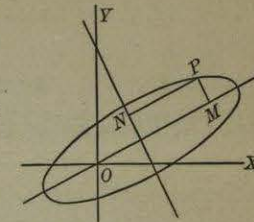

FIG. 30e.

The equation of the ellipse is $NP^2/9 + MP^2/1 = 1$, whence

$$\frac{(2x + y - 3)^2}{45} + \frac{(x - 2y)^2}{5} = 1.$$

Exercises

Make graphs of the following equations. Show that the curves are ellipses. Find their centers and semi-axes.

1. $x^2 + 2y^2 = 6$.
2. $4x^2 + y^2 - 8x + 4y + 7 = 0$.
3. $x^2 + 2y^2 = x + y$.
4. $3x^2 + 2y^2 - 6x + 8y = 1$.
5. $x^2 + 3y^2 + 6x - 6y = 1$.
6. $5x^2 + 2y^2 - 10x + 4y + 7 = 0$.

7. Find the equation of the ellipse with center $(1, -3)$ and semi-axes, parallel to OX and OY, whose lengths are 2 and 3 respectively.

8. Find the equation of the ellipse with center $(-2, 4)$ tangent to both coördinate axes.

9. Find the equation of the ellipse whose axes are the lines

$$x + y - 2 = 0, \quad x - y + 2 = 0,$$

and whose semi-axes along those lines have lengths equal to 1 and 4 respectively.

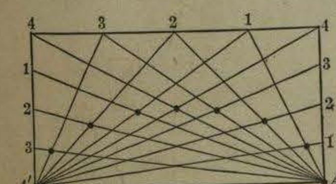

FIG. 30f.

10. Show that an oblique plane section of a right circular cylinder is an ellipse.

11. Three sides of a rectangle are divided into an equal number of parts and the points of division connected by straight lines with the opposite corners as shown in Fig. 30f. Show that the intersections of lines through like numbered

points are on an ellipse with axes equal in length to the sides of the rectangle.

12. Show that the locus of points, the sum of whose distances from two fixed points is constant, is an ellipse. Let the fixed points be $(-c, 0)$, $(+c, 0)$ and let the constant distance be $2a$.

Art. 31. The Parabola

Let LK, RS be perpendicular lines and MP, NP perpendiculars from any point P to them. If a is constant and NP considered positive when P is on one side of RS, negative when on the other, the locus of points P such that

$$MP^2 = a \cdot NP \qquad (31a)$$

is called a parabola. *A parabola is thus a locus of points the squares of whose distances from one of two perpendicular lines are proportional to their distances from the other.* The complete locus of such points is two parabolas, one on each side of RS.

To each value of NP correspond two values of MP differing only in sign. The curve is therefore symmetrical with respect to LK

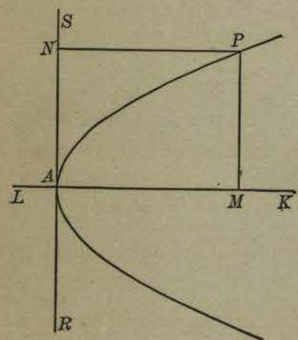

FIG. 31*a*.

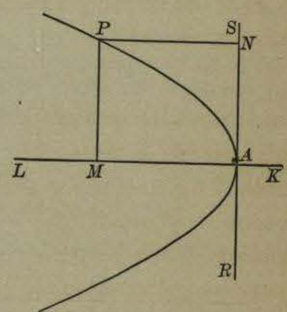

FIG. 31*b*.

which is called the axis of the parabola. The point A is called the vertex. The curve passes through A but, since $a \cdot NP$ is positive, it does not cross RS.

If LK is the x-axis, RS the y-axis, the equation of the curve is $y^2 = ax$. If a is positive x must be positive and the curve is on the

right of the y-axis as in Fig. 31*a*. If a is negative, x must be negative and the curve is on the left of the y-axis as in Fig. 31*b*.

If the axis of the parabola is parallel to OX and the vertex is

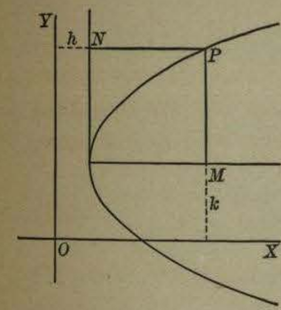

FIG. 31*c*.

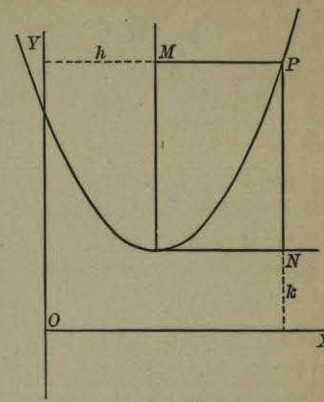

FIG. 31*d*.

(h, k) (Fig. 31*c*), $NP = x - h$, $MP = y - k$, and the equation of the parabola is

$$(y - k)^2 = a(x - h). \qquad (31b)$$

If (h, k) is the vertex and the axis is parallel to OY (Fig. 31*d*) the equation of the parabola is

$$(x - h)^2 = a(y - k). \qquad (31c)$$

If the axis of the parabola is not parallel to either coördinate axis, its equation can be obtained from (31*a*) by expressing MP and NP in terms of the coördinates of P.

Example 1. Show that $y^2 = 3x + 2y - 4$ is the equation of a parabola. Find its vertex and axis.

Transposing and completing the square, the equation becomes

$$(y - 1)^2 = 3x - 3 = 3(x - 1).$$

Comparing this with the equation

$$(y - k)^2 = a(x - h),$$

it is seen to represent a parabola for which

$$h = 1, \quad k = 1, \quad a = 3.$$

The vertex is the point $(1, 1)$ and the axis is the line $y = 1$.

Ex. 2. Find the equation of the parabola with vertex (1, 2) and axis $y = x + 1$, which passes through the point (3, 7).

The line through the vertex perpendicular to the axis is $x + y - 3 = 0$. If x, y are the coördinates of any point P (Fig. 31*e*), then

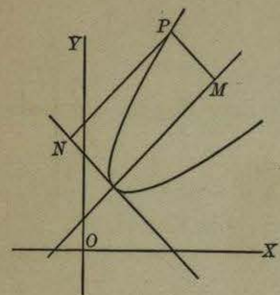

FIG. 31*e*.

$$MP = \frac{y - x - 1}{\pm\sqrt{2}},$$

$$NP = \frac{x + y - 3}{\pm\sqrt{2}}.$$

If P is a point on the parabola, $MP^2 = a \cdot NP$, whence

$$(y - x - 1)^2 = \pm a\sqrt{2}\,(x + y - 3).$$

Since the curve passes through (3, 7)

$$9 = \pm a\sqrt{2}\,(7).$$

This value of a substituted in the previous equation gives

$$7\,(y - x - 1)^2 = 9\,(x + y - 3)$$

as the equation of the parabola.

Ex. 3. An arch has the form of a parabola with vertical axis (Fig. 31*f*). If the arch is 10 feet high at the center and 30 feet wide at its base, find its height at a distance of 5 feet from one end.

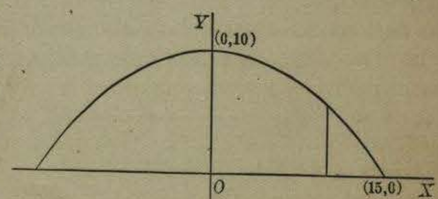

FIG. 31*f*.

Take the origin at the middle point of the base of the arch. The vertex is then (0, 10). The equation of the arch is consequently

$$(x - 0)^2 = a\,(y - 10).$$

The curve crosses the x-axis at (15, 0). Hence

$$15^2 = a\,(-10).$$

Substituting this value of a, the equation of the arch becomes

$$10\,x^2 = 225\,(10 - y).$$

At a point 5 feet from one end $x = \pm 10$. The corresponding value of y is 50/9, which is the height of the arch at that point.

Exercises

Make graphs of the following equations. Show that the curves are parabolas. Find their axes and vertices.

1. $y^2 = 8x - 4$.	4. $y = (x - 1)(x + 2)$.
2. $y^2 = -2x + 1$.	5. $x = y^2 - 3y$.
3. $y = x^2 - 2x + 3$.	6. $x^2 - 3x + 2y - 4 = 0$.

7. Find the equation of the parabola with horizontal axis and vertex at the origin, which passes through (3, 4).

8. Find the equation of the parabola with vertical axis and vertex at $(-2, 2)$, which passes through $(1, -3)$.

9. An arch in the form of a parabola with vertical axis is 29 feet across the bottom and its highest point is 8 feet above the base. What is the length of the beam placed horizontally across the arch 4 feet from the top?

10. A cable of a suspension bridge hangs in the form of a parabola with vertical axis. The roadway, which is horizontal and 240 feet long, is supported by vertical wires attached to the cable, the longest being 80 feet and the shortest 30 feet. Find the length of the supporting wire attached to the roadway 40 feet from the middle.

11. A point moves so that its distance from a fixed point is equal to its distance from a fixed line. Show that the locus described is a parabola.

12. Two sides, AB and BC, of a rectangle are divided into an equal number of parts and the points of division numbered as shown in Fig. 31*g*. Through the points of AB lines are drawn parallel to BC, and through those of BC lines are drawn passing through A. Show that the intersections of lines through like numbered points are on a parabola.

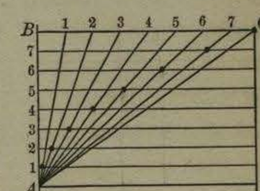

FIG. 31*g*.

Art. 32. The Hyperbola

Let KL and RS (Fig. 32*a*) be two straight lines intersecting in C, PM and PN the perpendiculars from any point P to these lines. Let MP be considered positive when P is on one side of KL, negative when on the other side. Similarly, let NP be positive when P is on one side of RS, negative when on the other side. A hyperbola is the locus of points P such that the product

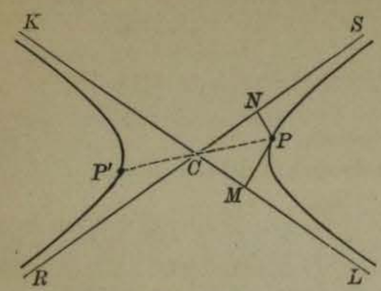

FIG. 32*a*.

$$MP \cdot NP = \text{constant}. \quad (32a)$$

If the signs of MP and NP are both changed the product is not changed. Hence if any point P lies on the curve, the point P' at equal distance on the other side of C is on the curve. The hyperbola is thus symmetrical with respect to C which is called the *center*. The curve consists of two parts in a pair of vertical angles determined by KL and RS. *The hyperbola is a locus of points the product of whose distances from two lines is constant.* The complete locus of such points is however two hyperbolas, one in each pair of vertical angles between the lines.

If MP is very large, NP must be very small and conversely. As it goes to an indefinite distance the curve thus approaches indefinitely near the lines KL and RS. They are called *asymptotes*.

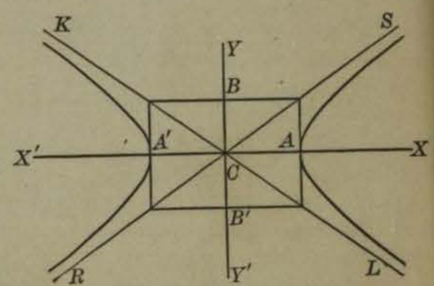

FIG. 32*b*.

Let C be the origin (Fig. 32*b*) and take as x-axis the line bisecting the vertical angles in which the curve lies. The curve crosses the x-axis at two points A', A. Construct the rectangle with sides through A' and A, having KL and RS as diagonals. Let $CA = a$, $CB = b$. The equations of KL and RS are then

$$y = \pm (b/a)\, x. \quad (32b)$$

Consequently, Fig. 32*a*,

$$MP = \frac{bx + ay}{\pm\sqrt{b^2 + a^2}}, \qquad NP = \frac{bx - ay}{\pm\sqrt{b^2 + a^2}}.$$

If P is a point on the hyperbola, $MP \cdot NP = $ const., whence

$$b^2x^2 - a^2y^2 = \pm (b^2 + a^2) \text{ const.} = k.$$

Since the hyperbola passes through $A\,(a, o)$

$$b^2a^2 = k.$$

Substituting this value of k and dividing by a^2b^2, the equation of the hyperbola becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \quad (32c)$$

Since the equation contains only squares of x and y the curve is symmetrical with respect to the coördinate axes. They are called the *axes* of the curve. The axis cutting the curve is called *transverse*, the one not cutting the curve is called *conjugate*. The distances $CA = a$, $CB = b$ are called the *semi-axes*. The points A' and A, where the transverse axis cuts the curve, are called *vertices*.

Equation (32*c*) represents the hyperbola referred to its axes, the x-axis being transverse. If the transverse axis is parallel to the x-axis and the center is (h, k) (Fig. 32*c*),

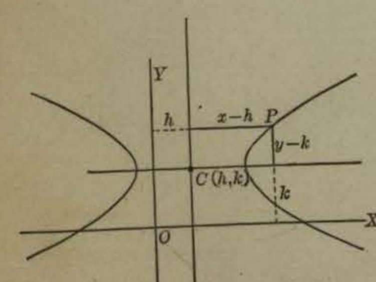

FIG. 32*c*.

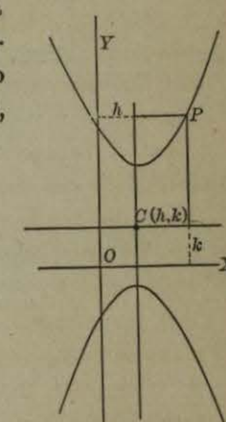

FIG. 32*d*.

the coördinates relative to the axes are $x - h$ and $y - k$. The equation of the hyperbola is then

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1. \tag{32d}$$

If the transverse axis is parallel to OY (Fig. 32*d*) and the center is (h, k) the equation of the hyperbola is

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1. \tag{32e}$$

If the axes of the curve are not parallel to the coördinate axes, its equation can be obtained by using the definition or by replacing x and y in equation (32*c*) by the distances of a point P from the axes of the curve.

Art. 33. The Rectangular Hyperbola

If the asymptotes of a hyperbola are perpendicular to each other it is called rectangular. In this case the asymptotes are usually

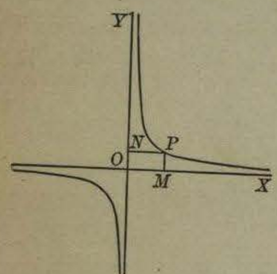

FIG. 33*a*.

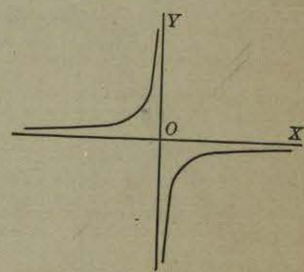

FIG. 33*b*.

taken as coördinate axes. The definition, $MP \cdot NP = \text{const.}$, gives the equation of the curve in the form

$$xy = k. \tag{33a}$$

If k is positive the curve lies in the first and third quadrants (Fig. 33*a*), if k is negative it lies in the second and fourth quadrants (Fig. 33*b*).

The axes of the rectangular hyperbola make angles of 45° with the asymptotes. The rectangle, Fig. 32*b*, is a square and $a = b$. If

the axes of the rectangular hyperbola are taken as axes of coördinates, its equation is then

$$x^2 - y^2 = a^2, \tag{33b}$$

the x-axis being transverse.

Example 1. Show that $9x^2 - 4y^2 + 18x + 16y - 43 = 0$ is the equation of a hyperbola. Find its center, axes and asymptotes.

The equation can be written

$$9(x^2 + 2x) - 4(y^2 - 4y) = 43.$$

Completing the squares,

$$9(x+1)^2 - 4(y-2)^2 = 36,$$

or

$$\frac{(x+1)^2}{4} - \frac{(y-2)^2}{9} = 1.$$

Comparing this with equation (32*d*), it is seen to represent a hyperbola with center $(-1, 2)$ and semi-axes, $a = 2$, $b = 3$. The transverse axis is the line $y = 2$. The conjugate axis is $x = -1$. The asymptotes are lines through the center with slopes $\pm b/a$. Their equations are consequently

$$y - 2 = \pm\tfrac{3}{2}(x+1).$$

Ex. 2. Show that $xy = 2x - 3y$ is the equation of a rectangular hyperbola. Find its center, axes and asymptotes.

The equation can be written

$$(x+3)(y-2) = -6.$$

The quantities $x + 3$ and $y - 2$ are coördinates of $P(x, y)$ with respect to axes through $(-3, 2)$ parallel to OX and OY (Fig. 33*c*). The curve is therefore a rectangular hyperbola with center $(-3, 2)$ and asymptotes $x = -3$ and $y = 2$. The axes pass through $(-3, 2)$ making angles of 45°

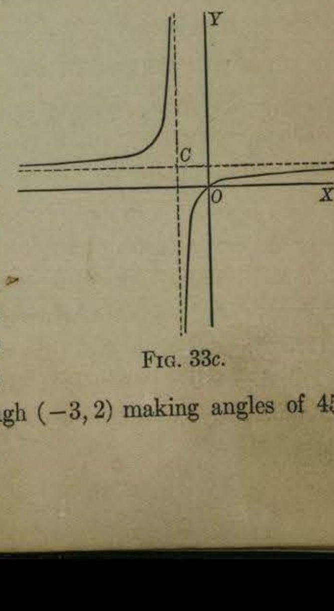

FIG. 33*c*.

Explanation: — but x will have value if $\frac{1}{5a^2}-\frac{4}{5b^2} \neq 0$, which according to the curve, can never be, so $\frac{1}{5a^2}+\frac{4}{5b^2}=0$ in order that x shall not be determined when $y=0$

with the asymptotes. Their equations are consequently

$$y - 2 = \pm(x + 3).$$

Ex. 3. Find the equation of the hyperbola with asymptotes $x - y = 1$ and $x = 2$, which passes through (3, 4).

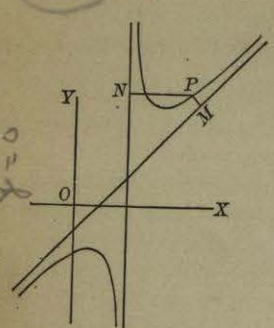

FIG. 33*d*.

Let $P\ (x,\ y)$ be a point on the curve. Then (Fig. 33*d*)

$$MP = \frac{x - y - 1}{\pm\sqrt{2}},\quad NP = x - 2.$$

The equation of the curve is $MP \cdot NP$ = constant. Consequently,

$$(x - y - 1)(x - 2) = \text{constant}.$$

Since the curve passes through (3, 4), the constant in this equation is $(3 - 4 - 1)(3 - 2) = -2$. The equation required is then

$$(x - y - 1)(x - 2) = -2.$$

Ex. 4. Find the equation of the hyperbola with center at the origin, transverse axis $y - 2x = 0$, which passes through (0, 2) and has the x-axis as an asymptote.

The conjugate axis, being perpendicular to the transverse axis at the center, is $x + 2y = 0$. In equation (32*c*) x and y can be replaced by the distances of a point on the hyperbola from the axes of the curve. In the present case these distances are $(x + 2y)/\sqrt{5}$ and $(y - 2x)/\sqrt{5}$. Hence the equation of the curve is

$$\frac{(x + 2y)^2}{5a^2} - \frac{(y - 2x)^2}{5b^2} = 1.$$

Since the curve passes through (0, 2) $16/5a^2 - 4/5b^2 = 1$. Since the x-axis is an asymptote, there must be no point of intersection of the curve and x-axis. When y is zero the equation becomes $x^2(1/5a^2 - 4/5b^2) = 1$. This will fail to determine a value of x if $1/5a^2 - 4/5b^2 = 0$. This and the previous equation solved simultaneously give $a^2 = 3$, $b^2 = 12$. The equation required is

therefore

$$\frac{(x + 2y)^2}{15} - \frac{(y - 2x)^2}{60} = 1.$$

Exercises

Show that the following equations represent hyperbolas. Find their centers, axes and asymptotes.

1. $4x^2 - 3y^2 + 12y = 24$.
2. $5x^2 - y^2 - 2y = 4$.
3. $xy + x - y = 3$.
4. $2x^2 - 3y^2 + 4x + 12y = 4$.
5. $x^2 - y^2 - 2x - 6y = 8$.
6. $2xy = 3x - 4$.
7. Find the equations of the two hyperbolas with center $(2, -1)$ and semi-axes, parallel to OX and OY, whose lengths are 1 and 4 respectively.
8. Find the equation of the hyperbola with center $(-2, 1)$, and axes parallel to the coördinate axes, passing through (0, 2) and $(1, -4)$.
9. Find the equations of the hyperbolas whose axes are the lines $3x + 2y = 0$, $2x - 3y = 0$ and whose semi-axes along those lines are equal to 2 and 5 respectively.
10. Show that the locus of a point, the difference of whose distances from two fixed points is constant, is a hyperbola. Let the fixed points be $(-c, 0)$, $(+c, 0)$ and let the difference of the distances be $2a$.
11. A point moves so that the product of the slopes of the lines joining it to $(-a, 0)$ and $(a, 0)$ is constant. Show that it describes an ellipse or a hyperbola.

Art. 34. The Second Degree Equation

An equation of the second degree in rectangular coördinates has the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{34a}$$

A, B, C, D, E, F being constants. The equations of the circle, ellipse, parabola and hyperbola are all of this kind. If the polynomial forming the left side of the equation can be resolved into a product of first degree factors, the equation is said to be *reducible*. If the polynomial cannot be so factored the equation is called *irreducible*.

Reducible Equations. — If the polynomial forming the left side of (34*a*) can be resolved into a product of first degree factors, the equation has the form

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0.$$

Since a product is zero when and only when one of its factors is zero, the equation is satisfied by all values of x and y such that either

$$a_1x + b_1y + c_1 = 0$$

or

$$a_2x + b_2y + c_2 = 0$$

and by no others. If the coefficients a_1, b_1, etc., are all real these equations represent straight lines. The locus of the equation is then a pair of straight lines (a single line if the factors are equal). If some of the coefficients are imaginary, the locus will usually be a single point whose coördinates make both factors vanish.

Example 1. Determine the locus of the equation $2x^2 - xy - 3y^2 = 0$.

The equation is equivalent to

$$(x + y)(2x - 3y) = 0.$$

The locus is two lines, $x + y = 0$ and $2x - 3y = 0$, passing through the origin.

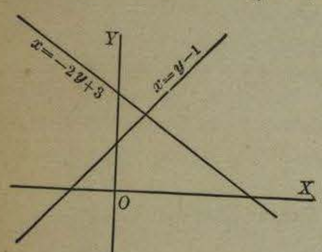

FIG. 34a.

Ex. 2. Determine the locus represented by the equation

$$x^2+xy-2y^2-2x+5y-3=0.$$

Arranged in powers of x the equation is

$$x^2+(y-2)x-(2y^2-5y+3) = 0.$$

Solving by the quadratic formula,

$$\begin{aligned} x &= -\tfrac{1}{2}(y-2) \pm \tfrac{1}{2}\sqrt{9y^2 - 24y + 16} \\ &= -\tfrac{1}{2}(y-2) \pm \tfrac{1}{2}(3y-4). \end{aligned}$$

There are then two solutions $x = y - 1$ and $x = -2y + 3$. The original equation is satisfied if either of these is satisfied. The locus is two straight lines.

Ex. 3. Determine the locus of the equation

$$x^2 + 3y^2 - 2x + 12y + 13 = 0.$$

When the squares are completed this becomes

$$(x-1)^2 + 3(y+2)^2 = 0.$$

The sum of squares of real numbers can only be zero when all are zero. The only real numbers satisfying this equation are then

$$x = 1, \quad y = -2.$$

The polynomial has imaginary factors, $(x-1) \pm (y-2)\sqrt{-3}$, but the locus has one real point.

Irreducible Equations. — It will be shown later (Art. 59) that the locus of an irreducible equation of the second degree is an ellipse, parabola, hyperbola or entirely imaginary. A circle is considered as an ellipse with axes of equal length.

By the second degree part of an equation of the second degree is meant the part

$$Ax^2 + Bxy + Cy^2 \tag{34b}$$

containing the terms of second degree. We shall now show how to determine by inspection of this second degree part whether a given second degree equation represents an ellipse, parabola or hyperbola.

An ellipse whose axes are the lines

$$A_1x + B_1y + C_1 = 0, \quad A_2x + B_2y + C_2 = 0$$

is represented by the equation

$$\frac{1}{a^2}\left(\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}}\right)^2 + \frac{1}{b^2}\left(\frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}}\right)^2 = 1.$$

The second degree part of this equation is

$$\text{(1)} \qquad \frac{(A_1x + B_1y)^2}{a^2(A_1^2 + B_1^2)} + \frac{(A_2x + B_2y)^2}{b^2(A_2^2 + B_2^2)}.$$

Since this is a sum of squares it has imaginary factors.

If the axis of a parabola is $A_1x + B_1y + C_1 = 0$, and the line through the vertex perpendicular to the axis is $A_2x + B_2y + C_2 = 0$, the equation of the curve is

$$\left(\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}}\right)^2 = a\left(\frac{A_2x + B_2y + C_2}{\pm\sqrt{A_2^2 + B_2^2}}\right).$$

The second degree part of this equation is

$$\text{(2)} \qquad \frac{(A_1x + B_1y)^2}{A_1^2 + B_1^2},$$

which is a complete square.

If the asymptotes of a hyperbola are

$$A_1x + B_1y + C_1 = 0, \quad A_2x + B_2y + C_2 = 0,$$

its equation is

$$\left(\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}}\right)\left(\frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}}\right) = \text{constant.}$$

The second degree part of the equation is

$$(3) \qquad \left(\frac{A_1x + B_1y}{\sqrt{A_1^2 + B_1^2}}\right)\left(\frac{A_2x + B_2y}{\sqrt{A_2^2 + B_2^2}}\right),$$

which is a product of real first degree factors.

Inspection of (1), (2) and (3) shows that the equations of ellipse, parabola and hyperbola are distinguished by the fact that *the second degree part has imaginary factors in case of the ellipse, is a complete square in case of the parabola, and has real and distinct factors in case of the hyperbola.*

Example 1. Show that $8x^2 - 8xy + 2y^2 = 2x - 3$ is the equation of a parabola.

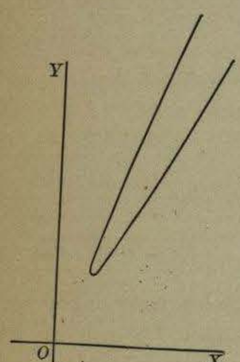

FIG. 34b.

Solving for y,

$$y = 2x \pm \tfrac{1}{2}\sqrt{4x - 6}.$$

Since y is an irrational function of x the equation is irreducible. That the curve is real is shown by plotting (Fig. 34b). The second degree part of the equation is

$$8x^2 - 8xy + 2y^2 = 2(2x - y)^2.$$

This being a square the curve is a parabola.

Ex. 2. Show that $x^2 + xy + y^2 = 3$ is the equation of an ellipse.

Solving for y,

$$y = \tfrac{1}{2}(-x \pm \sqrt{12 - 3x^2}).$$

The equation is irreducible and represents a real curve. The second degree part is

$$x^2 + xy + y^2 = (x + \tfrac{1}{2}y)^2 + \tfrac{3}{4}y^2.$$

This, being a sum of squares, has imaginary factors and the curve is an ellipse (Fig. 34c).

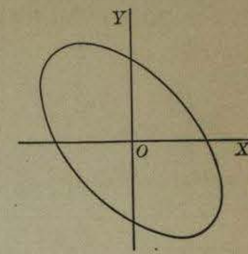

FIG. 34c.

Ex. 3. Show that $x^2 + xy = 2$ is the equation of a hyperbola.

The equation is irreducible and represents a real curve. The second degree part of the equation

$$x^2 + xy = x(x + y)$$

has real and distinct factors. The curve is therefore a hyperbola

Exercises

Plot and determine the nature of the loci represented by the following equations:

1. $x^2 + 2xy - 3y^2 = 0.$
2. $2x^2 + 3xy - y^2 = 0.$
3. $4x^2 + 12xy + 9y^2 = 0.$
4. $3x^2 - 2xy + 3y^2 = 0.$
5. $2x^2 - xy - y^2 - 4x + y + 2 = 0.$
6. $x^2 + 2xy + y^2 + 4x + 4y + 4 = 0.$
7. $xy = 7x.$
8. $x^2 - 2xy + y^2 = 4x.$
9. $x^2 + y^2 = 2x - 3y + 4.$
10. $x^2 + 2xy + 2y^2 = 5.$
11. $x^2 - xy = 4y.$
12. $4x^2 - 4xy + y^2 = 4x - 5y.$
13. $5x^2 + 6xy + 2y^2 - 4x - 2y + 1 = 0.$
14. $(x - y + 3)(2x - y - 1) = 4.$
15. $xy = 2x + 2y - 4.$
16. $x^2 - 2\sqrt{2}xy + 2y^2 = 4x.$
17. $2x^2 + 2y^2 - 4x + 6y = 7.$
18. $3x^2 + 2xy + 2y^2 - 4x - 4y = 4.$
19. $4x^2 + 3xy - 2y^2 + 4x + 7y - 6 = 0.$
20. $xy = 3y - 2x + 6.$

Art. 35. Locus Problems

A locus is often defined by a property of a moving point. The locus is the totality of points having the property. A pair of coördinate axes being given, the equation of the locus is an equation satisfied by the coördinates of every point on it and by no others. To find this equation choose as axes whatever perpendicular lines seem most convenient and let (x, y) be any point of the locus. In terms of x, y and any constant quantities occurring in the problem, express the property used as definition of the locus. The result will be an equation of the locus. In some cases this result can be reduced to a simpler form.

Example 1. The vertices A and B of a triangle ABC are fixed (Fig. 35a). Find the locus of the vertex C if $A + B = \frac{3}{4}\pi$.

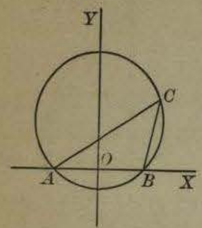

FIG. 35a.

The angle C will be

$$C = \pi - A - B = \frac{\pi}{4}.$$

Since this angle is constant the locus is a circle. To find its equation take the middle point of AB as origin and the line AB as x-axis. Let $AO = OB = a$. Then A and B are $(-a, 0)$ and $(a, 0)$. The definition of the locus is $A + B = \frac{3}{4}\pi$, whence

$$\tan(A + B) = -1 = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Now $\tan A$ is the slope of AC and $\tan B$ is the negative of the slope of BC. Hence

$$\tan A = \frac{y}{x + a}, \qquad \tan B = \frac{-y}{x - a}.$$

Substituting these values in the expression for $\tan(A + B)$, the equation of the circle is found to be

$$x^2 + y^2 = 2ay + a^2.$$

Ex. 2. A segment has its ends in the coördinate axes and determines with them a triangle of constant area. Find the locus of the middle point of the segment.

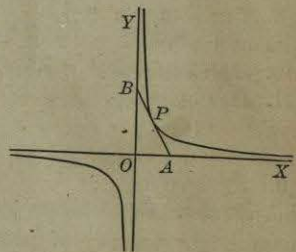

FIG. 35b.

Let the segment be AB (Fig. 35b). Let $OA = a$, $OB = b$. The area of the triangle OAB is

$$K = \tfrac{1}{2}ab,$$

K being constant. If x and y are the coördinates of the middle point P, then $a = 2x$, $b = 2y$ and

$$K = 2xy.$$

This is an equation satisfied by the coördinates of any point on the locus. Conversely, if the coördinates of any point satisfy this

equation, the segment AB whose intercepts are $OA = 2x$, $OB = 2y$ will have P as its middle point and will determine with the coördinate axes a triangle of area K. Therefore $K = 2xy$ is the equation of the locus. The curve is a rectangular hyperbola with the axes as asymptotes.

Exercises

1. A point moves so that the sum of the squares of its distances from the four sides of a square is equal to twice the area of the square. Find its locus.

2. A point moves so that its shortest distance from a fixed circle is equal to its distance from a fixed diameter of the circle. Find its locus.

3. In a triangle ABC, A and B are fixed. Find the locus of C, if $A - B = \frac{1}{4}\pi$.

4. A point moves so that the sum of the squares of its distances from the three sides of an equilateral triangle is equal to the square of one side of the triangle. Find its locus.

5. A point moves so that the square of its distance from the base of an isosceles triangle is equal to the product of its distances from the other two sides. What is its locus? Show that it passes through the vertices of the two base angles.

6. On a level plane the crack of a rifle and the thud of the bullet striking the target are heard at the same instant. Find the locus of the hearer.

7. A point moves so that the ratio of its distance from a fixed point to its distance from a fixed straight line is a constant e. Show that the locus is an ellipse if $e < 1$, a parabola if $e = 1$ and a hyperbola if $e > 1$.

8. AB and CD are two segments bisecting each other at right angles. Show that the locus of a point P which moves so that $PA \cdot PB = PC \cdot PD$ is a rectangular hyperbola.

9. OA and OB are fixed straight lines, P any point, and PM, PN the perpendiculars from P on OA, OB. Find the locus of P if the quadrilateral $OMPN$ has a constant area.

10. AB is a fixed diameter of a circle and AC is any chord; P and Q are two points on the line AC such that $QC = CP = CB$. Find the locus of P and Q as AC turns about A.