reflection that it would be impossible to construct a motor in which a vane would move continually in the same direction away from a fixed nozzle. The above discussion therefore gives but a rude approximation to the results obtainable under practical conditions. It shows truly, however, that the efficiency of a jet which is deflected normally from its path is but one-half of that obtainable when a complete reversal of direction is made.

Water wheels which act under the impulse of a jet consist of a series of vanes arranged around a circumference which by the motion are brought in succession before the jet. In this case the quantity of water which leaves the wheel per second is the same as that which enters it, so that W does not depend on the velocity of the vanes, as in the preceding case, but is a constant whose value is wq, where q is the quantity furnished per second. A close estimate of the efficiency of a series of such vanes can be made by considering a single vane and taking W as a constant. The water is supposed to impinge tangentially and the vane to move in the same line of direction as the jet (Fig. 158a). Then the work which is imparted in one second by the water to the moving vane is

$$k = (1 + \cos\beta) W \frac{(v-u)u}{g}$$

This becomes zero when $u = 0$ or when $u = v$, and it is a maximum when $u = \frac{1}{2}v$, or when the vane moves with one-half the velocity of the jet. Inserting this value of u,

$$k = \tfrac{1}{2}(1 + \cos\beta) W \frac{v^2}{2g}$$

and, dividing this by the theoretic energy of the jet, the efficiency of the vane is found to be

$$e = \tfrac{1}{2}(1 + \cos\beta)$$

When $\beta = 180°$, the jet merely glides along the surface without performing work and $e = 0$; when $\beta = 90°$, the jet is deflected normally to the direction of the motion and $e = \frac{1}{2}$; when $\beta = 0°$, a complete reversal of direction is obtained and the efficiency attains its maximum value $e = 1$.

These conclusions apply closely to the vanes of a water wheel which are so shaped that the water enters upon them tangentially in the direction of the motion. If the vanes are plane radial surfaces, as in simple paddle wheels, the water passes away normally to the circumference, and the highest obtainable efficiency is about 50 percent. If the vanes are curved backward, the efficiency becomes greater, and, neglecting losses in impact and friction, it might be made nearly unity, and the entire energy of the stream be realized, if the water could both enter and leave the vanes in a direction tangential to the circumference. The investigation shows that this is due to the fact that the water leaves the vanes without velocity; for, as the advantageous velocity of the vane is $\frac{1}{2}v$, the water upon its surface has the relative velocity $v - \frac{1}{2}v = \frac{1}{2}v$; then, if $\beta = 0°$, its absolute velocity as it leaves the vane is $\frac{1}{2}v - \frac{1}{2}v = 0$. If the velocity of the vanes is less or greater than half the velocity of the jet, the efficiency is lessened, although slight variations from the advantageous velocity do not practically influence the value of e.

Prob 159. A nozzle 0.125 feet in diameter, whose coefficient of discharge is 0.95, delivers water under a head of 82 feet against a series of small vanes on a circumference whose diameter is 18.5 feet. Find the most advantageous velocity of revolution of the circumference.

Art. 160. Revolving Vanes

When vanes are attached to an axis around which they move, as is the case in water wheels, the dynamic pressure which is effective in causing the motion is that tangential to the circumferences of revolution; or at any given point this effective pressure is normal to a radius drawn from the point to the axis. In Fig. 160 are shown two cases of a rotating vane; in the first the water passes outward or away from the axis, and in the second it passes inward or toward the axis. The reasoning, however, is general and will apply to both cases. At A, where the jet enters upon the vane, let v be its absolute velocity, V its velocity relative to the vane, and u the velocity of the point A; draw u normal to the radius r and construct the parallelogram of velocities as shown, α being the angle between the directions of u and v, and ϕ that between u and V. At B, where the water leaves the vane, let u_1 be the velocity of that point normal to the radius r_1, and V_1 the velocity of the water relative to the vane; then constructing

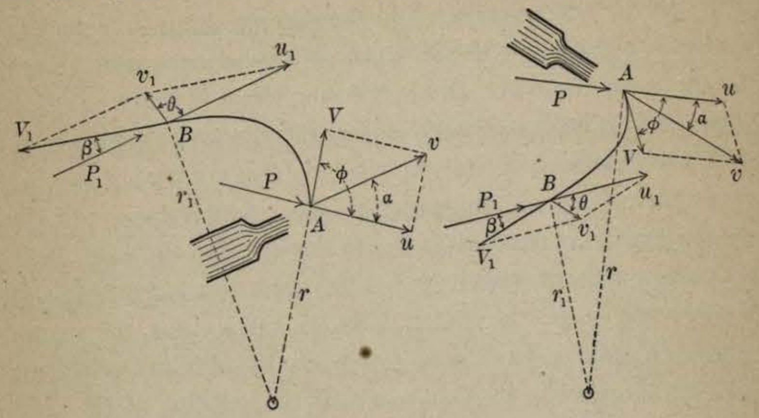

Fig. 160.

the parallelogram, the resultant of u_1 and V_1 is v_1, the absolute velocity of the departing water. Let β be the angle between V_1 and the reverse direction of u_1, and θ be the angle between the directions of v_1 and u_1.

The total dynamic pressure exerted in the direction of the motion will depend upon the values of the impulse of the entering and departing streams. The absolute impulse of the water before entering is $W \cdot v/g$, and that of the water after leaving is $W \cdot v_1/g$. Let the components of these in the directions of the motion of the vane at entrance and departure be designated by P and P_1; then

$$P = W\frac{v \cos\alpha}{g} \qquad P_1 = W\frac{v_1 \cos\theta}{g}$$

These, however, cannot be subtracted to give the resultant dynamic pressure, as was done in the case of motion in a straight line, because their directions are not parallel, and the velocities of their points of application are not equal. The resultant dynamic pressure is not important in cases of this kind, but the above values will prove useful in the next article in investigating the work that can be delivered by the vane.

If n be the number of revolutions around the axis in one second, the velocities u and u_1 are

$$u = 2\pi rn \qquad u_1 = 2\pi r_1 n$$

and accordingly the relation obtains

$$u_1/u = r_1/r \qquad \text{or} \qquad u_1 r = u r_1$$

which shows that the velocities of the points of entrance and exit are directly proportional to their distances from the axis. If r and r_1 are infinity, u equals u_1 and the case is that of motion in a straight line as discussed in Art. 158.

The relative velocities V_1 and V are connected with the velocities of rotation u_1 and u by a simple relation. To deduce it, imagine an observer standing on the outward-flow vane and moving with it; he sees a particle of weight w at A which to him appears to have the velocity V, while the same particle at B appears to have the velocity V_1; the difference of their kinetic energies, or $w(V_1^2 - V^2)/2g$, is the apparent gain of the wheel-energy. Again, consider an observer standing on the earth and looking down upon the vane; from his point of view the energy gained is $w(u_1^2 - u^2)/2g$. Now these two expressions for the gain of the wheel in energy must be equal, or

$$V_1^2 - V^2 = u_1^2 - u^2 \qquad (160)$$

and this is the formula by which V_1 is to be computed when V and the velocities of rotation are known. The same reasoning applies to the inward-flow vane by using the word "loss" instead of "gain," and the same formula results.

The given data for a revolving vane are the angles ϕ and β, the radii r and r_1, the velocity v, the number of revolutions per second, and the weight of water delivered to the vane per second. The value of $v \cos\alpha$, and hence that of P_1, is immediately known. From the speed of revolution the velocities u and u_1 are found. The relative velocity V is, from the triangle between u and v,

$$V = v \sin\alpha/\sin\phi$$

and by (160) the relative velocity V_1 is then found from

$$V_1^2 = u_1^2 - u^2 + V^2$$

Lastly, the value of $v_1 \cos\theta$, from the triangle between u_1 and V_1, is

$$v_1 \cos\theta = u_1 - V_1 \cos\beta$$

and accordingly the values of the dynamic pressures P and P_1 are fully determined. Numerical values of these, however, are never needed, but the work due to them is of much importance, as will be explained in the next article.

Prob. 160. Given $r = 2$ feet, $r_1 = 3$ feet, $\alpha = 45°$, $\psi = 90°$, $v = 100$ feet per second, and $n = 6$ revolutions per second. Compute the velocities u, u_1, V, and V_1.

Art. 161. Work derived from Revolving Vanes

The investigation in Art. 159 on the work and efficiency of a revolving vane supposes that all its points move with the same velocity, and that the water enters upon it in the same direction as that of its motion, or that $\alpha = 0$. This cannot in general be the case in water motors, as then the jet would be tangential to the circumference and no water could enter. To consider the subject further the reasoning of the last article will be continued, and, using the same notation, it will be plain that the work of a series of vanes arranged around a wheel may be regarded as that due to the impulse of the entering stream in the direction of the motion around the axis minus that due to the impulse of the departing stream in the same direction, or

$$k = Pu - P_1 u_1$$

Here P and P_1 are the pressures due to the impulse at A and B (Fig. 160), and inserting their values as found,

$$k = W \frac{uv \cos\alpha - u_1 v_1 \cos\theta}{g} \qquad (161)_1$$

This is a general formula applicable to the work of all wheels of outward or inward flow, and it is seen that the useful work k consists of two parts, one due to the entering and the other to the departing stream.

Another general expression for the work of a series of vanes may be established as follows: Let v and v_1 be the absolute veloc-

ities of the entering and departing water; the theoretic energy of this water is $W \cdot v^2/2g$, and when it leaves the wheel it still has the energy $W \cdot v_1^2/2g$. Neglecting losses of energy in impact and friction the work that can be derived from the wheel is

$$k = W \frac{v^2 - v_1^2}{2g} \qquad (161)_2$$

This is a formula of equal generality with the preceding, and like it is applicable to all cases of the conversion of energy into work by means of impulse or reaction. In both formulas, however, the plane of the vane is supposed to be horizontal, so that no fall occurs between the points of entrance and exit.

Formula (160) may be demonstrated in another way by equating the values of k in the preceding formulas; thus

$$uv \cos\alpha - u_1 v_1 \cos\theta = \tfrac{1}{2}(v^2 - v_1^2)$$

Now from the triangle at A between u and v

$$v^2 = V^2 - u^2 + 2\,uv \cos\alpha$$

and from the triangle at B between u_1 and v_1

$$v_1^2 = V_1^2 - u_1^2 + 2\,u_1 v_1 \cos\theta$$

Inserting these values of v^2 and v_1^2 the equation reduces to

$$V_1^2 - V^2 = u_1^2 - u^2$$

This shows that if u_1 be greater than u, as in the outward-flow vane, then V_1 is greater than V; if u_1 is less than u, as in an inward-flow vane, then V_1 is less than V.

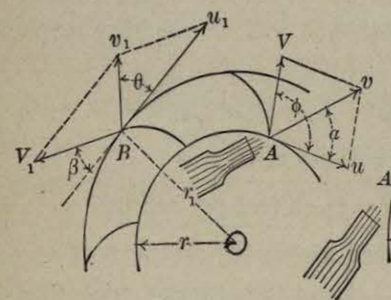

Fig. 161*a*.

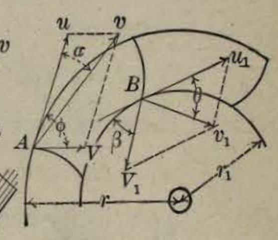

Fig. 161*b*.

The above principles will now be applied to the simple case of an outward-flow wheel driven by a fixed nozzle, as in Fig. 161*a*.

The wheel is so built that $r = 2$ feet, $r^1 = 3$ feet, $\alpha = 45°$, $\phi = 90°$, and $\beta = 30°$. The velocity of the water issuing from the nozzle is $v = 100$ feet per second, and the discharge per second is 2.2 cubic feet. It is required to find the work of the wheel and the efficiency when its speed is 337.5 revolutions per minute.

The theoretic work of the stream per second is the weight delivered per second multiplied by its velocity-head, or

$$k = 62.5 \times 2.2 \times 0.01555 \times 100^2 = 21\,380 \text{ foot-pounds}$$

which gives 38.9 theoretic horse-powers. The actual work of the wheel, neglecting losses in foam and friction, can be computed either from $(161)_1$ or $(161)_2$. In order to use the first of these, however, the velocities u, u_1, v_1, and the angle θ must be found, and to use the second, v_1 must be found; in each case V and V_1 must be determined.

The velocities u and u_1 are found from the given speed of 5.625 revolutions per second, thus:

$$u = 2 \times 3.1416 \times 2 \times 5.625 = 70.71 \text{ feet per second;}$$

$$u_1 = 1\tfrac{1}{2} \times 70.71 \qquad = 106.06 \text{ feet per second.}$$

The relative velocity V at the point of entrance is found from the triangle between V and v, which in this case is right-angled;

$$V = v \cos(\phi - \alpha) = v \cos 45° = 70.71 \text{ feet per second.}$$

The relative velocity V_1 at the point of exit is found from the relation (160), which gives $V_1 = u_1 = 106.06$ feet per second. And since u_1 and V_1 are equal, v_1 bisects the angle between V_1 and u_1, and accordingly

$$\theta = \tfrac{1}{2}(180° - \beta) = 75 \text{ degrees.}$$

The value of the absolute velocity v_1 then is

$$v_1 = 2\,u_1 \cos\theta = 54.90 \text{ feet per second,}$$

and $v_1^2/2g$ is the velocity-head lost in the escaping water.

The work of the wheel per second, computed either from $(161)_1$ or $(161)_2$, is now found to be $k = 14\,934$ foot-pounds or 27.2 horse-powers, and hence the efficiency, or the ratio of this work to the theoretic work, is $e = 0.699$. Thus 30.1 percent of the

energy of the water is lost, owing to the fact that the water leaves the wheel with such a large absolute velocity.

In this example the speed given, 337.5 revolutions per minute, is such that the direction of the relative velocity V is tangent to the vane at the point of entrance. For any other speed this will not be the case, and thus work will be lost in shock and foam. It is observed also that the approach angle α is one-half of the entrance angle ϕ; with this arrangement the velocities u and V are equal, as also u_1 and V_1. Had the angle β been made smaller the efficiency of the wheel would have been higher.

Prob. 161. Compute the power and efficiency for the above example if the angle β be 15° instead of 30°. Explain why β cannot be made very small.

Art. 162. Revolving Tubes

The water which glides over a vane can never be under static pressure, but when two vanes are placed near together and connected so as to form a closed tube, there may exist in it static pressure if the tube is filled. This is the condition in turbine wheels, where a number of such tubes, or buckets, are placed around an axis and water is forced through them by the static pressure of a head. The work in this case is done by the dynamic pressure exactly as in vanes, but the existence of the static pressure renders the investigation more difficult.

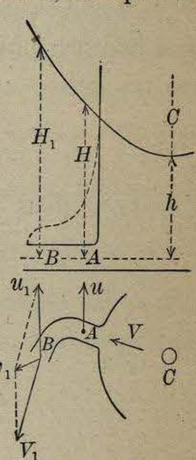

Fig. 162.

The simplest instance of a revolving tube is that of an arm attached to a vessel rotating about a vertical axis, as in Fig. 162. It was shown in Art. 29 that the water surface in this case assumes the form of a paraboloid, and if no discharge occurs, it is clear that the static pressures at any two points B and A are measured by the pressure-heads H_1 and H reckoned upwards to the parabolic curve, and, if the velocities of those points are u_1 and u, that

$$H_1 - \frac{u_1^2}{2g} = H - \frac{u^2}{2g} = h$$

Now suppose an orifice to be opened in the end of the tube and the flow to occur, while at the same time the revolution is continued. The velocities V_1 and V diminish the pressure-heads so that the piezometric line is no longer the parabola, but some curve represented by the lower broken line in the figure. Then, according to the theorem of Art. 31, that pressure-head plus velocity-head remains constant during steady flow, if no loss of energy occurs,

$$H_1 + \frac{V_1^2}{2g} - \frac{u_1^2}{2g} = H + \frac{V^2}{2g} - \frac{u^2}{2g} = h \qquad (162)$$

in which H_1 and H are the heads due to the actual static pressures. This is the theorem which gives the relation between pressure-head, velocity-head, and rotation-head at any point of a revolving tube or bucket. If the tube is only partly full, so that the flow occurs along one side, like that of a stream upon a vane, then there is no static pressure, and the formula becomes the same as (160).

An apparatus like Fig. 162, but having a number of arms from which the flow issues, is called a reaction wheel, since the dynamic pressure which causes the revolution is wholly due to the reaction of the issuing water. To investigate it, the general formula $(161)_1$ may be used. Making $u = 0$, the work done upon the wheel by the water is

$$k = W\frac{-u_1 v_1 \cos\theta}{g} = W\frac{u_1 V_1 \cos\beta - u_1^2}{g}$$

But since there is no static pressure at the point B, the value of V_1 is, from (162), or also from Art. 29,

$$V_1 = \sqrt{2gh + u_1^2}$$

The work that can be derived from the wheel now is

$$k = W\frac{u_1 \,\mathrm{co}\, \beta\sqrt{2gh + u_1^2} - u_1^2}{g}$$

This becomes nothing when $u_1 = 0$, or when $u_1^2 = 2gh \cot^2\beta$, and by equating the first derivative to zero it is found that k becomes a maximum when the velocity is given by

$$u_1^2 = \frac{gh}{\sin\beta} - gh$$

Inserting this advantageous velocity, the maximum work is

$$k = Wh(1 - \sin\beta)$$

and therefore the efficiency of the reaction wheel is

$$e = 1 - \sin\beta$$

When $\beta = 90°$, both u_1 and e become 0, for then the direction of the stream is normal to the circumference and no reaction can occur in the direction of revolution. When $\beta = 0$, the efficiency becomes unity, but the velocity u_1 becomes infinity. In the reaction wheel, therefore, high efficiency can only be secured by making the direction of the issuing water directly opposite to that of the revolution, and by having the speed very great. If $\beta =$ 19°.5 or $\sin\beta = \frac{1}{3}$, the advantageous velocity u_1 becomes $\sqrt{2gh}$ and e becomes 0.67. The effect of friction of the water on the sides of the revolving tube is not here considered, but this will be done in Art. 172.

Prob. 162*a*. Compute the theoretic efficiency of the reaction wheel when $\theta = 180°$, $\beta = 0°$, and $u_1 = \sqrt{2gh}$.

Prob. 162*b*. A reaction wheel has $\beta = 30°$, $r_1 = 0.302$ meters, and $h =$ 4.5 meters. Compute the most advantageous number of revolutions per minute. If the quantity of water delivered to the wheel is 1600 liters per minute, compute the power of the wheel in metric horse-powers and in kilowatts.

Prob. 162*c*. When l is in meters, v in meters per second, and p, p_1, and p_0 are in kilograms per square centimeter, the formulas $(157)_3$ for water hammer become

$$p = 0.0204\,(l/t)\,v + p_1 - p_0 \qquad p = 14.5\,v + p_1 - p_0$$

the first of which is to be used when t is greater than $0.001404l$ and the second when t is equal to or less than it, l being in meters.