ANALYTIC GEOMETRY

CHAPTER 1

ALGEBRAIC PRINCIPLES

Art. 1. Constants and Variables

In analytic geometry much use is made of algebra. Hence a brief review is here given of some algebraic principles and processes used in this book.

In a given investigation a quantity is *constant* if its value is the same throughout that work, and *variable* if it may have different values. It should be noted that a quantity that is constant in one problem may be variable in another. Thus, in discussing a particular circle the radius would be constant, but in a problem about a circular disk expanding under heat the radius would be variable.

A quantity whose value is to be determined is often called an *unknown*. Such a quantity may be either constant or variable. In some cases it is not even known in advance whether it is constant or variable.

Real Numbers. — The simplest constants are numbers. The process of counting gives whole numbers. Division and subtraction give fractions and negative numbers. Whole numbers and fractions, whether positive or negative, are called *rational* numbers. A number, like $\sqrt{2}$, that can be expressed to any required degree of accuracy, but not exactly, by a fraction, is called *irrational*. Rational and irrational numbers, whether positive or negative, are called *real*.

The *absolute value* of a real number is the number without its algebraic sign. The absolute value of x is sometimes written $|x|$. Thus, $|-2| = |+2| = 2$.

Graphical Representation. — Real numbers can be represented graphically by the points of a straight line. Upon any point O of a line mark the number 0. Choose a unit of length. On one side of O mark positive numbers, on the other negative numbers, making the number at each point equal in absolute value to the distance from O to the point. The result is a *scale* on the line. When the line is horizontal, as in Fig. 1, it is usual, but not necessary, to lay off the positive numbers on the right of O, the negative numbers on the left.

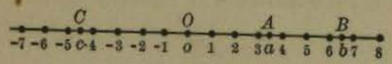

FIG. 1.

The point A representing the number a divides the scale into two parts. On one side of A, called the positive, lie all numbers greater than a; on the other side, called the negative, lie all numbers less than a. At a point B on the positive side of A is located a number b greater than a, at a point C on the negative side of A is a number c less than a.

The distance between two points of the scale is equal to the difference of the numbers at those points. This is obvious if the numbers are both positive. Thus

$$AB = OB - OA = b - a.$$

It is still true if one or both are negative. Thus, since c is negative, $CO = -c$ and

$$CB = CO + OB = -c + b = b - c.$$

Imaginary Quantities. — The extraction of roots sometimes leads to expressions like $\sqrt{-1}$ or $a + b\sqrt{-1}$, where a and b are real numbers. These expressions are called *imaginary*. This means merely that such expressions are not real numbers. It should not be inferred that imaginaries cannot be used or that they have no meaning. A quantity may have a meaning in one problem and not in another. For example, in determining the number of workmen needed in a certain undertaking the answer $3\frac{2}{3}$ would be absurd since $3\frac{2}{3}$ workmen cannot exist. In determining the area of a field the answer

-10 acres would be meaningless since there is no negative area. In determining the ratio of two lengths the answer $\sqrt{-2}$ is imaginary since the result must be a real number. But in still other problems, notably in work with alternating currents, an interpretation can be given to the process of extracting the square root of a negative number and then such results are entirely real.

Art. 2. Equations

An equation is the expression of equality between two quantities. An *identical* equation is one in which the equality is true for all values of the variables. Thus, in

$$(x - y)^2 + 4xy = (x + y)^2$$

the two sides are equal whatever values be assigned to x and y.

In many equations, however, the equality is true only for certain values of the variables; thus $x^2 + x = 2$ is an equation not true for all values of x, but only when $x = +1$ or -2.

Two or more equations are called simultaneous if all are satisfied at the same time. Equations often occur that are not simultaneous. Thus if $x^2 = 1$, then $x = 1$, or $x = -1$, but not both simultaneously.

A solution of an equation is a set of values of the variables satisfying the equation. Thus $x = 3$, $y = 4$ is one solution of the equation $x^2 + y^2 = 25$. A solution of a set of simultaneous equations is a set of values of the variables satisfying all of the equations.

Equivalent Equations. — Sets of equations are called *equivalent* if they have the same solutions. Thus the pair of simultaneous equations

$$x^2 + xy + y^2 = 4, \qquad x^2 - xy + y^2 = 2$$

is equivalent to

$$x^2 + y^2 = 3, \qquad xy = 1$$

(obtained by adding and subtracting the original equations) in the sense that any values of x and y, satisfying both equations of one pair, satisfy both equations of the other pair. Similarly, $(x + y)(x - 2y) = 0$ is equivalent to the two equations

$$x + y = 0, \qquad x - 2y = 0$$

in the sense that if x and y satisfy the equation $(x + y)(x - 2y) = 0$, then either $x + y = 0$ or $x - 2y = 0$; and, conversely, if x and y satisfy either of the latter equations, they satisfy the former.

The main problem in handling equations is to replace an equation or set of equations by a simpler or more convenient equivalent set. To solve an equation or set of equations is merely to find a particular equivalent set of equations.

Degree of Equation. — The equations of algebra usually have the form of polynomials equated to zero. By a polynomial is meant an expression, such as $x^3 + x^2 - 2$ or $x^2y + 3xy - y^2$, containing only positive integral powers and products of the variables.

The degree of a term like x^3 or $3xy$ is the sum of the exponents of the variables in that term. Thus the degree of x^3 is three, that of $3xy$ is two. The degree of a polynomial is that of the highest term in it. Thus, the polynomials given above are both of the third degree.

If a polynomial is equated to zero, or if two polynomials are equated to each other, the degree of the resulting equation is that of the highest term in it. For example, $x^2 + y^2 - x = 0$ and $xy = 1$ are both equations of the second degree.

Exercises

1. Determine which of the following equations are identities:

$$(a)\ x^m x^n = x^{m+n}, \quad (b)\ \frac{x}{y} + \frac{y}{x} = 2, \quad (c)\ \frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy}.$$

2. Expand $(x + y)^6$ by the binomial theorem. Is the resulting equation an identity?

3. Show that $x = \sqrt{2}$ is a solution of the equation

$$x^5 + 2x^3 - x^2 - 8x + 2 = 0.$$

4. Show that $x = -1$, $y = 2$ is a solution of the simultaneous equations

$$x^4 + 6xy + y^4 = 5, \quad x^2 + y^2 = 5.$$

5. Show that the pair of simultaneous equations

$$x^3 + y^3 = 2, \quad x + y = 1$$

is equivalent to the pair

$$x^2 - xy + y^2 = 2, \quad x + y = 1.$$

6. Find a set of three equations equivalent to

$$(x^2 - 1)(x^3 + 2) = 0.$$

Explain in what sense the three are equivalent to the one.

7. Is $x^2 - 4xy + 3y^2 = 0$ equivalent to the pair of simultaneous equations $x = y$, $x = 3y$?

8. The symbol $\sqrt{2}$ is generally used to represent the positive square root of 2. Is $x = \sqrt{2}$ equivalent to $x^2 = 2$?

9. Show that $\sqrt{x+1} + \sqrt{x-2} = 3$ is equivalent to $x = 3$.

10. The solution of the simultaneous equations

$$x + y = 3, \quad xy = 1$$

can be written

$$x = \tfrac{1}{2}(3 \pm \sqrt{5}), \quad y = \tfrac{1}{2}(3 \mp \sqrt{5}).$$

What do these mean? How many solutions are there?

11. What is the degree of the equation $(x + y)^4 = 3xy$?

12. If x and y are the variables, what is the degree of $ax^2 = bxy$?

Art. 3. Equations in One Variable

Quadratic Equations. — The quadratic equation

$$ax^2 + bx + c = 0$$

can be solved by completing the square. Transposing c, dividing by a and adding $b^2/4a^2$ to both sides, the equation becomes

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}.$$

Extracting the square root and solving for x,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If the expression under the radical is positive, the square root can be extracted and real values are obtained for x. If it is negative, no real square root exists and the values of x are imaginary.

Solution by Factoring. — Another method for solving quadratic equations is factoring. Thus

$$x^2 + 5x - 6 = 0$$

is equivalent to

$$(x-1)(x+6)=0.$$

Since a product can only be zero when one of its factors is zero, the above equation is satisfied only when $x = 1$ or $x = -6$.

If the quadratic cannot be factored by inspection, it can still be factored by completing the square. Thus

$$3x^2 - 2x + 1 = 3(x^2 - \tfrac{2}{3}x + \tfrac{1}{3}) = 3[(x - \tfrac{1}{3})^2 + \tfrac{2}{9}]$$
$$= 3(x - \tfrac{1}{3} - \tfrac{1}{3}\sqrt{-2})(x - \tfrac{1}{3} + \tfrac{1}{3}\sqrt{-2}).$$

The solutions of the equation $3x^2 - 2x + 1 = 0$ are then

$$x = \tfrac{1}{3}(1 \pm \sqrt{-2}).$$

In this way any equation can be solved if the expression equated to zero can be factored. For example, to solve the equation

$$x^3 + x^2 - 2 = 0$$

write it in the form

$$x^3 - 1 + x^2 - 1 = 0.$$

Since $x^3 - 1$ and $x^2 - 1$ both have $x - 1$ as a factor, the equation is equivalent to

$$(x-1)(x^2 + 2x + 2) = 0.$$

The solutions are consequently

$$x = 1 \text{ and } x = -1 \pm \sqrt{-1}.$$

Exercises

Solve the following equations:

1. $2x^2 + 3x - 2 = 0.$
2. $x^2 + 4x - 5 = 0.$
3. $3x^2 + 5x + 1 = 0.$
4. $x^2 + x + 1 = 0.$
5. $(x^2 - 1)(x^2 - 2) = 0.$
6. $\dfrac{(x^2 - 1)}{(x^2 - 4)} = 0.$
7. $\dfrac{1}{x} + \dfrac{1}{x-1} + \dfrac{1}{x+1} = 0.$

Solve by factoring

8. $x^2 - 3x - 1 = 0.$
9. $2x^2 + x - 2 = 0.$
10. $x^2 - x + 1 = 0.$
11. $x^3 - 2x^2 - x + 2 = 0.$
12. $x^3 - 1 = 0.$
13. $x^4 = 1.$

14. Solve the equation $x^4 + 1 = 0$ by reducing it to the form $(x^2 + 1)^2 - 2x^2 = 0.$

15. Solve the equation $x^4 + x^2 + 4 = 0$ by the method of the last example.

16. Factor $4x^2 + 4xy - y^2$ by completing the square of the first two terms.

Art. 4. Factors and Roots

It has been shown above that the roots of an equation can be found if the factors of the polynomial equated to zero are known. Conversely, if the roots are known the factors can be found. This is done by the use of the following theorem: *If r is a root of a polynomial equation in one variable x, then $x - r$ is a factor of the polynomial.* To prove this, let

$$P = ax^n + bx^{n-1} + \cdots + px + q$$

be a polynomial of the nth degree in which $a, b, \ldots, p, q$ are constants. If r is a root of the equation given by equating this polynomial to zero,

$$ar^n + br^{n-1} + \cdots + pr + q = 0.$$

Since subtracting zero from a quantity does not change its value,

$$P = ax^n + bx^{n-1} + \cdots + px + q - (ar^n + br^{n-1} + \cdots + pr + q)$$
$$= a(x^n - r^n) + b(x^{n-1} - r^{n-1}) + \cdots + p(x - r).$$

Each term on the right side of this equation is divisible by $x - r$. Hence the polynomial, P, has $x - r$ as a factor, which was to be proved.

Number of Roots. — It can be shown that any polynomial equation in one unknown has a root, real or imaginary. Assuming this, it follows that any polynomial of the nth degree in one variable is the product of n first degree factors. In fact, if r_1 is a root of $P = 0$, then

$$P = (x - r_1)Q,$$

Q being the quotient obtained by dividing P by $x - r_1$. Similarly, if r_2 is a root of $Q = 0$,

$$Q = (x - r_2)R.$$

Hence

$$P = (x - r_1)(x - r_2)R.$$

In the same way R can be factored, etc. Now each time a factor $x - r$ is divided out the degree of the quotient is one less. After taking out n factors, what is left will be of zero degree, that is, a constant. If a is the constant

$$P = a(x - r_1)(x - r_2) \ldots (x - r_n).$$

Hence P is the product of n first degree factors, $a(x - r_1)$, $(x - r_2)$, etc.

Since a product can only be zero when one of its factors is zero, it follows that the roots of $P = 0$ are $r_1, r_2, \ldots, r_n$. It is thus shown that *an equation of the nth degree has n roots*. Some of these r's may be equal and so the equation may have less than n distinct roots.

Rational Roots. — Though every polynomial equation in one unknown has a root, no very definite method can be given for finding it. If nothing in the particular equation suggests a better method, it is customary to try first to find a whole number or fraction that is a root of the equation. Such roots are found by trial. Some methods that may be useful are shown in the following examples.

Example 1. Solve the equation $4x^3 + 4x^2 - x - 1 = 0$.

Since x is a factor of all the terms in this equation except the last, -1, it follows that any integral value of x must be a divisor of -1. The only integral roots possible are then ± 1. By trial it is found that $x = -1$ satisfies the equation. Hence $x + 1$ is a factor of the polynomial. Factoring, the equation becomes $(x+1)(4x^2-1)=0$. The roots are consequently -1, and $\pm\frac{1}{2}$.

Ex. 2. Solve the equation $27x^3 + 9x^2 - 12x - 4 = 0$.

Proceeding as in the last example it is found that the equation has no integral root. Suppose a fraction p/q (reduced to its lowest terms) satisfies the equation. Substituting and multiplying by q^3,

$$27p^3 + 9p^2q - 12pq^2 - 4q^3 = 0.$$

Since all the terms but the last are divisible by p, and p and q have no common factor, -4 must be divisible by p. For the same reason 27 must be divisible by q. Any fractional root must then be equal to a divisor of 4 divided by a divisor of 27. It is found by trial that

$\frac{2}{3}$ is a root. Hence $x - \frac{2}{3}$ is a factor. Dividing and factoring the quotient, the equation is found to be

$$27(x - \tfrac{2}{3})(x + \tfrac{2}{3})(x + \tfrac{1}{3}) = 0.$$

The roots are consequently $\pm\frac{2}{3}$ and $-\frac{1}{3}$.

Exercises

Solve the following equations:

1. $x^3 - 2x^2 - x + 2 = 0$.
2. $3x^3 - 7x^2 - 8x + 20 = 0$.
3. $4x^3 - 8x^2 - 35x + 75 = 0$.
4. $8x^3 - 28x^2 + 30x - 9 = 0$.
5. $x^3 - 4x^2 - 2x + 5 = 0$.
6. $x^3 + 4x^2 + 4x + 3 = 0$.
7. $4x^4 + 8x^3 + 3x^2 - 2x - 1 = 0$.
8. $6x^4 - 11x^3 - 37x^2 + 36x + 36 = 0$.
9. $3x^4 - 17x^3 + 41x^2 - 53x + 30 = 0$.
10. $2x^4 - 9x^3 - 9x^2 + 57x - 20 = 0$.

Art. 5. Approximate Solution of Equations

If the equation has no whole numbers or fractions as roots, any real roots can still be found approximately. The method depends on the following theorem: *Between two values of x for which a polynomial has opposite signs must be a value for which it is zero.* To show this suppose when $x = a$ the polynomial is positive and when $x = b$ it is negative. Let x, beginning with the value a, gradually change. The value of the polynomial changes gradually. When x reaches b the polynomial is negative. There must have been an instant when it ceased to be positive and began to be negative. Now a number can only change gradually from positive to negative by going through zero. There is consequently a value of x between a and b for which the polynomial is zero.

The theorem can be illustrated by a figure. Let x be the number at the point M in a scale OX (Fig. 5), and let the perpendicular MP have a measure equal to the value of the polynomial for that value of x, it being drawn above OX when the value is positive, below when negative. As M moves along OX from A to B, the point P describes a curve. Since the curve is

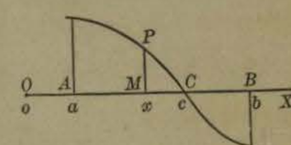

FIG. 5.

above at A and below at B, it must cross the axis at some intermediate point C. At that point the value of the polynomial is zero.

Example 1. Find the roots of $x^3 + 3x^2 - 1 = 0$ accurate to one decimal place.

By substitution the following pairs of values are found:

$x =$	$-3,$	$-2,$	$-1,$	$0,$	$+1,$
$x^3 + 3x^2 - 1 =$	$-1,$	$+3,$	$+1,$	$-1,$	$+3.$

The polynomial changes sign between $x = -3$ and $x = -2$, between $x = -1$ and $x = 0$ and between $x = 0$ and $x = 1$. There is consequently a root of the equation in each of these intervals. To find the root between 0 and 1, make an enlarged table for this region.

$x =$	$0,$	$0.5,$	$0.6,$	$1,$
$x^3 + 3x^2 - 1 =$	$-1,$	$-.125,$	$+.296,$	$3.$

It is thus seen that the root is between 0.5 and 0.6. When $x = 0.55$ the polynomial is positive. Hence the root lies between 0.5 and 0.55. The value 0.5 is therefore correct to one decimal. In the same way the value -2.9 is found for the root between -2 and -3, and -0.7 for the one between -1 and 0. Since the equation can have only three roots this completes the list.

Ex. 2. Solve the equation $x^3 + x - 3 = 0$.

Since $x^3 + x$ increases with x it can equal 3 for only one real value of x. To two decimals this root is found to be 1.21. The polynomial then has $x - 1.21$ as an approximate factor. Dividing by this the quotient is

$$x^2 + 1.21x + 2.46.$$

The solutions obtained by equating this to zero are

$$x = -.6 \pm 1.4\sqrt{-1}.$$

Exercises

Find to one decimal the roots of the following equations:

1. $x^3 - 3x^2 + 1 = 0$.
2. $x^3 + 3x - 7 = 0$.
3. $x^3 + x^2 + x - 1 = 0$.
4. $x^4 - 3x^3 + 3 = 0$.
5. $x^4 + x - 1 = 0$.
6. $x^5 - 3x - 1 = 0$.

Art. 6. Inequalities

An inequality expresses that one quantity is greater than (>) or less than (<) another. Thus,

$$x^2 + 1 > 2x \quad \text{and} \quad (x - 1)(x + 2) < 0$$

are inequalities. The first of these is an identical inequality (true for all values of x), the second is not. As in equations, terms can be shifted (with change of sign) from one side of an inequality to the other and inequalities having the same sign (> or <) can be added but not subtracted. Both sides of an inequality can be multiplied or divided by a positive quantity, but the sign must be changed (> to < and < to >) when an inequality is multiplied or divided by a negative quantity.

The main problem in inequalities is to determine for what values of the variable an inequality holds. How this is done is best shown by an example.

Example. Find the values of x for which

$$\frac{5x^2 - x - 3}{x^2(2 - x)} > 1.$$

This is equivalent to

$$\frac{5x^2 - x - 3}{x^2(2 - x)} - 1 > 0$$

or

$$\frac{(x + 1)(x - 1)(x + 3)}{x^2(2 - x)} > 0.$$

The problem is to determine the values of x for which the expression on the left is positive. Since x^2 is always positive, the sign of the expression is determined by the signs of the other four factors. The values of x making one of these factors zero are $-3, -1, 1, 2$. Mark

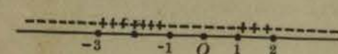

FIG. 6.

these values on a scale (Fig. 6). If $x < -3$ the three factors in the numerator are all negative, and $(2 - x)$ is positive. The whole expression, having an odd number of negative factors, is negative.

If x is between -3 and -1, there are two negative factors, $x + 1$ and $x - 1$, and the whole expression is positive. If x is between -1 and $+1$, the only negative factor is $(x - 1)$ and the expression is negative. If x is between 1 and 2, all the factors are positive and the whole expression is positive. If $x > 2$ there is one negative factor, $(2 - x)$, and the expression is negative.

The expression is positive when x is between -3 and -1, or between 1 and 2. These conditions are expressed by the inequalities

$$-3 < x < -1 \quad \text{and} \quad 1 < x < 2.$$

The original inequality is equivalent to these two, in the sense that it holds when one of these does and conversely.

Exercises

Find the values of x satisfying the following inequalities:

1. $x^2 + x - 2 > 0$.
2. $x^3 > x$.
3. $x^3 - 2x^2 + 2x - 1 < 0$.
4. $x^3 - 3x - 1 < 0$.
5. $\frac{1}{x} + \frac{1}{x-2} + \frac{1}{x+2} > 0$.
6. Show that $x^2 - 3x + 3 > 0$ is true for all values of x.
7. Find for what values of x, the value of y is real in the equation $x^2 + xy + y^2 = 1$.
8. Find the values of x satisfying both the inequalities

$$x^2 > x, \qquad x^2 > 2.$$

Art. 7. Simultaneous Equations

Simultaneous equations in more than one unknown are solved by a process called *elimination*. This is a name applied to any process by which equations are found equivalent to the given equations but some of which contain fewer unknowns. By a continuation of this process equations may eventually be obtained each containing a single unknown and these can be solved by the methods already given. In other cases it may not be possible to solve the equations completely but they may be reduced to a simpler form. If nothing in the equations indicates a simpler way, there are three general methods that may be useful:

(1) Multiply the equations by constants or variables and add or subtract to get rid of an unknown or to obtain a simpler equation.

(2) Solve one of the equations for one of the unknowns and substitute this value in each of the other equations.

(3) Between one of the equations and each of the others eliminate the same unknown. Proceed with the new equations in the same way until finally (if possible) one of the unknowns is found. Then determine the other unknowns by substituting this value in the previous equations.

However the solutions be found they should be checked by substitution in each of the original equations.

Example 1. Solve the simultaneous equations

$$\begin{aligned} x + y + z &= 2, \\ 2x - y + 3z &= 9, \\ 3x + 2y - z &= -1. \end{aligned}$$

Adding the second to the first and twice the second to the third,

$$\begin{aligned} 3x + 4z &= 11, \\ 7x + 5z &= 17. \end{aligned}$$

Subtracting 5 times the first from 4 times the second of these equations, there is found $13x = 13$, whence $x = 1$. This value substituted in either of the preceding equations gives $z = 2$. The values of x and z substituted in either of the original equations give $y = -1$. The solution is $x = 1$, $y = -1$, $z = 2$. These values check when substituted in the original equations.

Ex. 2. Solve the equations

$$\begin{aligned} x^2 + y^2 - 2x + 4y &= 21, \\ x^2 + y^2 + x - y &= 12. \end{aligned}$$

Subtraction gives $5y - 3x = 9$. Hence $y = \frac{3}{5}(x + 3)$. This value substituted in the second equation gives

$$17x^2 + 32x - 132 = 0.$$

The roots of this are 2 and $-\frac{66}{17}$. The corresponding values of y are 3 and $-\frac{9}{17}$. The solutions are $x = 2$, $y = 3$ and $x = -\frac{66}{17}$, $y = -\frac{9}{17}$. These values check when substituted in the original equations.

Exercises

Solve the following simultaneous equations:

1. $4x - 5y + 6 = 0,$
   $7x - 9y + 11 = 0.$
2. $x + 2y - z + 3 = 0,$
   $2x - y - 5 = 0,$
   $x + 2z - 8 = 0.$
3. $x + 2y + z = 0,$
   $x - y - z = 1,$
   $2x + y - z = 0.$
4. $x + 2y + 3z = 3,$
   $x - 2y + 3z = 1,$
   $x + 4y + 9z = 6.$
5. $\frac{1}{x} + \frac{1}{y} = 1,$
   $\frac{1}{y} + \frac{1}{z} = 2,$
   $\frac{1}{z} + \frac{1}{x} = 4.$
6. $x^2 + y^2 + 2x = 0,$
   $y = 3x + 4.$
7. $h^2 + k^2 - 8h + 4k + 20 = r^2,$
   $h^2 + k^2 + 6h + 2k + 10 = r^2,$
   $h^2 + 8h + 16 = r^2.$
8. $x^2 + 4y^2 = 5,$
   $xy = -1.$
9. $x = \frac{1}{y} + \frac{1}{z},$
   $y = \frac{1}{z} + \frac{1}{x},$
   $z = \frac{1}{x} + \frac{1}{y}.$
10. $x^2 + y^2 + z^2 = 6,$
    $x + y + z = 2,$
    $2x - y + 3z = 9.$

Art. 8. Special Cases

Inconsistent Equations. — Sometimes equations are inconsistent, that is, have no simultaneous solution. This is usually shown by the equations requiring the same expression to have different values. For example, take the equations

$$x + y + z = 1,$$
$$2x + 3y + 4z = 5,$$
$$x + 2y + 3z = 3.$$

Elimination of x between the first and second and first and third gives

$$y + 2z = 3, \quad y + 2z = 2.$$

Any solution of the original equations must satisfy these. Since the expression $y + 2z$ cannot equal both 2 and 3, there is no solution.

Dependent Equations. — Sometimes the solutions of part of the equations all satisfy the remaining equations. These last give no added information. Such equations are called dependent.

For example, take the equations

$$x + y = 1, \quad x^2 - y^2 + x + 3y = 2.$$

Substituting $1 - x$ for y in the second equation, it becomes $2 = 2$. All the solutions of the first equation satisfy the second. The two equations are equivalent to one equation $x + y = 1$. They have an infinite number of simultaneous solutions.

Number of Solutions. — In general, definite solutions are expected if the number of equations is equal to the number of unknowns. Thus, two equations usually determine two unknowns, three equations determine three unknowns, etc. This is, however, not always the case. The equations may be inconsistent and have no solution or may be dependent and have an infinite number of solutions. If the equations determine definite solutions, the number of solutions is expected to equal the product of the degrees of the equations. Special circumstances may, however, change this number. It can be shown that unless the number of solutions is infinite it cannot exceed the product of the degrees of the equations.

If there are fewer equations than unknowns, the unknowns will not be determined. In this case, if the equations are consistent, there will be an infinite number of solutions.

If there are more equations than unknowns, usually there will be no solution. In particular cases, however, there may be solutions. To determine whether there is a solution, solve part of the equations and substitute the values found in the remaining equations. If any of them satisfy all of the equations, there is a solution, otherwise there is none.

Homogeneous Equations. — If all the terms of an equation have the same degree the equation is called *homogeneous*. A set of homogeneous equations can often be solved for the ratios of the variables when there are not enough equations to determine the exact values.

For example, take the homogeneous equations

$$x - y - z = 0, \quad 3x - y - 2z = 0.$$

Solving for x and y

$$x = \tfrac{1}{2}z, \quad y = -\tfrac{1}{2}z.$$

Any value can be assigned to z and the values of x and y can then be determined from these equations. Let $z = 2k$. The solution is then

$$x = k, \quad y = -k, \quad z = 2k.$$

Since k is arbitrary, x, y, z have any values proportional to $1, -1, 2$. This result can be written $x:y:z = 1:-1:2$.

Exercises

Determine whether the following equations have no solutions, definite solutions, or an infinite number of solutions:

1. $x + y + z = 1,\quad 2x - 2y + 5z = 7,\quad 2x - 3y + 4z = 5.$
2. $x + 2y + z = 0,\quad 2x + y - z = 5,\quad 5x + 7y + 4z = 2.$
3. $x + 2y + 1 = 0,\quad 3x - y - 4 = 0,\quad 2x + 3y + 1 = 0.$
4. $x + y - z = 4,\quad x + 2y + 3z = 2,\quad 5x + 8y + 7z = 14.$
5. $x + y - 5 = 0,\quad 3x + 2y - 12 = 0,\quad 2x + y - 6 = 0.$
6. $x - y + 2z = 1,\quad 3x + y - z = 5,\quad 3x + 2y - 3z = 2.$
7. $3x + y + 1 = 0,\quad 2x + 3y - 4 = 0,\quad x + 2y - 3 = 0.$
8. $x - y = 0,\quad x^2 - y^2 = 1.$
9. $(x - y)^2 + (y - z)^2 + (z - x)^2 = 1,\quad x^2 + y^2 + z^2 = 2,\quad (x + y)^2 + (y + z)^2 + (z + x)^2 = 3.$
10. $w + x + y + z = 1,\quad w + 2x - 3y - 4z = 2,\quad w - x - 2y - 3z = 3,\quad w - 5x + 2y + 3z = 3.$

Find values to which the variables in the following equations are proportional:

11. $x + y - 2z = 0,\quad 3x - y - 4z = 0.$
12. $x + 2y + z = 0,\quad 4y + 3z = 0.$
13. $x + y - z = 0,\quad x^2 + y^2 - 5z^2 = 0.$
14. $x^2 + y^2 = 2z^2,\quad y^2 = xz.$

Art. 9. Undetermined Coefficients

It is often necessary to reduce a given expression or equation to a required form. This form is indicated by an expression or equation having letters for coefficients and the reduction is made by calculating the values of these coefficients.

In this work frequent use is made of the following theorem: *If two polynomials in one variable are equal for all values of the variable, the coefficients of the same power of the variable in the two polynomials are equal.* To show this, suppose, for all values of x,

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n.$$

Then for all values of x

$$(a_0 - b_0) + (a_1 - b_1)x + \cdots + (a_n - b_n)x^n = 0.$$

If the coefficients in this equation are not zero, by Art. 4, it cannot have more than n distinct roots. Hence the coefficients must all be zero and $a_0 = b_0$, $a_1 = b_1$, etc., which was to be proved.

To reduce an expression to a given form, equate the expression to the given form, clear of fractions or radicals, and determine the unknown coefficients by the above theorem.

Example 1. To find the coefficients a and b such that

$$\frac{x}{(x-1)(x+3)} = \frac{a}{x-1} + \frac{b}{x+3}$$

clear of fractions, getting,

$$x = a(x+3) + b(x-1) = (a+b)x + 3a - b.$$

If this equation holds for all values of x,

$$a + b = 1, \quad 3a - b = 0.$$

Hence $a = \frac{1}{4}$, $b = \frac{3}{4}$. Conversely, if a and b have these values, the above equations are identically satisfied. Therefore

$$\frac{x}{(x-1)(x+3)} = \frac{1}{4(x-1)} + \frac{3}{4(x+3)}.$$

In many cases the expression can be more easily changed to the required form by simple algebraic processes. This is particularly the case with second degree expressions where completing the square may give the required result.

Example 2. To reduce the expression

$$1 + 4x - 2x^2$$

to the form $a - b(x - c)^2$, it can be written

$$1 - 2(x^2 - 2x) = 3 - 2(x-1)^2,$$

which is the result required.

In reducing equations to a required form it should be noted that multiplying an equation by a number gives an equivalent equation. Thus, $x + y = 1$ and $2x + 2y = 2$ are equivalent. Two equations are then equivalent when corresponding coefficients are proportional.

Example 3. To find k such that $x + 8y - 1 = 0$ and $3x - y - 3 + k(x + 3y - 1) = 0$ are equivalent, write the last equation in the form

$$(3 + k)x + (3k - 1)y - (k + 3) = 0.$$

This is equivalent to $x + 8y - 1 = 0$ if corresponding coefficients are proportional, that is, if

$$\frac{3 + k}{1} = \frac{3k - 1}{8} = \frac{k + 3}{1}.$$

These equations are satisfied by $k = -5$.

Exercises

Reduce the following expressions to the forms indicated:

1. $2x^2 + 3x + 4 = (ax + b)^2 + c.$
2. $3 + 2x - x^2 = b - (x - a)^2.$
3. $x^2 + xy + y^2 = a(x + my)^2 + b(y - mx)^2.$
4. $\dfrac{x + 1}{x(x - 2)} = \dfrac{a}{x} + \dfrac{b}{x - 2}.$
5. $\dfrac{1}{(x + 3)(x^2 + 1)} = \dfrac{ax + b}{x^2 + 1} + \dfrac{c}{x + 3}.$

Reduce the following equations to the forms indicated:

6. $3x - 4y = 5, \quad y = mx + b.$
7. $2x + 3y = 4, \quad \dfrac{x}{a} + \dfrac{y}{b} = 1.$
8. $3x^2 + 2y^2 - 6x + 4y = 1, \quad \dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1.$
9. $x^2 - 4y^2 - 4x + 8y = 4, \quad \dfrac{(x - h)^2}{a^2} - \dfrac{(y - k)^2}{b^2} = 1.$
10. $3x - y + 5 = 0, \quad (x + y - 1) + k(x - y + 3) = 0.$

Art. 10. Functions

It is often desirable to state that one quantity is determined by another. For this purpose the word *function* is used. A quantity

y is called a function of x if values of x determine values of y. Thus, if $y = 1 - x^2$, then y is a function of x, for a value of x determines a value of y. Similarly, the area of a circle is a function of its radius; for, the length of radius being given, the area of the circle is determined.

It is not necessary that a value of the variable determine a single value of the function. It may be that a limited number of values are determined. Thus, y is a function of x in the equation

$$x^2 - 2xy + y^2 + x = 1.$$

To each value of x correspond two definite values of y obtained by solving a quadratic equation.

If a single value of the function corresponds to each value of the variable, the function is called *single valued*. If several values of the function correspond to the same value of the variable the function is called *many valued*.

Kinds of Functions. — Any expression containing a variable is a function of that variable, for, a value of the variable being given, a value of the expression is determined. Such a function is called *explicit*. Thus $\sqrt{x^2 + 1}$ is an explicit function of x. Similarly, if $y = \sqrt{x^2 + 1}$, then y is an explicit function of x.

If x and y are connected by an equation not solved for y, then y is called an *implicit* function of x. For example, y is an implicit function of x in the equation

$$x^2 + y^2 + 2x + y = 1.$$

Also x is an implicit function of y.

Explicit and implicit do not denote properties of the function but merely of the way it is expressed. An implicit function is rendered explicit by solving. For example, the above equation is equivalent to

$$y = \tfrac{1}{2}(-1 \pm \sqrt{5 - 8x - 4x^2}).$$

A *rational* function is one representable by an algebraic expression containing no fractional powers of variable quantities. For example,

$$\frac{x\sqrt{5}+3}{x^3+2}$$

is a rational function of x.

An *irrational* function is one represented by an algebraic expression which cannot be reduced to a rational form. For example, $\sqrt{x+1}$ is an irrational function of x.

A function is called *algebraic* if it can be expressed explicitly or implicitly by a finite number of algebraic operations (addition, subtraction, multiplication, division, raising to integral powers, and extraction of integral roots). All the functions previously mentioned are algebraic.

Functions that are not algebraic are called *transcendental*. For example, $x^{\sqrt{2}}$ and 2^x are transcendental functions of x.

The terms rational, irrational, algebraic, and transcendental denote properties of the function itself and do not depend on the way the function is expressed.

Notation.—A particular function of x is represented by the notation $f(x)$, which should be read function of x, or f of x, not f times x. For example, $f(x) = \sqrt{x^2+1}$, means that $f(x)$ is the definite function $\sqrt{x^2+1}$. Similarly, $y = f(x)$ means that y is a definite (though perhaps unknown) function of x.

The f in the symbol of a function should be considered as representing an operation to be performed on the variable. Thus, if $f(x) = \sqrt{x^2+1}$, f represents the operation of squaring the variable, adding 1, and extracting the square root of the result. If x is replaced by any other quantity, the same operation is to be performed on that quantity. For example, $f(2)$ is the result of performing the operation f on 2. With the above value of f,

$$f(2) = \sqrt{2^2+1} = \sqrt{5}.$$

Similarly,

$$f(y+1) = \sqrt{(y+1)^2+1} = \sqrt{y^2+2y+2}.$$

If it is necessary to consider several functions in the same discussion, they are distinguished by subscripts or accents or by the

use of different letters. Thus, $f_1(x)$, $f_2(x)$, $f_3(x)$, $f'(x)$, $f''(x)$, $f'''(x)$ (read f-one of x, f-two of x, f-three of x, f-prime of x, f-second of x, f-third of x), $g(x)$ represent (presumably) different functions of x.

Functions of Several Variables. — A quantity u is called a function of several variables if values of u are determined by values of those variables. For example, the volume of a cone is a function of its altitude and the radius of its base; for the volume is determined by the altitude and radius of base. This is indicated by the notation

$$v = f(h, r),$$

which should be read, v is a function of h and r, or v is f of h and r.

Similarly, the volume of a rectangular parallelopiped is a function of the lengths of its three edges. If a, b, and c are the lengths of the edges, this is expressed by the equation

$$v = f(a, b, c),$$

which should be read, v is a function of a, b, and c, or v is f of a, b, c.

Independent and Dependent Variables. — In most problems there occur a number of variable quantities connected by equations. Arbitrary values can be assigned to some of these quantities and the others are then determined. Those taking arbitrary values are called *independent* variables; those determined are called *dependent* variables. Which are taken as independent and which as dependent variables is usually a matter of convenience. The number of independent variables is, however, fixed by the equations.

Example. The radius r, altitude h, volume v, and total surface S of a cylinder are connected by the equations

$$v = \pi r^2 h, \quad S = 2\pi r^2 + 2\pi r h.$$

Any two of these four quantities can be taken as independent variables and the other two calculated in terms of them. If, for example, v and r are taken as the independent variables, h and S have the values

$$h = \frac{v}{\pi r^2}, \quad S = 2\pi r^2 + \frac{2v}{r}.$$

Exercises

1. If $f(x) = x^2 - 3x + 2$, show that $f(1) = f(2) = 0$.
2. If $f(x) = x + \frac{1}{x}$, find $f(x + 1)$. Also find $f(x) + 1$.
3. If $f(x) = \sqrt{x^2 - 1}$, find $f(2x)$. Also find $2f^2(x)$.
4. If $f(x) = \frac{x + 2}{2x - 3}$, find $f\left(\frac{1}{x}\right)$. Also find $\frac{1}{f(x)}$.
5. If $\psi(x) = x^4 + 2x^2 + 3$, show that $\psi(-x) = \psi(x)$.
6. If $\phi(x) = x + \frac{1}{x}$, show that $[\phi(x)]^2 = \phi(x^2) + 2$.
7. If $F(x) = \frac{1 - x}{1 + x}$, show that $F(a)\,F(-a) = 1$.
8. If $f_1(x) = 2^x$, $f_2(x) = x^2$, find $f_1[f_2(y)]$. Also find $f_2[f_1(y)]$.
9. If $f(x, y) = x^2 + 2xy - 5$, show that $f(1, 2) = 0$.
10. If $F(x, y) = x^2 + xy + y^2$, show that $F(x, y) = F(y, x)$.
11. If $f(x, y) = x^3 + 3x^2y + y^3$, show that $f(x, vx) = x^3 f(1, v)$.
12. If a, b, c are the sides of a right triangle how many of them can be taken as independent variables?
13. Express the radius and area of a sphere in terms of the volume taken as independent variable.
14. Given $u = x^2 + y^2$, $v = x + y$, determine x and y as functions of the independent variables u and v.
15. If x, y, z satisfy the equations

$$x + y + z = 6,$$
$$x - y + 2z = 5,$$
$$2x + y - z = 1,$$

show that none of them can be independent variables.

16. The equations

$$x + y + z = 6,$$
$$x - y + 2z = 5,$$
$$2x + 4y + z = 13$$

are dependent. Show that any one of the quantities x, y, z can be taken as independent variable.

17. If u, v, x, y are connected by the equations

$$u^2 + uv - y = 0, \qquad uv + x - y = 0,$$

show that u and x cannot both be independent variables.

CHAPTER 2

RECTANGULAR COÖRDINATES

Art. 11. Definitions

Scale on a Line. — In Art. 1 it has been shown that real numbers can be attached to the points of a straight line in such a way that the distance between two points is equal to the difference (larger minus smaller) of the numbers located at those points.

The line with its associated numbers is called a scale. Proceeding along the scale in one direction (to the right in Fig. 11*a*) the

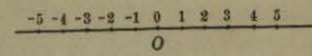

FIG. 11*a*.

numbers increase algebraically. Proceeding in the other direction the numbers decrease. The direction in which the numbers increase is called positive, that in which they decrease is called negative.

Coördinates of a Point. — In a plane take two perpendicular scales $X'X$, $Y'Y$ with their zero points coincident at O (Fig. 11*b*). It is customary to draw $X'X$, called the *x-axis*, horizontal with its positive end on the right, and $Y'Y$, called the *y-axis*, vertical with its positive end above. The point O is called the origin. The axes divide the plane into four sections called *quadrants*. These are numbered I, II, III, IV, as shown in Fig. 11*b*.

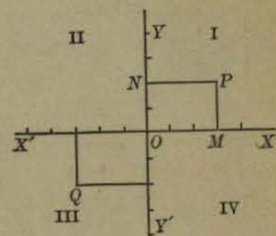

FIG. 11*b*.

From any point P in the plane drop perpendiculars PM, PN to the axes. Let the number at M in the scale $X'X$ be x and that at N in the scale $Y'Y$ be y. These num-