Exercises

1. If $f(x) = x^2 - 3x + 2$, show that $f(1) = f(2) = 0$.
2. If $f(x) = x + \frac{1}{x}$, find $f(x+1)$. Also find $f(x) + 1$.
3. If $f(x) = \sqrt{x^2 - 1}$, find $f(2x)$. Also find $2f(x)$.
4. If $f(x) = \frac{x+2}{2x-3}$, find $f\left(\frac{1}{x}\right)$. Also find $\frac{1}{f(x)}$.
5. If $\psi(x) = x^4 + 2x^2 + 3$, show that $\psi(-x) = \psi(x)$.
6. If $\phi(x) = x + \frac{1}{x}$, show that $[\phi(x)]^2 = \phi(x^2) + 2$.
7. If $F(x) = \frac{1-x}{1+x}$, show that $F(a)\,F(-a) = 1$.
8. If $f_1(x) = 2^x$, $f_2(x) = x^2$, find $f_1[f_2(y)]$. Also find $f_2[f_1(y)]$.
9. If $f(x, y) = x^2 + 2xy - 5$, show that $f(1, 2) = 0$.
10. If $F(x, y) = x^2 + xy + y^2$, show that $F(x, y) = F(y, x)$.
11. If $f(x, y) = x^3 + 3x^2y + y^3$, show that $f(x, vx) = x^3 f(1, v)$.
12. If a, b, c are the sides of a right triangle how many of them can be taken as independent variables?
13. Express the radius and area of a sphere in terms of the volume taken as independent variable.
14. Given $u = x^2 + y^2$, $v = x + y$, determine x and y as functions of the independent variables u and v.
15. If x, y, z satisfy the equations

$$\begin{aligned} x + y + z &= 6,\\ x - y + 2z &= 5,\\ 2x + y - z &= 1, \end{aligned}$$

show that none of them can be independent variables.

16. The equations

$$\begin{aligned} x + y + z &= 6,\\ x - y + 2z &= 5,\\ 2x + 4y + z &= 13 \end{aligned}$$

are dependent. Show that any one of the quantities x, y, z can be taken as independent variable.

17. If u, v, x, y are connected by the equations

$$u^2 + uv - y = 0, \qquad uv + x - y = 0,$$

show that u and x cannot both be independent variables.

CHAPTER 2

RECTANGULAR COÖRDINATES

Art. 11. Definitions

Scale on a Line. — In Art. 1 it has been shown that real numbers can be attached to the points of a straight line in such a way that the distance between two points is equal to the difference (larger minus smaller) of the numbers located at those points.

The line with its associated numbers is called a scale. Proceeding along the scale in one direction (to the right in Fig. 11*a*) the

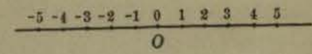

FIG. 11*a*.

numbers increase algebraically. Proceeding in the other direction the numbers decrease. The direction in which the numbers increase is called positive, that in which they decrease is called negative.

Coördinates of a Point. — In a plane take two perpendicular scales $X'X$, $Y'Y$ with their zero points coincident at O (Fig. 11*b*). It is customary to draw $X'X$, called the *x-axis*, horizontal with its positive end on the right, and $Y'Y$, called the *y-axis*, vertical with its positive end above. The point O is called the origin. The axes divide the plane into four sections called *quadrants*. These are numbered I, II, III, IV, as shown in Fig. 11*b*.

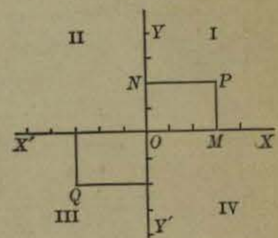

FIG. 11*b*.

From any point P in the plane drop perpendiculars PM, PN to the axes. Let the number at M in the scale $X'X$ be x and that at N in the scale $Y'Y$ be y. These numbers x and y are called the *rectangular coördinates* of P with respect to the axes $X'X$, $Y'Y$. The number x is called the *abscissa*, the number y is called the *ordinate* of P.

If the axes are drawn as in Fig. 11*b*, the abscissa x is equal in magnitude to the distance PN from P to the y-axis, is positive for points on the right and negative for points on the left of the y-axis. The ordinate y is equal in magnitude to the distance PM from P to the x-axis, is positive for points above and negative for points below the x-axis. For example, the point P in Fig. 11*b* has coördinates $x = 3$, $y = 2$, while Q has the coördinates $x = -3$, $y = -2$.

Notation. — The point whose coördinates are x and y is represented by the symbol (x, y). To signify that P has the coördinates x and y the notation $P\ (x, y)$ is used. For example, $(1, -2)$ is the point $x = 1$, $y = -2$. Similarly, $A\ (-2, 3)$ signifies that the abscissa of A is -2, and its ordinate 3.

Plotting. — The process of locating a point whose coördinates are given is called *plotting*. It is convenient for this purpose to use coördinate paper, that is, paper ruled with two sets of lines as in Fig. 11*c*. Two of these perpendicular lines are taken as axes. A certain number of divisions (one in Figs. 11*c*, *d* and *e*) are taken as representing a unit of length. This unit should be long enough to make the diagram of reasonable size but not so long that any points to be plotted fail to lie on the paper. By counting the rulings from the axes it is easy to locate the point having given coördinates (approximately if it does not fall at an intersection).

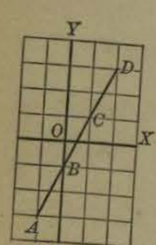

FIG. 11*c*.

Example 1. Show graphically that the points $A\ (-1, -3)$, $B\ (0, -1)$, $C\ (1, 1)$ and $D\ (2, 3)$ lie on a straight line.

The points are plotted in Fig. 11*c*. By applying a ruler to the figure it is found that a straight line can be drawn through the points.

Ex. 2. Plot the points $P_1\ (-2, -1)$, $P_2\ (3, 4)$ and find the distance between them.

The points are plotted in Fig. 11*d*. Let the horizontal line through P_1 cut the vertical line through P_2 in R. From the figure it is seen that $P_1R = 5$, $RP_2 = 5$. Consequently,

$$P_1P_2 = \sqrt{P_1R^2 + RP_2^2} = \sqrt{50}.$$

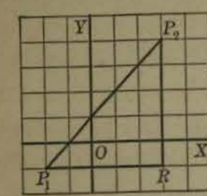

FIG. 11*d*.

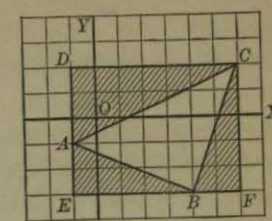

FIG. 11*e*.

Ex. 3. Plot the points $A\ (-1, -1)$, $B\ (4, -3)$, $C\ (6, 2)$ and find the area of the triangle ABC.

The points are plotted in Fig. 11*e*. The triangle ABC is part of a rectangle $CDEF$ with sides 5 and 7 and area $5 \times 7 = 35$. ADC, AEB and BFC are right triangles whose areas are (one-half base times altitude) $10\frac{1}{2}$, 5 and 5 respectively. The area of ABC is then

$$CDEF - ADC - AEB - BFC = 35 - 20\tfrac{1}{2} = 14\tfrac{1}{2}.$$

Exercises

1. Plot the points $A\ (4, 4)$, $B\ (-4, 4)$, $C\ (-4, -4)$, $D\ (4, -4)$ and $E\ (\sqrt{2}, \sqrt{3})$.

2. What are the algebraic signs of the coördinates in each of the four quadrants?

3. If a point lies on the x-axis what is its ordinate? If it lies on the y-axis what is its abscissa? Where are all the points for which $x = 1$? for which $y = -2$?

4. Let a and b be any given numbers. How is (a, b) related to each of the points $(-a, b)$, $(a, -b)$, $(-a, -b)$, (b, a), $(a + 1, b)$, $(a, b - 2)$?

5. Plot the points $(0, -2)$, $(1, 1)$, $(2, 4)$, $(-1, -5)$ and verify from the figure that they lie on a line.

6. Show graphically that the points $(5, 0)$, $(3, -4)$, $(-4, 3)$ and $(0, -5)$ lie on a circle. Find its center and radius.

7. Construct the point equidistant from $(-3, 4)$, $(5, 3)$ and $(2, 0)$. Determine its coördinates by measurement.

8. Plot the points $A\ (1, 2)$ and $B\ (3, 4)$. Let the horizontal line

through A meet the vertical line through B in P. What are the coördinates of P? What are the lengths of AP and BP? Calculate the distance AB.

9. Find the distance between A $(-2, -3)$ and B $(3, -4)$.

10. Find the area of the triangle formed by the points $(-3, 4)$, $(5, 3)$ and $(2, 0)$.

Art. 12. Segments

In this book the term line (meaning straight line) is used only when referring to the infinite line extending indefinitely in both directions. The part of a line between two points will be called a *segment.*

In many cases a segment is regarded as having a definite direction. The symbol AB is used for the segment beginning at A and ending at B. The segment beginning at B and ending at A is written BA.

The value of a segment may be any one of three things that should be carefully distinguished. Whenever the symbol AB occurs in an equation it must be understood which of these things is meant.

(1) In many cases the value AB means the length of the segment. In this case $AB = 3$ means that AB has a length of three units and $AB = CD$ means that AB and CD have equal lengths.

(2) In other cases the symbol AB represents the length together with a sign positive or negative according as the segment is directed one way or the other along the line. For example, the x-coördinate of a point is equal to the segment NP (Fig. 11*b*) considered positive when drawn to the right, negative when drawn to the left. The value of the segment is in this case a number with a positive or negative sign.

(3) Certain physical quantities, such as velocities, include in their description the direction of the lines along which they occur as well as their magnitudes and directions one way or the other along those lines. Two segments representing such quantities are equal only when they have the same length and direction. The value of such a segment (called a vector) is not a number but a number and direction.

Segments Parallel to a Coördinate Axis. — In most cases the value of a segment parallel to a coördinate axis will be the number equal

in magnitude to the length of the segment and positive or negative according as the segment is drawn in the positive or negative direc-

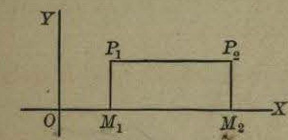

FIG. 12*a*.

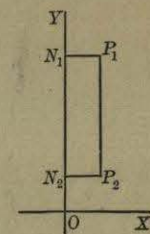

FIG. 12*b*.

tion of the axis. That is, horizontal segments are positive when drawn to the right, negative when drawn to the left; vertical segments are positive when drawn upward, negative when drawn downward.

Let P_1 (x_1, y_1) and P_2 (x_2, y_2) be the end points of a segment P_1P_2. If the segment is parallel to the x-axis (Fig. 12*a*)

$$P_1P_2 = M_1M_2 = x_2 - x_1, \tag{12}$$

for the difference $x_2 - x_1$ is equal in absolute value to the distance M_1M_2 and is positive when M_2 is on the right of M_1, negative when on the left. Similarly, if P_1P_2 is parallel to the y-axis (Fig. 12*b*), then

$$P_1P_2 = N_1N_2 = y_2 - y_1. \tag{12}$$

Therefore *in length and sign a segment parallel to a coördinate axis is represented by the difference obtained by subtracting the coördinate of the beginning from the coördinate of the end of the segment.*

Art. 13. Projection

The projection of a point A on a line MN is the foot of the perpendicular from A to MN. The projection of a segment AB is the segment $A'B'$ of MN intercepted between perpendiculars from the ends of AB. Since parallels intercept proportional distances on two lines, the lengths of two segments on a line have the same ratio as their projections. Furthermore if two segments have the same direction along a line their projections have the same direction.

For example, in Fig. 13*a* or 13*b*, in both magnitude and sign,

$$AB : BC = A'B' : B'C'.$$

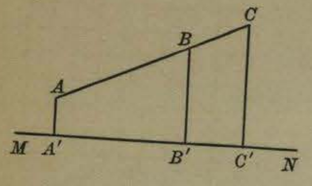

FIG. 13*a*.

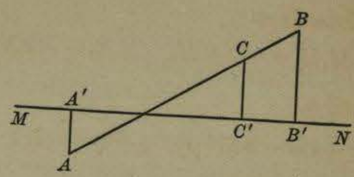

FIG. 13*b*.

In using this formula it should be noted that if AB and BC are distances, their projections must be distances and if AB and BC have algebraic signs their projections must have algebraic signs. For example, if the segments are projected on the coördinate axes and the values of the projections determined by equation (12), AB and BC must be considered opposite in sign when they have opposite directions.

Example 1. Find the middle point of the segment joining $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Let $P(x, y)$ be the middle point. Project on the axes (Fig. 13*c*).

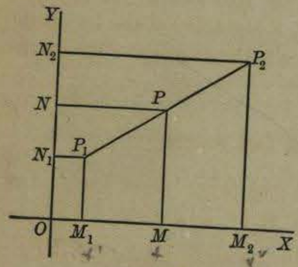

FIG. 13*c*.

Since P is the middle point of P_1P_2, in both length and direction $P_1P = PP_2$. Hence

$$M_1M = MM_2, \quad N_1N = NN_2,$$

and so

$$x - x_1 = x_2 - x, \quad y - y_1 = y_2 - y.$$

Consequently,

$$x = \tfrac{1}{2}(x_1 + x_2), \quad y = \tfrac{1}{2}(y_1 + y_2).$$

The sum of several quantities divided by their number is called the average. Hence *each coördinate of the middle point is the average of the corresponding coördinates of the ends.*

Ex. 2. Given $P_1(-1, 1)$, $P_2(3, 4)$, find the point $P(x, y)$, on P_1P_2 produced, which is twice as far from P_1 as from P_2 (Fig. 13*d*).

Since P_1P is twice as long as PP_2 and they have opposite directions,

$$P_1P = -2\,PP_2.$$

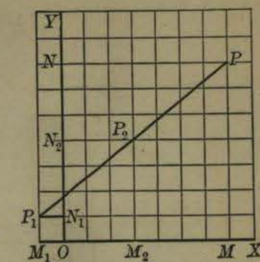

FIG. 13*d*.

Consequently,

$$M_1M = -2\,MM_2, \quad N_1N = -2\,NN_2,$$

and so

$$x + 1 = -2(3 - x), \quad y - 1 = -2(4 - y).$$

Hence $x = 7$, $y = 7$.

Exercises

1. On the line through $A(-3, -4)$ and $B(4, 2)$ find the point two-fifths of the way from A to B.
2. The points $A(-1, -4)$, $B(0, -1)$ and $C(2, 5)$ lie on a line. Find the point P, on AC produced, such that $AB = CP$.
3. On the line through $A(1, -1)$, $B(3, 5)$ find two points each of which is twice as far from A as from B.
4. The points $A(-4, 9)$, $B(2, 0)$, $C(4, -3)$, $D(0, 3)$ lie on a line. Find the ratio of the segments AB and CD. Do these segments have the same or opposite directions?
5. On the line through $A(2, -1)$ parallel to the line through $B(1, 1)$ and $C(4, 5)$ find two points each of which is at the distance BC from A.
6. Given three points, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, find the middle point Q of P_1P_2, then find the point R one-third of the way from Q to P_3. Show that each of its coördinates is the average of the corresponding coördinates of P_1, P_2 and P_3.

Art. 14. Distance between Two Points

Let the points be $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ (Fig. 14*a* or 14*b*). Let the projections of P_1P_2 on the axes be M_1M_2 and N_1N_2. Let the lines P_1N_1 and P_2M_2 intersect in R. In the right triangle P_1RP_2

$$P_1P_2 = \sqrt{\overline{P_1R}^2 + \overline{RP_2}^2}.$$

Now $P_1R = M_1M_2$ and the distance between two points of a scale is equal to the positive difference of their coördinates. Consequently, $\overline{P_1R}^2 = (x_2 - x_1)^2$. Similarly, $\overline{RP_2}^2 = (y_2 - y_1)^2$. Therefore

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \quad (14)$$

In this formula, since $x_2 - x_1$ is squared it is immaterial whether it is written $x_2 - x_1$ or $x_1 - x_2$. Similarly, instead of $y_2 - y_1$ can

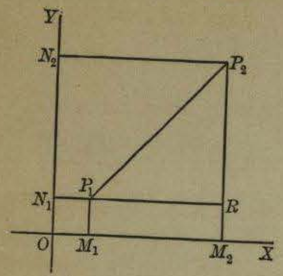

FIG. 14a.

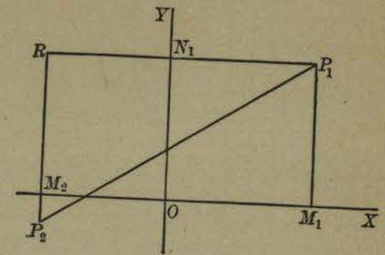

FIG. 14b.

be put $y_1 - y_2$. The formula expresses that *the distance between two points is equal to the square root of the sum of the squares of the differences of corresponding coördinates.*

Example 1. Find the distance between the points $A\ (-1, 1)$ and $B\ (1, -2)$.

By the formula

$$AB = \sqrt{(1+1)^2 + (-2-1)^2} = \sqrt{13}.$$

Ex. 2. Find the points 2 units distant from the x-axis and 5 units distant from the point $A\ (1, 2)$.

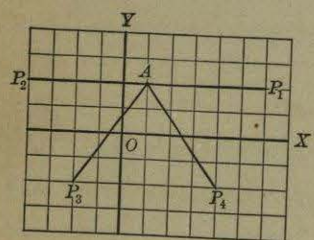

FIG. 14c.

Let $P\ (x, y)$ be such a point (Fig. 14c). Since $PA = 5$

$$(x-1)^2 + (y-2)^2 = 25.$$

Since the point is two units distant from the x-axis

$$y = \pm 2.$$

Substitution of 2 for y in the previous equation gives $x = 6$ or -4. Substitution of -2 for y gives $x = 4$ or -2. There are consequently four points $P_1\ (6, 2)$, $P_2\ (-4, 2)$, $P_3\ (-2, -2)$, $P_4\ (4, -2)$ which satisfy the conditions.

Ex. 3. Find a point equidistant from the three points, $A\ (9, 0)$, $B\ (-6, 3)$ and $C\ (5, 6)$.

Let $P\ (x, y)$ be the point required (Fig. 14d). Since $PA = PB$

$$\sqrt{(x-9)^2 + y^2} = \sqrt{(x+6)^2 + (y-3)^2}.$$

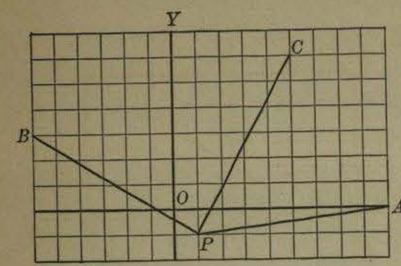

FIG. 14d.

Squaring and cancelling,

$$5x - y = 6.$$

Similarly, since $PA = PC$,

$$2x - 3y = 5.$$

Solving simultaneously gives $x = 1, y = -1$. The required point is therefore $(1, -1)$.

Exercises

1. Find the perimeter of the triangle whose vertices are (2, 3), (−3, 3) and (1, 1).
2. Show that the points (1, −2), (4, 2) and (−3, −5) are the vertices of an isosceles triangle.
3. Show that the points (0, 0), (3, 1), (1, −1) and (2, 2) are the vertices of a parallelogram.
4. Given A (2, 0), B (1, 1), C (0, 2) show that the distances AB, BC and AC satisfy the equation $AB + BC = AC$. What do you conclude about the points?
5. Show that (6, 2), (−2, −4), (5, −5), (−1, 3) are on a circle whose center is (2, −1).
6. It can be shown that four points form a quadrilateral inscribed in a circle if the product of the diagonals is equal to the sum of the products of the opposite sides. Assuming this, show that (−2, 2), (3, −3), (1, 1) and (2, 0) lie on a circle.
7. Find the coördinates of two points whose distances from (2, 3) are 4 and whose ordinates are equal to 5.
8. Find a point on the x-axis which is equidistant from (0, 4) and (−3, −3).
9. Find the center of the circle passing through (0, 0), (−3, 3) and (5, 4).
10. Given A (0, 0), B (1, 1), C (−1, 1), D (1, −2), find the point in which the perpendicular bisector of AB cuts the perpendicular bisector of CD.
11. Find the foot of the perpendicular from (1, 2) to the line joining (2, 1) and (−1, −5).

Art. 15. Vectors

A vector is a segment of definite length and direction. The point P_1 is the beginning, the point P_2 the end of the vector P_1P_2. The direction of the vector along its line is often indicated by an arrow as shown in Fig. 15*a*.

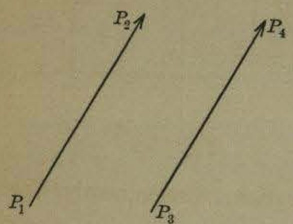

FIG. 15*a*.

Two vectors are called equal if they have the same length and direction. For example, in Fig. 15*a*, $P_1P_2 = P_3P_4$. If two vectors have the same length but opposite directions, either is called the *negative* of the other. For example, $P_1P_2 = -P_4P_3 = -P_2P_1$.

Let the projections of P_1P_2 on the coördinate axes be M_1M_2 and N_1N_2 (Fig. 15*b*). The *x-component* of the vector P_1P_2 is defined as the length of M_1M_2 or the negative of that length, according as M_1M_2 has the positive or negative direction along the x-axis. Similarly, the *y-component* is the distance N_1N_2 or the negative of that distance, according as N_1N_2 is drawn in the positive or negative direction along the y-axis. For example, in Fig. 15*b*, the components of P_1P_2 are 3 and 4, while those of P_2P_3 are -7 and 3.

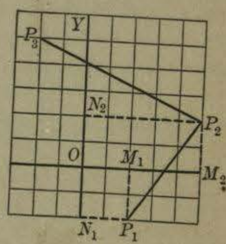

FIG. 15*b*.

Let the points be $P_1(x_1, y_1)$, $P_2(x_2, y_2)$. In Art. 12 it has been shown that in magnitude and sign M_1M_2 and N_1N_2 are represented by $x_2 - x_1$ and $y_2 - y_1$. Hence $x_2 - x_1$ is the x-component and $y_2 - y_1$ is the y-component of P_1P_2. That is, the components of a vector are obtained by subtracting the coördinates of the beginning from the corresponding coördinates of the end of the vector.

If two vectors are equal their components are equal. For let $P_1P_2 = P_3P_4$ (Fig. 15*c*). Then, since by definition of equality P_1P_2 and P_3P_4 have the same length and direction, the triangles

P_1RP_2, P_3SP_4 are equal and corresponding sides have the same direction. Consequently, in both length and sign,

$$M_1M_2 = P_1R = P_3S = M_3M_4,$$
$$N_1N_2 = RP_2 = SP_4 = N_3N_4.$$

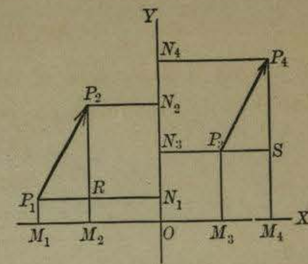

FIG. 15*c*.

Conversely, if the components are equal, the triangles are equal and corresponding sides have the same direction. Consequently, the vectors are equal.

Notation. — In this book the vector whose components are a and b will be represented by the symbol $[a, b]$. To signify that the vector u has the components a and b, the notation $u\,[a, b]$ will be used. For example, $P_1P_2 = [-2, 3]$ means that the vector P_1P_2 has an x-component equal to -2 and a y-component equal to 3. If the points P_1 and P_2 are (x_1, y_1) and (x_2, y_2), the components of P_1P_2 are $x_2 - x_1$ and $y_2 - y_1$ and

$$P_1P_2 = [x_2 - x_1,\quad y_2 - y_1].$$

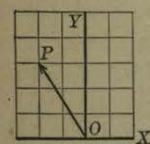

FIG. 15*d*.

Example 1. Construct a vector equal to $[-2, 3]$.

The vector OP from the origin to the point $P(-2, 3)$ is

$$OP = [-2-0, 3-0] = [-2, 3].$$

Hence OP is a vector of the kind required (Fig. 15*d*).

Ex. 2. Show that the points A (1, 3), B (2, 1), C (3, 4), D (4, 2) form the vertices of a parallelogram.

The vectors AB and CD are equal. For

$$AB = [2-1, 1-3] = [1, -2],$$
$$CD = [4-3, 2-4] = [1, -2].$$

That is, AB and CD are parallel and have the same length. Consequently $ABCD$ is a parallelogram (Fig. 15*e*).

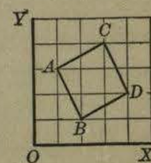

FIG. 15*e*.

Ex. 3. Find the area of the triangle PP_1P_2, given

$$PP_1 = [a_1, b_1], \quad PP_2 = [a_2, b_2].$$

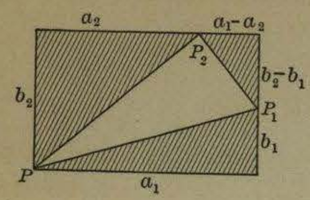

FIG. 15*f*.

In Fig. 15*f* the sides of the shaded triangles have their lengths marked on them. The area of the triangle PP_1P_2 is equal to the area of the whole rectangle less the sum of the areas of the shaded triangles. Hence

$$PP_1P_2 = a_1b_2 - \tfrac{1}{2}\, a_1b_1 - \tfrac{1}{2}\, a_2b_2 - \tfrac{1}{2}\,(a_1 - a_2)\,(b_2 - b_1) = \tfrac{1}{2}\,(a_1b_2 - a_2b_1).$$

This result is shown for a particular figure. By drawing other figures it will be found that the result is always correct if the angle from PP_1 to PP_2 is positive (that is, drawn in the counter-clockwise direction). If that angle is negative the formula gives the negative of the area.

Exercises

1. If the vectors AB and CD are equal show that AC and BD are equal.
2. If $AB = A_1B_1$, $BC = B_1C_1$, show that $AC = A_1C_1$.
3. The components of a vector are a, b. Show that the components of its negative are $-a$, $-b$.
4. Show that the vector $[a, b]$ is equal to the vector from the origin to the point (a, b).
5. Construct vectors equal to $[2, 3]$, $[-2, 3]$, $[-2, -3]$, and $[2, -3]$.
6. Show that the points $P\,(-1, 2)$, $Q\,(1, -2)$, $R\,(3, 4)$, $S\,(5, 0)$ are the vertices of a parallelogram.
7. A vector equal to $[-3, 4]$ begins at the point $(1, -2)$. What are the coördinates of its end?
8. The points $(1, 2)$, $(-2, -1)$, $(3, -2)$ are the vertices of a triangle. Find the vectors from the vertices to the middle points of the opposite sides.
9. Given $A\,(2, 3)$, $B\,(-4, 5)$, $C\,(-2, 3)$, find D such that $AB = CD$.
10. The middle point of a certain segment is $(1, 2)$ and one end is $(-3, 5)$. Find the coördinates of the other end.
11. By showing that the area of the triangle ABC is zero show that the points $A\,(0, -2)$, $B\,(1, 1)$ and $C\,(3, 7)$ lie on a line.
12. Find the area of the quadrilateral whose vertices are the points $(-2, -3)$, $(1, -2)$, $(3, 4)$, $(-1, 5)$.

13. Show that the vectors $[a_1, b_1]$, $[a_2, b_2]$ are parallel if and only if $a_1/a_2 = b_1/b_2$. In this way show that the line through $(-1, 2)$ and $(3, 4)$ is parallel to that through $(1, -1)$ and $(9, 3)$.

Art. 16. Multiple of a Vector

If u is a vector and r a real number, the symbol ru is used to represent a vector r times as long as u and having the same direction if r is positive, but the opposite direction if r is negative.

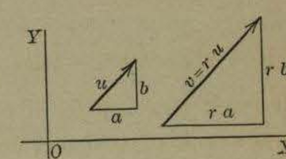

FIG. 16.

In Fig. 16, let $u = [a, b]$ and $v = ru$. Since u and v are parallel their components have lengths proportional to the lengths of u and v. Also corresponding components have the same or opposite signs according as u and v have the same or opposite directions. Hence the components of v are ra and rb. That is,

$$v = [ra, rb].$$

Hence *to multiply a vector by a number, multiply each of its components by that number.*

Example 1. Given $P_1\,(-2, 3)$, $P_2\,(1, -4)$, find a vector having the same direction as P_1P_2 but 3 times as long.

The vector required is

$$3\,P_1P_2 = 3\,[3, -7] = [9, -21].$$

Ex. 2. Find the point on the line joining $P_1\,(2, 3)$, $P_2\,(1, -2)$ and one-third of the way from P_1 to P_2.

By hypothesis $P_1P = \frac{1}{3}\,P_1P_2$, that is,

$$[x - 2, y - 3] = \tfrac{1}{3}\,[-1, -5] = [-\tfrac{1}{3}, -\tfrac{5}{3}].$$

Hence $x - 2 = -\frac{1}{3}$, $y - 3 = -\frac{5}{3}$, and consequently $x = \frac{5}{3}$, $y = \frac{4}{3}$.

Ex. 3. Show that the segment joining the points $A\,(1, -1)$, $B\,(5, 7)$ is parallel to and twice as long as that joining $C\,(4, 3)$, $D\,(2, -1)$.

The vectors are

$$AB = [4, 8], \quad CD = [-2, -4].$$

Hence $AB = -2\,CD$. Consequently the segments are parallel and the first is twice as long as the second.

Art. 17. Addition and Subtraction of Vectors

Sum of Two Vectors. — Draw a vector equal to v, beginning at the end of u. The vector from the beginning of u to the end of v is called the sum of u and v.

Let $u = [a_1, b_1]$, $v = [a_2, b_2]$. From the diagram it is seen that the components of $u + v$ are $a_1 + a_2$ and $b_1 + b_2$. This is true not

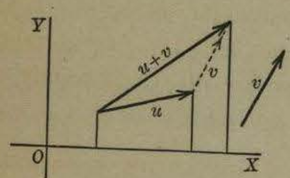

FIG. 17a.

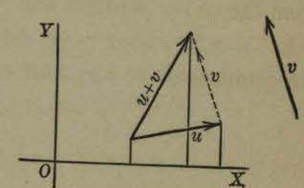

FIG. 17b.

only in Fig. 17a but also in Fig. 17b, for there a_2 is negative and $a_1 + a_2$ is in absolute value equal to the difference of the lengths of the projections. Hence

$$u + v = [a_1 + a_2, \quad b_1 + b_2],$$

that is, *vectors are added by adding corresponding components.*

Example 1. Show that $u + v = v + u$.

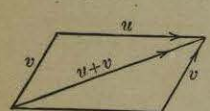

FIG. 17c.

The sum $u + v$ results when v is put at the end of u, while $v + u$ is given by putting u at the end of v. The two are equal since both are equal to the vector diagonal of the parallelogram whose sides are u and v (Fig. 17c).

Ex. 2. Show that $(u + v) + w = u + (v + w)$.

In the expression $(u + v) + w$ the sum of u and v is added to w, while in $u + (v + w)$, u is added to the sum of v and w. In

Fig. 17d let $u = AB$, $v = BC$, $w = CD$. Then

$$(u + v) + w = AC + CD = AD,$$
$$u + (v + w) = AB + BD = AD.$$

The two sums are consequently equal.

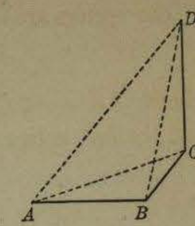

FIG. 17d.

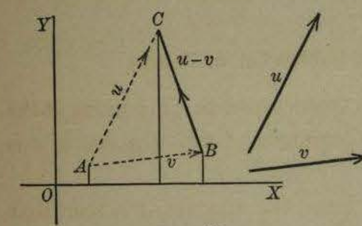

FIG. 17e.

Difference of Two Vectors.— Draw vectors equal to u and v, beginning at the same point (Fig. 17e). The vector BC from the end of v to the end of u is called the difference $u - v$.

Let $u = [a_1, b_1]$, $v = [a_2, b_2]$. It is seen from the diagram that the components of $u - v$ are $a_2 - a_1$ and $b_2 - b_1$. Consequently,

$$u - v = [a_2 - a_1, \quad b_2 - b_1],$$

that is, *vectors are subtracted by subtracting corresponding components.*

Example 1. Show that $u + (-v) = u - v$.

In Fig. 17e, $-v = BA$.

Hence

$$u + (-v) = AC + BA = BA + AC = BC = u - v.$$

Ex. 2. Two segments equal to $[2, -3]$ and $[3, -1]$ extend from the point A (2, 2). Find their ends and the vector connecting their ends.

In Fig. 17f, by hypothesis,

$$OA = [2, 2], \quad AP_1 = [2, -3], \quad AP_2 = [3, -1].$$

Hence

$$OP_1 = OA + AP_1 = [2 + 2, 2 - 3] = [4, -1],$$
$$OP_2 = OA + AP_2 = [2 + 3, 2 - 1] = [5, 1].$$

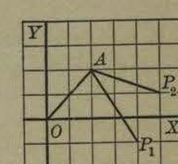

FIG. 17f.

Consequently P_1 and P_2 are (4, −1) and (5, 1). Also

$$P_1P_2 = AP_2 - AP_1 = [3 - 2, -1 + 3] = [1, 2].$$

Ex. 3. If weights w_1, w_2, w_3, etc., are located at the points P_1, P_2, P_3, etc., the center of gravity of these weights can be defined as the point P satisfying the equation

$$w_1PP_1 + w_2PP_2 + w_3PP_3 + \text{etc.} = 0.$$

Find the center of gravity of three weights of 2, 3 and 5 pounds placed at the points $P_1(-2, 1)$, $P_2(1, -3)$, $P_3(4, 5)$ respectively.

The center of gravity P satisfies the equation

$$2\,PP_1 + 3\,PP_2 + 5\,PP_3 = 0,$$

that is,

$$2[-2-x, 1-y] + 3[1-x, -3-y] + 5[4-x, 5-y]$$
$$= [19 - 10x, 18 - 10y] = 0.$$

Consequently the coördinates of the center of gravity are $x = 1.9$ and $y = 1.8$.

Exercises

1. Given $P(1, -3)$, $Q(7, 1)$, $R(-1, 1)$, $S(2, 3)$, show that PQ has the same direction as RS and is twice as long.

2. Given $P_1(2, -3)$, $P_2(-1, 2)$, find the point on P_1P_2 which is twice as far from P_1 as from P_2. Also find the point on P_1P_2 produced which is twice as far from P_1 as from P_2.

3. Find the points P and Q on the line through $P_1(2, -1)$, $P_2(-4, 5)$ if $P_1P = -\frac{2}{3}PP_2$, $P_1Q = -\frac{2}{3}P_1P_2$.

4. One end of a segment is $(2, -5)$ and a point one-fourth of the distance to the other end is $(-1, 4)$. Find the coördinates of the other end.

5. Given the three points $A(-3, 3)$, $B(3, 1)$, $C(6, 0)$ on a line, find the fourth point D on the line such that $AD:DC = -AB:BC$.

6. Show that the line through $(-4, 5)$, $(-2, 8)$ is parallel to that through $(3, -1)$, $(9, 8)$.

7. If the vectors AC and AB satisfy the equation $AC = rAB$, where r is a real number, show that A, B, C lie on a line. In this way show that $(2, 3)$, $(-4, -7)$ and $(5, 8)$ lie on a line.

8. Given $A(1, 1)$, $B(2, 3)$, $C(0, 4)$, find the point D such that $BD = 2\,DC$. Show that $AD = \frac{1}{3}[AB + 2\,AC]$.

9. In Ex. 8 find the point P such that $PA + PB + PC = 0$.

10. If v is any vector let v^2 mean the square of the length of v. Given $v_1 = [2, 5]$, $v_2 = [10, -4]$, show that $(v_1 + v_2)^2 = v_1^2 + v_2^2$ and consequently that v_1 and v_2 are perpendicular to each other.

11. Show that the vectors from the vertices of a triangle to the middle points of the opposite sides have a sum equal to zero.

12. Find the center of gravity of two weights of 1 and 4 pounds placed at the points $(3, -4)$ and $(2, 7)$ respectively.

13. Show that the center of gravity of four equal weights placed at the vertices of a quadrilateral is the middle point of the segment joining the middle points of two opposite sides of the quadrilateral.

Art. 18. Slope of a Line

Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ be two points of the line MN. The ratio

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{18a}$$

is called the *slope* of the line MN. The same value of the slope is obtained whatever *pair* of points on the line is used. For, if

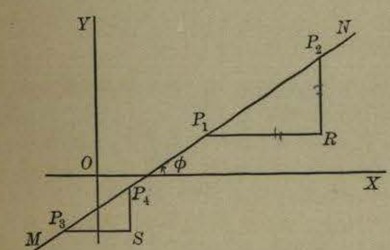

FIG. 18*a*.

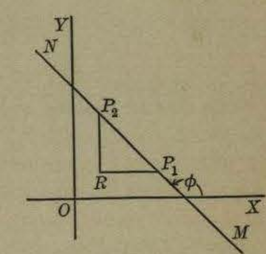

FIG. 18*b*.

P_1, P_2 and P_3, P_4 are pairs of points on the same line, the triangles P_1RP_2 and P_3SP_4 are similar and

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{RP_2}{P_1R} = \frac{SP_4}{P_3S} = \frac{y_4 - y_3}{x_4 - x_3}.$$

Let ϕ be the angle measured from the positive direction of the x-axis to the line MN. This angle is considered positive when measured in the counter-clockwise direction, negative when measured in the other direction. The tangent of this angle is RP_2/P_1R. Accordingly

$$m = \tan\phi. \tag{18b}$$

Hence the slope of a line is equal to the tangent of the angle from the positive direction of the x-axis to the line.

Since the x-axis is horizontal, the steeper the line the nearer is the angle to 90° and consequently the greater the slope. The slope is thus a measure of the steepness of the line.

If the line extends upward to the right, as in Fig. 18*a*, the components P_1R and RP_2 have the same sign and the slope is positive. If the line extends upward to the left, as in Fig. 18*b*, the components have opposite signs and the slope is negative.

Parallel Lines. — *If two lines are parallel, they have the same slope;* for the angles ϕ_1 and ϕ_2 (Fig. 18*c*) are then equal and their slopes, $\tan\phi_1$ and $\tan\phi_2$, are equal. Conversely, if the slopes are equal, the angles are equal and the lines are parallel.

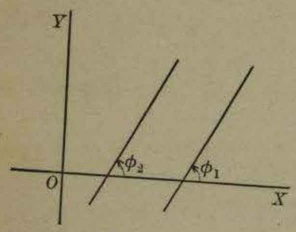

FIG. 18*c*.

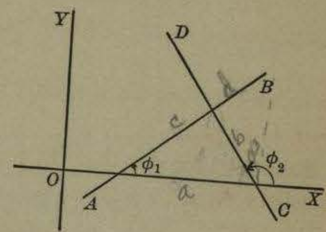

FIG. 18*d*.

Perpendicular Lines. — In Fig. 18*d*, let the lines AB and CD be perpendicular. Since the exterior angle is equal to the sum of the two opposite interior angles

$$\phi_2 = \phi_1 + 90°.$$

Consequently $\tan\phi_2 = -\cot\phi_1 = -\dfrac{1}{\tan\phi_1}.$

Conversely, if this relation holds, ϕ_1 and ϕ_2 differ by 90° and the lines are perpendicular. If m_1 and m_2 are the slopes, $m_1 = \tan\phi_1$, $m_2 = \tan\phi_2$, and

$$m_2 = -\frac{1}{m_1}. \tag{18c}$$

Two lines are perpendicular if and only if the slope of one is equal to the negative reciprocal of the slope of the other.

Angle between Two Lines. — Let two lines L_1 and L_2 make with the x-axis the angles ϕ_1 and ϕ_2 (Fig. 18*e*). If β is the angle from L_1 to

L_2 (positive when measured in the counter-clockwise direction), then

$$\beta = \phi_2 - \phi_1.$$

Hence

$$\tan\beta = \tan(\phi_2 - \phi_1) = \frac{\tan\phi_2 - \tan\phi_1}{1 + \tan\phi_1\tan\phi_2}.$$

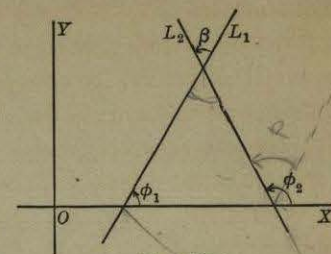

FIG. 18*e*.

If m_1 and m_2 are the slopes of the lines, $m_1 = \tan\phi_1$, $m_2 = \tan\phi_2$, whence

$$\tan\beta = \frac{m_2 - m_1}{1 + m_1m_2}. \tag{18d}$$

In using this formula it should be remembered that β is measured from L_1 to L_2.

Example 1. Show that the line through A (1, −3), B (5, −1) is parallel to that through C (−3, −2), D (−1, −1).

$$\text{Slope of } AB = \frac{-1+3}{5-1} = \frac{1}{2},$$

$$\text{slope of } CD = \frac{-1+2}{-1+3} = \frac{1}{2}.$$

Since the slopes are equal the lines are parallel.

Ex. 2. Show that the line through A (2, 3), B (−4, 5) is perpendicular to that through C (1, 1), D (2, 4).

$$\text{Slope of } AB = \frac{5-3}{-4-2} = -\frac{1}{3},$$

$$\text{slope of } CD = \frac{4-1}{2-1} = 3.$$

Since either slope is the negative reciprocal of the other, the lines are perpendicular.

Ex. 3. Find the angles of the triangle formed by the points A (−1, 2), B (1, 1), C (3, 4).

The triangle is shown in Fig. 18*f*. The slopes of AB, BC, CA are found to be $-\frac{1}{2}$, $\frac{3}{2}$ and $\frac{1}{2}$ respectively. Between two lines such as AB and AC are two positive angles less than 180°. One of these is interior, the other exterior to the triangle. From the figure the positive interior angle A is seen to extend from AB to AC. Consequently,

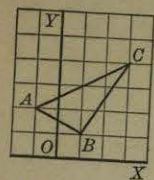

FIG. 18f.

$$\tan A = \frac{\frac{1}{2} - (-\frac{1}{2})}{1 + \frac{1}{2}(-\frac{1}{2})} = \frac{4}{3}.$$

In the same way it is found that $\tan B = -8$, $\tan C = \frac{4}{7}$. The angles of the triangle are therefore the positive angles less than 180° whose tangents are $\frac{4}{3}$, -8 and $\frac{4}{7}$. The angles A and C are acute, B is obtuse.

Exercises

1. Find the slopes of the sides of the triangle formed by the points $(-1, 2)$, $(1, 3)$ and $(2, -4)$.

2. Find the angle from the x-axis to the line joining $(1, 1)$ and $(-4, 5)$.

3. The angles from the x-axis to three lines are 45°, 120° and −30° respectively. Find the slopes of the lines.

4. The sides of a triangle have slopes equal to $\frac{1}{2}$, 1 and 2. Show that the triangle is isosceles.

5. If the slopes of AB and BC are equal, show that A, B, C lie on a line. In this way show that the points $(4, 1)$, $(1, 2)$ and $(-5, 4)$ lie on a line.

6. Show that the line through $(1, -3)$ and $(-1, 3)$ is parallel to that through $(3, -5)$ and $(0, 4)$.

7. Show that the lines determined by the pairs of points $(2, 3)$, $(3, 5)$ and $(1, 7)$, $(3, 6)$ are perpendicular to each other.

8. Show that the vectors $[a, b]$ and $[c, d]$ are parallel if $b/a = d/c$ and perpendicular if $b/a = -c/d$.

9. Find the interior angles of the triangle formed by the points $(2, 3)$, $(-4, 5)$ and $(1, -2)$.

10. Lines join the point $(-2, 2)$ to the points $(-3, 1)$, $(0, 2)$ and $(1, -1)$. Show that one of these bisects the angle between the other two.

11. Two lines have slopes equal to 2 and −3. Find the slopes of the two bisectors of the angles between these lines.

12. Find the angle between the lines joining the origin to the two points of trisection of the segment joining $(-2, 3)$ and $(1, -7)$.

13. The slope of AB is $\frac{1}{2}$. Find the slope of CD if the angle from AB to CD is 30°.

14. By showing that the angles CAD and CBD are equal show that the points $A\,(6, 11)$, $B\,(-11, 4)$, $C\,(-4, -13)$, $D\,(1, -14)$ lie on a circle.

15. A point is 7 units distant from the origin and the slope of the line joining it to $(3, 4)$ is $\frac{1}{2}$. Find its coördinates.

16. A point is equidistant from $(2, 1)$ and $(-4, 3)$ and the slope of the line joining it to $(1, -1)$ is $\frac{2}{3}$. Find its coördinates.

Art. 19. Graphs

In many cases corresponding values of two related quantities are known. Each pair of values can be taken as the coördinates of a point. The totality, or locus, of such points is called a *graph*. This graph exhibits pictorially the relation of the quantities represented. There are three cases differing in the accuracy with which intermediate values are known.

Statistical Graphs. — Sometimes definite pairs of values are given but there is no information by which intermediate values can be even approximately inferred. In such cases the values are plotted and consecutive points connected by straight lines to show the order in which the values are given.

Example 1. The temperature at 6 A.M. on ten consecutive days at a certain place is given in the following table:

Jan.	1	2	3	4	5	6	7	8	9	10
	19°	27°	11°	40°	34°	28°	36°	18°	42°	38°

These values are represented by points in Fig. 19*a*. Points representing temperatures on consecutive days are connected by straight lines. At times between those marked no estimate of the temperature can be made.

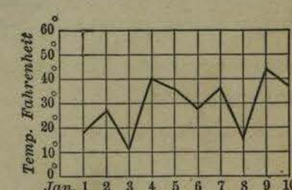

FIG. 19*a*.

Physical Graphs. — In other cases it is known that in the intervals between the values given the variables change nearly proportionally. In such cases the points are plotted and as smooth a curve as possible is drawn through them. Points of this curve between those plotted are assumed to represent approximately corresponding values of the variables.

Ex. 2. The observed temperatures θ of a vessel of cooling water at times t, in minutes, from the beginning of observation were

$t = 0$	1	2	3	5	7	10	15	20
$\theta = 92°$	85.3°	79.5°	74.5°	67°	60.5°	53.5°	45°	39.5°

These values are plotted in Fig. 19*b*. Since the temperature decreases gradually and during the intervals given at nearly constant rates a smooth curve drawn through these points will represent approximately the relation of temperature and time throughout the experiment.

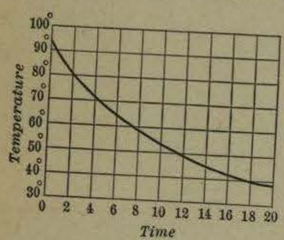

FIG. 19*b*.

The Graph of an Equation. — In other cases the equation connecting the variables is known. The graph then consists of all points whose coördinates satisfy the equation. Arbitrary values are assigned to either of the variables, the corresponding values of the other variable calculated, and the resulting points plotted. When the points have been plotted so closely that between consecutive points the curve is nearly straight, a smooth curve is drawn through them.

Ex. 3. Plot the graph of the equation $y = x^2$.

In the following table values are assigned to x and the corresponding values of y calculated:

$x =$ −5	−4	−3	−2	−1	0	1	2	3	4	5
$y =$ 25	16	9	4	1	0	1	4	9	16	25

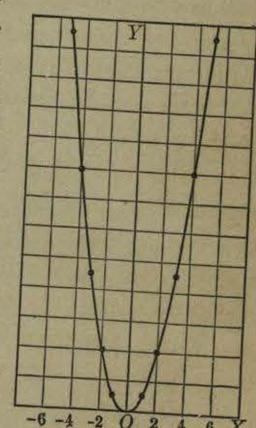

FIG. 19*c*.

The points are plotted in Fig. 19*c*. The part of the curve shown extends from $x = -5$ to $x = +5$. The whole curve extends to an indefinite distance. Horizontal lines cut the

the curve at equal distances right and left of the y-axis. This is expressed by saying that the curve is symmetrical with respect to the y-axis.

Ex. 4. The force of gravitation between two bodies varies inversely as the square of their distance apart, that is, $F = k/d^2$, k being constant. Assuming $k = 10$, plot the curve representing the relation of force and distance.

In the following table values are assigned to d and the corresponding values of F calculated:

$d = 0$	0.5	1	1.5	2	3	5	10
$F = \infty$	40	10	4.4	2.5	1.1	0.4	0.1

The distance d cannot be negative. For very large values of d, F is very small. Hence when d is large the curve very nearly coincides with the horizontal axis. When d is very small F is very large and the curve very nearly coincides with the vertical axis (Fig. 19*d*).

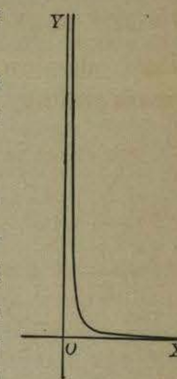

FIG. 19*d*.

Units. — Before a graph can be plotted a length must be chosen on each axis to represent a unit measure of the quantity represented by that coördinate. If the quantities represented by x and y are of different kinds, for example temperature and time, these lengths can be chosen independently. If, however, these quantities are of the same kind it is usually more convenient to have the same unit of length along both axes. This is also the case in graphing mathematical equations where no physical interpretation is given to x and y. Other things being equal, a large curve is better than a small one. Such units of length should then be chosen as to make the curve spread both vertically and horizontally very nearly over the region available for it. If several graphs are to be compared, the same unit lengths should be used for all.

Exercises

1. The price of steel rails each year from 1892 to 1907 was as follows, in dollars per gross ton:

30 28 24 24 28 19 18 28 32 27 28 28 28 28 28 28

Make a graph showing the relation of year and price. (Let x be the number of years beyond 1892 and y the amount that the price exceeds 18 dollars.)

2. The amplitudes of successive vibrations of a pendulum set in motion and left free were

Number of vibration	1	2	3	4	5	6	7
Amplitude	69	48	33.5	23.5	16.5	11.5	8

Plot points representing these pairs of values and draw a smooth curve through them. Do points of this curve between those plotted have any physical meaning?

3. The atomic weight, W, and specific heat, S, of several chemical elements are shown in the following table:

$W =$ 7	9.1	11	12	23	28	39	55	56	108	196
$S =$ 0.94	0.41	0.25	0.147	0.29	0.177	0.166	0.122	0.112	0.057	0.032

Make a graph showing the relation of specific heat and atomic weight.

4. The table below gives the maximum length of spark between needle points of an alternating current under standard conditions of needles, temperature and barometric pressure.

L = length in millimeters. V = electromotive force in kilovolts.

$V =$ 10	15	20	25	30	35	40	45	50	60	70	80
$L =$ 12	18	25	33	41	51	62	75	90	118	149	180

Make a curve showing the relation of voltage and length of spark.

5. In the table below are given the maximum vapor pressures of water at various temperatures, where T = temperature in degrees Centigrade and P = pressure in centimeters of mercury:

$T =$ 0	10	20	30	40	50	60	70	80	90	100
$P =$ 0.46	0.91	1.74	3.15	5.49	9.20	14.9	23.3	35.5	52.5	76

Make a graph showing the relation of temperature and vapor pressure.

6. The rate for letter postage is two cents for each ounce or fraction. Make a graph showing the relation of weight and cost of postage. (The cost for 1.5 ounces is the same as that for 2, etc.)

7. Make a graph showing the number of centimeters y in x inches.

8. Make a graph showing the relation between the side and the area of a square.

9. According to Boyle's law the pressure and volume of a gas at constant temperature are connected by the equation pv = constant. Assuming the constant equal to 10, construct the curve representing the relation of pressure and volume. Can p or v be negative?

10. Plot the curve $y = x^2 - 2x + 3$. Show that it passes through the point (1, 2).

11. On the same diagram plot the curves $y = x^4$ and $y = (x + 1)^4$. How are they related? Show that they both pass through the point $(-\frac{1}{2}, \frac{1}{16})$.

12. On the same diagram plot the curves $x = 1/y^2$ and $x = 1/(y-2)^2$. How are they related? Show that both pass through (1, 1).

13. On the same diagram plot the graphs of $x + y = 2$ and $2x - 3y = 1$. What are they? Find their point of intersection. (It must have coördinates satisfying both equations.)

Art. 20. Equation of a Locus

If a locus and an equation are such that (1) *every point on the locus has coördinates that satisfy the equation and* (2) *every point whose coördinates satisfy the equation lies on the locus, then the equation is said to represent the locus and the locus to represent the equation.*

Usually a locus is defined by a property possessed by each of its points. Thus a circle is the locus of points at a constant distance from a fixed point. To find the equation of a locus express this property by means of an equation connecting the coördinates of each locus point. The graph is constructed by using the definition or by plotting from the equation.

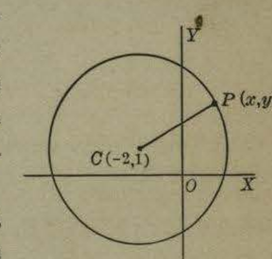

FIG. 20a.

Example 1. Find the equation of a circle with center C (−2, 1) and radius equal to 4.

Let P (x, y) (Fig. 20a) be any point on the circle. By the definition of a circle, $CP = 4$, and this is equivalent to

$$\sqrt{(x+2)^2 + (y-1)^2} = 4,$$

which is the equation required. By squaring this can be reduced to the form

$$(x+2)^2 + (y-1)^2 = 16.$$

Ex. 2. A curve is described by a point P (x, y) moving in such a way that its distance from the x-axis equals its distance from the point Q (1, 1). Construct the curve and find its equation.

The curve is constructed by locating points such that $MP = QP$. Since no point below the x-axis can be equally distant from Q and the x-axis, the curve lies entirely above the x-axis. Hence y is positive and equal to the distance MP. Also $QP = \sqrt{(x-1)^2 + (y-1)^2}$. Consequently, if $P\ (x, y)$ is any point on the curve,

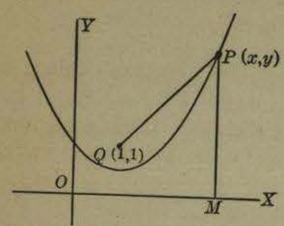

FIG. 20b.

$$y = \sqrt{(x-1)^2 + (y-1)^2}.$$

Conversely, if this equation is satisfied, the point P is equidistant from Q and the x-axis and so lies on the curve. It is therefore the equation of the curve. Since y cannot be negative the equation is equivalent to

$$y^2 = (x-1)^2 + (y-1)^2,$$

and consequently to

$$x^2 - 2x - 2y + 2 = 0.$$

Exercises

1. What loci are represented by the equations, (a) $x = 3$, (b) $y = -2$, (c) $y = 2x$, (d) $x^2 + y^2 = 1$?

2. The point $P\ (x, y)$ is equidistant from (1, 2) and (3, −4). What is the locus of P? Find its equation.

3. The point $P\ (x, y)$ is twice as far from the x-axis as from the y-axis. What is the locus of P? (Two parts.) Find its equation.

4. The slope of the line joining $P\ (x, y)$ to (−2, 3) is equal to 3. What is the locus of P? Find its equation.

5. Given $A\ (-3, 1)$, $B\ (2, 0)$, $P\ (x, y)$. If the slopes of AB and BP are equal what is the locus of P? Find its equation.

6. The point $P\ (x, y)$ is equidistant from the y-axis and the point (−3, 4). Construct the locus of P. Find its equation.

7. Given $A\ (2, 3)$, $B\ (-1, 1)$, $C\ (2, -3)$, $P\ (x, y)$. If P moves along the line through A perpendicular to BC, show that

$$PC^2 - PB^2 = AC^2 - AB^2.$$

Find the equation of the line described by P.

8. Find the equation of a circle with center (−1, 2) and radius 5.

9. Find the equation of the circle whose diameter is the segment joining (2, −3) and (−1, 4).

10. Given $P_1\ (1, 3)$, $P_2\ (4, -1)$, find the equation of the locus described by $P\ (x, y)$ if the sum of its distances from P_1 and P_2 is 5. Free the equation of radicals and show that the result is

$$(4x + 3y - 13)^2 = 0.$$

Make a graph of this equation. Do all points of this graph belong to the locus?

11. Given $A\ (2, -2)$, $B\ (6, 0)$, $C\ (7, 3)$, $P\ (x, y)$. If the angle from AB to AC is equal to the angle from PB to PC, show that the locus of P is the circle through A, B, C. Calculate the tangents of the two angles, equate them and so get the equation of the circle.

Art. 21. Point on a Locus

If a point lies on a locus its coördinates satisfy the equation of the locus. Hence to ascertain whether a point lies on a given locus, substitute its coördinates in the equation of the locus and find out whether the equation is satisfied.

Example 1. Show that the curve $x^2 + y^2 = 2x$ passes through the origin.

Substituting the values $x = 0$, $y = 0$ the equation becomes $0 = 0$. Since the equation is satisfied the curve passes through the origin. In the same way it could be shown that any locus, represented by a polynomial equation with no constant term, passes through the origin.

Ex. 2. If the curve $x^2 + y^2 = 2ax$ passes through the point (2, −1) find the value of the constant a.

The coördinates 2, −1 must satisfy the equation. Hence

$$2^2 + (-1)^2 = 2a(2).$$

Consequently, $a = \frac{5}{4}$.

Ex. 3. If the locus of $y = mx + b$ passes through (−1, 1) and (3, 2), find the values of m and b.

The conditions required are

$$1 = -m + b, \quad 2 = 3m + b.$$

The solution of these equations is $m = \frac{1}{4}$, $b = \frac{5}{4}$.

Intersection of Curves. — An intersection of two curves must have coördinates satisfying both equations. Hence to find the points of intersection, solve the equations simultaneously. All solutions should be checked by substitution.

Example 1. Plot the curves represented by the equations $y^2 = 2x$ and $x^2 = 2y$ and find their points of intersection.

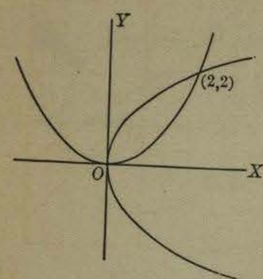

FIG. 21*a*.

The curves are shown in Fig. 21*a*. From the figure it is seen that there are two real points of intersection. Squaring the first equation and substituting the value of x^2 from the second,

$$y^4 = 4x^2 = 8y.$$

Consequently, $y\ (y^3 - 8) = 0$. The two real solutions are $y = 0, 2$. The corresponding values of x are $x = \frac{1}{2}y^2 = 0,\ 2$. The points of intersection are then $(0, 0)$ and $(2, 2)$. Substitution shows that the coördinates of these points satisfy both equations.

Ex. 2. Plot the curves $xy = 2$, $x^2 + y^2 = 4x$, and find their points of intersection.

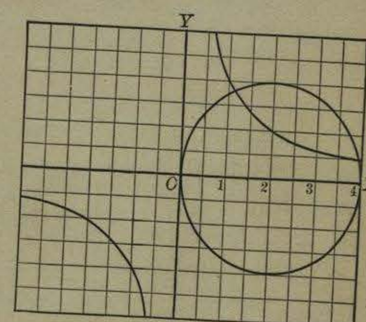

FIG. 21*b*.

The curves are shown in Fig. 21*b*. It is seen that there are two points of intersection. Elimination of y gives

$$x^4 - 4x^3 + 4 = 0.$$

The equation can only be solved approximately. The figure indicates that the solutions are near $x = 1.2$ and $x = 4$. By substitution the following values are found:

$x =$	1.1	1.2	3.9	4
$x^4 - 4x^3 + 4 =$	0.14	−0.84	−1.9	4

There is a root between 1.1 and 1.2 and another between 3.9 and 4. When $x = 1.15$ the expression $x^4 - 4x^3 + 4$ is found to be negative.

Hence, to one decimal, the value 1.1 is a root. Similarly 3.9 is found to be the other root. The corresponding values of y are $2/x = 1.8$ and 0.6. The points of intersection are then approximately

$$(1.1,\ \ 1.8),\ \ (3.9,\ \ 0.6).$$

Ex. 3. Plot the loci of the three equations

$$x + y = 1,\quad 2x + 3y = 5,\quad x^2 + y^2 = 13,$$

and show that they pass through a point.

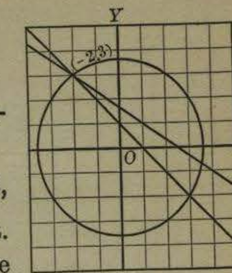

FIG. 21*c*.

The graphs are shown in Fig. 21*c*. The solution of the first two equations is $x = -2,\ y = 3$. These values satisfy the third equation. Consequently all the loci pass through the point $(-2, 3)$.

Art. 22. Tangent Curves

If at a point of intersection two curves have the same direction, they are called tangent. Let the curves AB and CD be tangent at P (Fig. 22*a*). CD can be considered as the limiting position of a curve $C'D'$ cutting AB in two points P_1 and P_2 close together. The equations of AB and $C'D'$ have two simultaneous solutions that are nearly the same. As $C'D'$ approaches CD these solutions approach equality. One might then expect that if the equations of AB and CD are solved simultaneously two of the solutions will be equal and that, conversely, equal roots occurring in the simultaneous solution of two equations indicate tangency. This should however be checked by the graphs as there are other circumstances that result in coincident solutions.

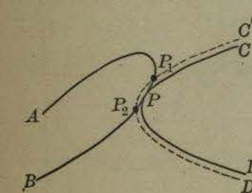

FIG. 22*a*.

Example 1. Show that the line $x + y = 2$ and the circle $x^2 + y^2 = 2$ are tangent and find the point of tangency.

Eliminating y,

$$x^2 + (2 - x)^2 - 2 = 0.$$

This equation is equivalent to $2(x-1)^2 = 0$ and so has two equal roots $x = 1$. The corresponding value of y is $2 - x = 1$. The line and circle intersect in only one point (1, 1). They must therefore be tangent at (1, 1).

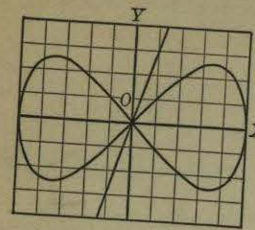

FIG. 22b.

Ex. 2. Plot the graphs of $y^2 = x^2 - x^4$ and $y = 3x$ and find their intersections.

The graphs are shown in Fig. 22b. The first is a curve like a horizontal figure 8. The second is a straight line through the origin. Eliminating y,

$$x^4 + 8x^2 = 0.$$

This equation has two roots $x = 0$. Two parts of the curve pass through the origin and both cut the line at that point. The curve and line are not however tangent.

Exercises

1. A, B, C, D, E being any fixed numbers, show that the curve whose equation is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$$

passes through the origin.

2. If x_1, y_1, m are constant show that the locus of the equation

$$y - y_1 = m(x - x_1)$$

passes through the point (x_1, y_1).

3. The curve $x^2/a^2 + y^2/b^2 = 1$ passes through the points (0, 1) and (2, 0). Find the values of a and b and plot the curve.

4. The curve $y^2 = ax + b$ passes through the points (0, −1) and (1, 2). Find the values of a and b and plot the curve.

Plot the following pairs of loci and find their points of intersection:

5. $3x + 2y = 1$, $2x - y = 0$.

6. $x^2 + y^2 = 10$, $2y - 3x = 3$.

7. $x^2 - y^2 = 3$, $x^2 + xy + y^2 = 7$.

8. $y^2 = x + 1$, $xy = 2$.

9. $xy^2 = 1$, $2x - y = 3$.

10. $x^2 - 2y^2 = 7$, $2y + 3x = 7$.

11. $2x^2 + y^2 = 6$, $x - y = 1$.

12. $x^2y + 3y = 4$, $x + 2y = 3$.

13. $y = 2x^3$, $y = 3x^2 - 5$.

14. $x^2 + 4y = 0$, $x^2 + y^2 + 6x + 7 = 0$.

15. Show that the following curves have a common point:

$$x^2 + y^2 = 10, \quad y^3 = x + 4, \quad y = \frac{2}{x+5}.$$