sultant is the initial impulse F minus the final reaction F, or simply zero; in this case, however, there may be a couple which tends to twist the pipe, unless the impulse and reaction lie in the same straight line.

The dynamic pressure developed in a unit of length of the curve will now be found. Let the pipe at A in Fig. 156a have the length δl, and let θ be nearly o°, so that its value is the elementary angle $\delta\theta$. Then in the above formula P' becomes the elementary radial pressure δP_1, and

$$\delta P_1 = 2 \sin \tfrac{1}{2}\delta\theta \cdot F = F\delta\theta$$

Now since $\delta\theta = \delta l/R$, where R is the radius of the curve, the dynamic pressure developed in the distance δl is $F\delta l/R$, and that for a unit of length is F/R. Accordingly, by Art. 153, this pressure is

$$P_1 = \frac{F}{R} = \frac{2wa}{R}\,\frac{v^2}{2g}$$

The unit-pressure p' is found by dividing P_1 by a, and the corresponding head h_1 is found by dividing p' by w; hence

$$p' = \frac{2\,w}{R}\,\frac{v^2}{2g} \qquad \text{and} \qquad h_1 = \frac{2}{R}\,\frac{v^2}{2g}$$

are the values for one unit of length of the curve. The dynamic pressure-head h_1 is developed in every unit of length of the pipe. It is not known how these influence the static pressure or how they affect piezometers. Nor is it known whether they combine so that the dynamic pressure becomes greater with the distance from the beginning of the curve. Undoubtedly, however, a part of h_1 is expended in causing cross-currents whereby impact results and some of the static head is lost. This loss should be proportional to h_1 and proportional to the length l of the curve, or, if d is the diameter of the pipe,

$$h''' = m_1\frac{l}{R}\,\frac{v^2}{2g} = m_1\frac{d}{R}\cdot\frac{l}{d}\,\frac{v^2}{2g} = f_1\frac{l}{d}\,\frac{v^2}{2g}$$

in which the curvature factor f_1 depends upon the ratio R/d. This investigation appears to indicate that pipes of the same diameter and of different curvatures give the same loss of head, if the central angle is the same; but, as seen in Art. 91, certain experiments seem to point to the conclusion that the loss per linear unit is greatest in the pipe having the longest radius.

The same reasoning applies approximately to the curves of conduits, canals, and rivers. In any length l there exists a radial dynamic pressure P_1, acting toward the outer bank and causing currents in that direction, which, in connection with the greater velocity that naturally there exists, tends to deepen the channel on that side. This may help to explain the reason why rivers run in winding courses. At first the curve may be slight, but the radial flow due to the dynamic pressure causes the outer bank to scour away; this increases the velocity v_2 and decreases v_1 (Fig. 156b), and this in turn hastens the scour on the outer and allows material to be deposited on the inner side. Thus the process continues until a state of permanency is reached, and then the existing forces tend to maintain the curve. The cross-currents which the radial pressure produces have been actually seen in experiments devised by Thomson,* and it is found that they move in the manner shown in the above figure, the motion toward the outer bank being in the upper part of the section, while along the wetted perimeter they flow toward the inner bank. When the slope is small and the mean velocity low, the influence of the cross-currents is relatively greater than for higher slopes, and this is probably one of the reasons why the sharpest curves are found in streams of slight slope. Perhaps another reason for this is that at very low velocities the law of flow is different, the head varying as the first power of the velocity (Art. 124).

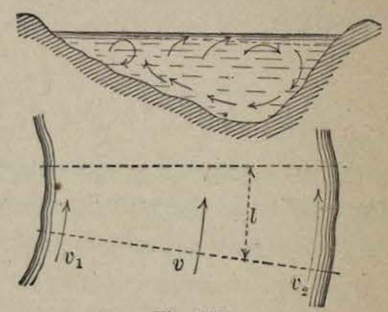

Fig. 156b.

The elevation of the water on the outer bank of a bend is higher than on the inner. This is only a partial consequence of the radial dynamic pressure, as in straight streams it is also seen that the water surface is curved, the highest elevation being where the velocity is greatest. In this case cross-currents are also ob-

* Proceedings Royal Society of London, 1878, p. 356.

served which move near the surface toward the center of the stream, and near the bottom toward the banks, their motion being due to the disturbance of the static pressure consequent upon the varying water level.

Prob. 156. The mean velocity in a pipe is 9 feet per second. If it be laid on a curve of 3 feet radius, show that the dynamic pressure-head for each foot in length of the pipe is 0.84 feet. If the radius of the curve be 6 feet, what is the dynamic pressure-head? What is the dynamic pressure-head for each case when the mean velocity is 3 feet per second?

Art. 157. Water Hammer in Pipes

When a valve in a pipe is closed while the water is flowing, the velocity of the water is retarded as the valve descends, and thus a dynamic pressure is produced. When the valve is closed quickly, this dynamic pressure may be much greater than that due to the static pressure, and it is then called "water hammer" or "water ram." Pipes have often been known to burst under this cause, and hence the determination of the maximum dynamic pressure of the water hammer is a matter of importance. Fig. 157*a* illustrates the phenomena of water hammer for the closing

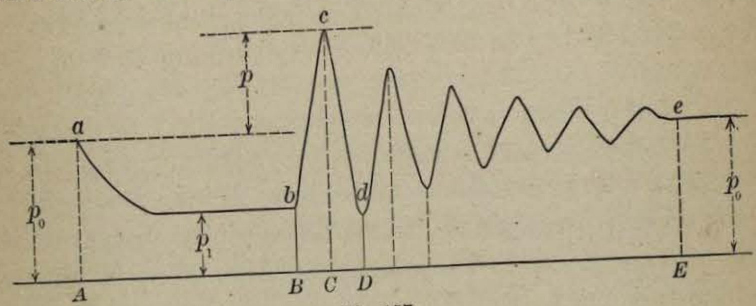

Fig. 157*a*.

of a valve at the end of a pipe where the water issues through a nozzle. At the entrance there is supposed to be a gage which registers the static unit-pressure p_1 while the flow is in progress, and the static unit-pressure p_0 when there is no flow. The abscissas represent time, and at B the valve begins to close. After a short interval of time BC the gage registers the unit-pressure Cc; after another short interval the unit-pressure has decreased

to Dd, and a series of oscillations follows until finally the disturbance ceases. A diagram of this kind may be autographically drawn by suitable mechanism connected with the pressure gage, and such were made in the experiments conducted by Carpenter,* as also in those of Fletcher.†

Let p represent the excess of maximum dynamic unit-pressure over the static unit-pressure when there is no flow; that is, the difference of the ordinates Cc and Ee. This is the excess unit-pressure due to the water hammer, and it is required to determine an expression for its value. It is first to be noted that the actual dynamic unit-pressure produced by the retardation of the velocity is the difference of the ordinates Cc and Bb and this difference is $p + p_0 - p_1$. The dynamic pressure on the area a of the cross-section of the pipe is then $(p + p_0 - p_1)a$, and for brevity this may be represented by P. If this pressure be regarded as varying uniformly from 0 up to P during the time t in which the valve closes, its mean value is $\frac{1}{2}P$ and its total impulse during this time is $\frac{1}{2}Pt$. If l be the length of the pipe, w the weight of a cubic unit of water, and v the velocity during the flow, the total weight of water in the pipe is wal and its impulse is $wal \cdot v/g$. Equating these expressions of the impulse there is found $P = 2\,walv/gt$, and replacing P by its value, there results

$$p = \frac{2wl}{gt}v + p_1 - p_0 \qquad (157)_1$$

as the excess dynamic unit-pressure due to closing the valve in the time t. This formula, having been deduced without considering the fact that time is required for the transmission of stress through water, cannot be regarded as applicable to all cases.

In Art. 5 it was shown that the velocity with which any disturbance is propagated through water is about 4670 feet per second, and this velocity may be represented by u. Now let the pipe of length l have an open valve at the end, and let the water be flowing through every section with the velocity v. Then the

* Transactions American Society of Mechanical Engineers, 1894, vol. 15.
† Engineering News, 1898, vol. 39, p. 323.

time l/u must elapse after the valve begins to close before the velocity begins to be checked at the upper end of the pipe, and the further time of l/u must elapse before the pressure due to this retardation can be transmitted back to the valve. The total time $2l/u$ is then required before the gage at the valve can indicate the pressure due to the retardation of the velocity in the length l. Hence, if the time in which the valve closes be equal to or less than $2l/u$, the time t in the above formula is to be replaced by $2l/u$, and thus

$$p = \frac{wu}{g}v + p_1 - p_0 \qquad (157)_2$$

is the maximum excess dynamic unit-pressure that can occur in the pipe. This depends upon the velocity of the water and upon the initial and final static pressures.

The subject of water hammer in pipes is one of the most difficult in hydromechanics, and the above investigation cannot be regarded as final. Formula $(157)_1$ is probably correct only for a certain law of valve closing. Formula $(157)_2$, however, is certainly correct, for it may be proved by other methods, one of which is as follows: When the water is in motion, the kinetic energy in a length δl at the gage is $wa\delta l \cdot v^2/2g$; when it is brought to rest under the unit-stress S, its stress energy is $a\delta l \cdot S^2/2E$, if E be the modulus of elasticity of the water.* Equating these expressions, and substituting $p + p_0 - p_1$ for S, there results for the excess dynamic unit-pressure

$$p = \left(\frac{Ew}{g}\right)^{\frac{1}{2}} v + p_1 - p_0$$

and this reduces to $(157)_2$ if E be replaced by wu^2/g, which is its value according to formula (5).

When v is in feet per second, and p_0, p_1, and p are in pounds per square inch, these formulas become

$$p = 0.027\,(l/t)v + p_1 - p_0 \qquad p = 63\,v + p_1 - p_0 \qquad (157)_3$$

the first of which is to be used when t is greater than $0.000428l$ and the second when t is equal to or less than it, l being in feet.

* Merriman's Mechanics of Materials (New York, 1911), p. 306.

From the first of these formulas the value of t, when $p = 0$, is found to be

$$t = 0.027\frac{lv}{p_0 - p_1}$$

which is the time of valve closing in order that there may be no water hammer. For example, let p_0 be 83 and p_1 be 58 pounds per square inch, l be 1903 feet, and v be 5 feet per second, then t is 10.3 seconds. To prevent the effects of water hammer, it is customary to arrange valves so that they cannot be closed very quickly, and the last formula furnishes the means of estimating the time required in order that no excess of dynamic pressure over the static pressure p_0 may occur.

The elaborate experiments of Joukowsky at Moscow in 1898 * have fully confirmed the truth of formula $(157)_2$. Horizontal pipes of 2, 4, and 6 inches diameter, with lengths of 2494, 1050, and 1066 feet, were used, and the valve at the end was closed in 0.03 seconds. Ten autographic recording gages were placed along the length of a pipe, and it was found that substantially the same dynamic pressure was produced at each, but that the time length of a wave was the shorter the farther the distance of a gage from the valve; this wave length is shown in the above figure by the distance BD. The following is a comparison of the observed

For the 4-inch Pipe			For the 6-inch Pipe		
Velocity	Observed	Computed	Velocity	Observed	Computed
0.5	31	31	0.6	43	38
1.9	115	118	1.9	106	118
2.9	168	183	3.0	173	189
4.1	232	258	5.6	369	353
9.2	519	580	7.5	426	472

values of $p + p_0 - p_1$ for a few of these experiments with the values computed from $(157)_3$. It is seen that the observed are less than the computed values except in one instance, and Joukowsky

* Stoss in Wasserleitungsröhren, St. Petersburg, 1900. Translation from the Memoirs of the St. Petersburg Academy of Sciences.

concludes that, owing to the influence of the metal of the pipes, the velocity u with which stress is transmitted in the water is about 4200 instead of 4670 feet per second. This conclusion may be applied in practice by using $59v$ instead of $63v$ in $(157)_3$.

Fig. 157a shows the waves of pressure for a case where the valve is closed in a time greater than $2l/u$. Fig. 157b shows

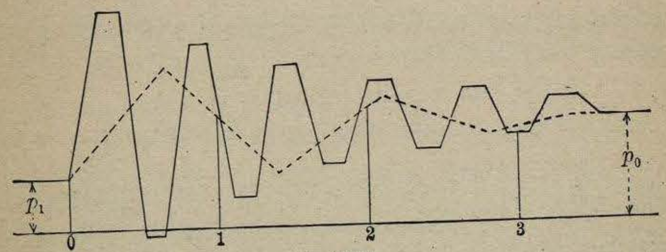

Fig. 157b.

the oscillations for two cases, the broken line being for $t = 0.7$ seconds and the full line for $t = 0.3$ seconds, both cases referring to a pipe for which the time $2l/u$ is about 0.6 seconds. It is seen that the crests of the waves are flat when the time of closing the valve is less than $2l/u$, and diagrams of this kind only were drawn in the experiments of Joukowsky.

In computing the thickness of water pipes it is customary to add 100 pounds per square inch to the static pressure in order to allow for the influence of water hammer. This is equivalent, if p_1 is zero, to making $p_0 + 100$ equivalent to $63v$; when v is 3 feet per second, then p_0 is 89 pounds per square inch. Since these values of v and p are larger than the usual ones for a city water supply, the customary practice is on the safe side for this case, but it would not give sufficient security for the high velocities often used in pipe lines for power plants. When a wave of dynamic pressure travels toward a dead end of a pipe, the water hammer at that end may be two or three times as great as the maximum pressure given by the formula.

In the case of a water power plant supplied from a pipe or long penstock, a "surge tank"* may be placed near the lower end in order to prevent sudden changes in pressure due to sudden

* Transactions American Society of Mechanical Engineers, 1908, p. 443.

changes in load on the wheels and the consequent fluctuations of velocity within the feeding pipe.

Prob. 157. The pressure-head at the entrance to a nozzle is 400 feet when there is no flow and 200 feet when the water is flowing. The pipe is 1500 feet long and the velocity in it is 4 feet per second when the nozzle is in operation. Compute the excess dynamic pressure when the valve is closed in 0.7 seconds and also when it is closed in 0.3 seconds.

Art. 158. Moving Vanes

A vane is a plane or curved surface which moves in a given direction under the dynamic pressure exerted by an impinging jet or stream. The direction of the motion of the vane depends upon the conditions of its construction; for example, the vanes of a water wheel can only move in a circumference around its axis. The simplest case for consideration, however, is that where the motion is in a straight line, and this alone will be considered in this article. The plane of the stream and vane is to be taken as horizontal, so that no direct action of gravity can influence the action of the jet.

Let a jet with the velocity v impinge upon a vane which moves in the same direction with the velocity u, and let the velocity of the jet relative to the surface at the point of exit make an angle β with the reverse direction of u, as shown in Fig. 158a. The velocity of the stream relative to the surface is $v - u$, and the dynamic pressure is the same as if the surface were at rest and the stream moving with the absolute velocity $v - u$. Hence formula $(154)_1$ directly applies, replacing v by $v - u$ and θ by $180° - \beta$, and the dynamic pressure is

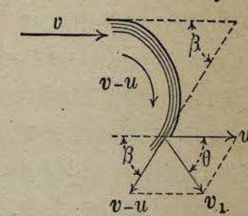

Fig. 158a.

$$P = (1 + \cos\beta) W \frac{v - u}{g}$$

In this formula W is not the weight of the water which issues from the nozzle, but that which strikes and leaves the vane, or $W = wa(v - u)$; for under the condition here supposed the vane moves continually away from the nozzle, and hence does not receive all the water which it delivers.

Another method of deducing the last equation is as follows: At the point of exit let lines be drawn representing the velocities $v - u$ and u; then, completing the parallelogram, the line v_1 is the absolute velocity of the departing jet (Art. 28). Let θ be the angle which v_1 makes with the direction of u, and β as before the angle between $v - u$ and the reverse direction of u. Then the dynamic pressure on the vane is that due to the absolute impulse of the entering and departing streams: the former of these is $W \cdot v/g$ and the latter is $W \cdot v_1 \cos\theta/g$. Hence the resultant dynamic pressure in the direction of the motion of the vane is the difference of these impulses, or

$$P = W\frac{v - v_1 \cos\theta}{g}$$

But from the triangle between v_1 and u

$$v_1 \cos\theta = u - (v - u)\cos\beta$$

Inserting this, the value of the dynamic pressure is

$$P = (1 + \cos\beta) W \frac{v - u}{g}$$

which is the same as that found before. If $\beta = 180°$, there is no pressure, and if $\beta = 0°$, the stream is completely reversed, and P attains its maximum value. The dynamic pressure may be exerted with different intensities upon different parts of the vane, but its total value in the direction of the motion is that given by the formula.

Usually the direction of the motion is not the same as that of the jet. This case is shown in Fig. 158*b*, where the arrow marked F designates the direction of the impinging jet, and that marked P the direction of the motion of the vane, α being the angle between them. The jet having the velocity v impinges upon the vane at A, and in passing over it exerts a dynamic pressure P which causes it to move with the velocity u. At A let lines be drawn representing the intensities and directions of v and u, and let the parallelogram of velocities be formed as shown; the line

V then represents the velocity of the water relative to the vane at A. The stream passes over the surface and leaves it at B with the same relative velocity V, if not retarded by friction or foam. At B let lines be drawn to represent u and V, and let β be the angle which V makes with the reverse direction of u; let the parallelogram be completed, giving v_1 for the absolute velocity of the departing water, and let θ be the angle which it makes with u. The total pressure in the direction of the motion is now to be regarded as that caused by the components in that direction of the initial and the final impulse of the water. The impulse of the stream before striking the vane is $W \cdot v/g$ and its component in the direction of the motion is $W \cdot v \cos\alpha/g$. The impulse of the stream as it leaves the vane is $W \cdot v_1/g$ and its component in the direction of the motion is $W \cdot v_1 \cos\theta/g$. The difference of these components is the resultant dynamic pressure in the given direction, or

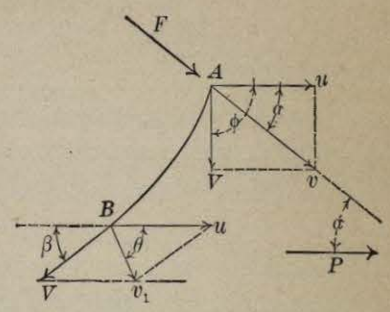

Fig. 158*b*.

$$P = W\frac{v \cos\alpha - v_1 \cos\theta}{g} \tag{158}$$

This is a general formula for the dynamic pressure in any given direction upon a vane moving in a straight line, if α and θ be the angles between that direction and those of v and v_1. If the surface be at rest, v and v_1 are equal and the formula reduces to $(154)_2$.

If it be required to find the numerical value of P, the given data are the velocities v and u and the angles α and β. The term $v_1 \cos\theta$ is hence to be expressed in terms of these quantities. From the triangle at B between v_1 and u, there is found

$$v_1 \cos\theta = u - V \cos\beta$$

and substituting this, the formula becomes

$$P = W\frac{v \cos\alpha - u + V\cos\beta}{g}$$

which is often a more convenient form for discussion. The value of V is found from the triangle at A between u and v, thus:

$$V^2 = u^2 + v^2 - 2uv\cos\alpha$$

and hence the dynamic pressure P is fully determined in terms of the given data.

In order that the stream may enter tangentially upon the vane, and thus prevent foam, the curve of the vane at A should be tangent to the direction of V. This direction can be found by expressing the angle ϕ in terms of the given angle α. Thus from the relation between the sides and angles of the triangle included between u, v, and V there is found

$$\sin(\phi - \alpha)/\sin\phi = u/v$$

which is easily reduced to the form

$$\cot\phi = \cot\alpha - \frac{u}{v\sin\alpha}$$

from which ϕ can be computed when u, v, and α are given. For example, if u be equal to $\frac{1}{2}v$, and if α be $30°$, then $\cot\phi$ is 0.732, whence the angle ϕ should be $53\frac{3}{4}°$, in order that the jet may enter without impact. If the angle made by the vane with the direction of motion be greater or less than this value, some loss due to impact will result at the given speed.

Prob. 158. Given u = 86.6 and v = 100.0 feet per second, and $\alpha = 30°$. What should be the value of the angle ψ in order that no loss by impact may occur? Draw the parallelogram showing the velocities u, v, and V.

Art. 159. Work derived from Moving Vanes

The work imparted to a moving vane by the energy of the impinging water is equal to the product of the dynamic pressure P, which is exerted in the direction of the motion and the space through which it moves. If u be the space described in one second, or the velocity of the vane, the work per second is

$$k = Pu$$

This expression is now to be discussed in order to determine the value of u which makes k a maximum.

When the vane moves in a straight line in the same direction as the impinging jet and the water enters it tangentially, as shown in Fig. 154*b*, the work imparted is found by inserting for P its value from $(154)_1$. If a be the area of the cross-section of the jet and w the weight of a cubic unit of water, the weight W is $wa(v - u)$, and then

$$k = (1 + \cos\beta)W\frac{(v-u)u}{g} = (1 + \cos\beta)wa\frac{(v-u)^2u}{g}$$

The value of u which renders k a maximum is obtained by equating to zero the derivative of k with respect to u, or

$$\frac{\delta k}{\delta u} = (1 + \cos\beta)\frac{wa}{g}(v^2 - 4vu + 3u^2) = 0$$

from which the value of u is $\frac{1}{3}v$; and accordingly

$$k = \tfrac{8}{27}(1 + \cos\beta)wa\frac{v^3}{2g}$$

is the maximum work that can be done by the vane in one second. The theoretic energy of the impinging jet is

$$K = W\frac{v^2}{2g} = wa\frac{v^3}{2g}$$

and the efficiency of the vane is the ratio of the effective work of the vane to the theoretic energy of the water, or

$$e = k/K = \tfrac{8}{27}(1 + \cos\beta)$$

If $\beta = 180°$, the jet glides along the vane without producing work and $e = 0$; if $\beta = 90°$, the water departs from the vane normal to its original direction and $e = \frac{8}{27}$; if $\beta = 0°$, the direction of the stream is reversed and $e = \frac{16}{27}$.

It appears from the above that the greatest efficiency which can be obtained by a vane moving in a straight line under the impulse of a jet of water is $\frac{16}{27}$; that is, the effective work is only about 59 percent of the theoretic energy attainable. This is due to two causes: first, the quantity of water which reaches and leaves the vane per second is less than that furnished by the nozzle or mouthpiece from which the water issues; and, secondly, the water leaving the vane still has an absolute velocity of $\frac{1}{3}v$. A vane moving in a straight line is therefore a poor arrangement for utilizing energy, and it will also be seen upon