as high as at present. It is supposed that these tides rolled over the low lands and moved great rocks from place to place. The greatest velocity of such a wave is $\sqrt{gd}$, where d is the depth of the water. What is the probable weight and size of the largest rock that such a current would move?

Art. 136. Influence of Dams and Piers

When a dam is built across a stream, it is often desired to compute its height so that the water level may stand at a given elevation. Thus in the figures, CC represents the surface of the stream before the construction of the dam, the depth of the water being D, and it is required to find the height G of the dam so that

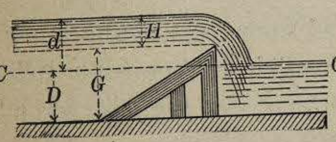

Fig. 136a.

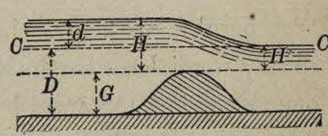

Fig. 136b.

the water surface may be raised the vertical distance d. There are two cases, the first where the crest is above the original water level CC, and the second where it is below that level; in both cases the discharge q must be known in order to compute the height of the dam.

When the crest is not submerged, as in Fig. 136a, it is seen that the value of G is $D + d - H$, where H is the head on the crest. Now from Art. 64 the value of q is $\text{M}b(H + 1\frac{1}{3}h)^{\frac{3}{2}}$, where b is the length of the crest and h is the head due to velocity of approach. Hence there results

$$G = D + d + 1\tfrac{1}{3}h - (q/\text{M}b)^{\frac{2}{3}} \qquad (136)_1$$

in which M is to be taken from Art. 69. For example, let the discharge be 18 000 cubic feet per second, let the width of the stream above the dam be 600 feet, and the width on the crest be 525 feet; also let D and d be 8.5 and 6.0 feet, and let M be 3.33. The mean velocity of approach is

$$v = \frac{18\,000}{600 \times 14.5} = 2.1 \text{ feet per second}$$

whence the velocity-head is $h = 0.0155 \times 2.1^2 = 0.07$ feet. Then from the formula there results $G = 9.9$ feet, which is the

required height of the dam. In many cases it will be unnecessary to consider velocity of approach, and h may be omitted from the formula; if this be done for the example in hand, the value of G is 9.8 feet.

When it is desired to raise the water level only a short distance, the crest of the dam will be submerged. For this case Fig. 136b gives $H = D + d - G$ and $H' = D - G$. By inserting these heads in formula $(67)_2$ and neglecting velocity of approach, there is found

$$G = D + \tfrac{2}{3}d - \tfrac{2}{3}q/\text{M}b\sqrt{d} \qquad (136)_2$$

Here the coefficient M lies between 3.09 and 3.37, depending on the value of the ratio H'/H, and as a mean 3.1 may be used. For example, let $q = 400$ cubic feet per second, $D = 4$, $d = 1$, $b = 50$ feet; then G is found to be 2.95 feet. The value of H is then 2.05 feet and that of H' is 1.05, whence H'/H is 0.5 closely, and from Art. 67 the value of M is 3.11, which indicates that the assumed value is close enough. Accordingly 3.0 feet may be taken as the height of the submerged dam.

When bridge piers are built in a stream, its cross-section is diminished and the water level up-stream from the piers stands at a greater height than before. The most common problem is to find how high the water will rise when the original width B is to be contracted to the width b. Let D (Fig. 136c) be the mean depth of the water before the building of the piers, H the rise in the water level, and q the discharge of the stream. Then the discharge q may be regarded as consisting of two parts, first that passing over a weir of breadth B under the head H, and second that passing through the submerged orifice of breadth b and height D under the head H. Hence, from Arts. 64 and 51,

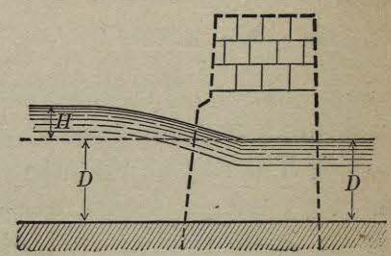

Fig. 136c.

$$c\sqrt{2g}\left(\tfrac{2}{3}B(H+h)^{\frac{3}{2}} + bD(H+h)^{\frac{1}{2}}\right) = q \qquad (136)_3$$

in which h is the head due to the velocity of approach. The coefficient of discharge c for weirs and orifices is about 0.6, but here it is much larger, since there is no crest. From experiments by Weisbach on a small round pier, c appears to be over 0.9, and from other discussions it appears in some cases to be a little lower than 0.8. Its value in any event depends upon the shape of the piers and their cutwaters, and probably the best that can now be done is to take it as 0.9 for piers with round ends and at 0.8 for piers with triangular cutwaters.

As an example of the determination of c, take the case of a flood in the Gungal River,* where $B = 650$, $b = 578$, and $D = 35$ feet and $q = 477\,800$ cubic feet per second, and where it was observed that the height H was 3.6 feet. The mean velocity above the piers was $v = 477\,800/38.6 \times 650 = 19.0$ feet per second, whence the velocity-head $h = 5.61$ feet. Inserting all these data in the formula and solving for c, there is found $c = 0.79$. This is an unusual case where the velocity was very high, and the piers had sharp cutwaters.

As an example of the determination of the height H, take the case of a bridge over the Weser,† where $B = 593$, $b = 315$, $D = 16.4$ feet, and $q = 46\,550$ cubic feet per second. As nothing is known about the shape of the piers, c may be taken as 0.8; then formula $(136)_3$ reduces to

$$(H+h)^{\frac{3}{2}} + 13.1\,(H+h)^{\frac{1}{2}} = 18.3$$

from which $H + h$ is found by trial to be 1.55 feet. Now, assuming H as 1.2 feet, the mean velocity above the piers is found to be 4.3 feet per second, whence h is 0.29 feet. Accordingly $H = 1.55 - 0.29 = 1.26$ feet, and with this value the velocity above the pier is found to be 4.44 feet per second, whence a better value of h is 0.31 feet. This gives $H = 1.24$ feet, which may be regarded as the final result for the height of the backwater.

Prob. 136. A river 940 feet wide has a mean depth of 4.1 feet and a mean velocity of 3.3 feet per second. Ten piers, each 12 feet wide, are to be built across it. Compute the probable rise of backwater caused by the piers. Compute also the probable rise during a flood which increases the mean depth to 18.5 feet and the mean velocity to 5.8 feet per second.

* Proceedings, Institution of Civil Engineers, 1868, vol. 27, p. 222.

† D'Aubuisson's Treatise on Hydraulics, Bennett's translation (New York, 1857), p. 189.

Art. 137. Steady Non-uniform Flow

In Arts. 112–133 the slope of the channel, its cross-section, and its hydraulic radius have been regarded as constant. If these are variable in different reaches of the stream, the case is one of non-uniformity, and this will now be discussed. The flow is still regarded as steady, so that the same quantity of water passes each section per second, but its velocity and depth vary as the slope and cross-section change. Let there be several reaches $l_1, l_2, \cdots l_n$, which have the falls $h_1, h_2, \cdots h_n$, the water sections being $a_1, a_2, \cdots a_n$, the hydraulic radii $r_1, r_2, \cdots r_n$, and the velocities $v_1, v_2, \cdots v_n$. The total fall $h_1 + h_2 + \cdots + h_n$ is expressed by h. Now the head corresponding to the mean velocity in the first section is $v_1^2/2g$. The theoretic effective head for the last section is $h + v_1^2/2g$, while the actual velocity-head is $v_n^2/2g$. The difference of these is the head lost in friction; or by (125),

$$h + \frac{v_1^2}{2g} - \frac{v_n^2}{2g} = \frac{l_1 v_1^2}{c_1^2 r_1} + \frac{l_2 v_2^2}{c_2^2 r_2} + \cdots + \frac{l_n v_n^2}{c_n^2 r_n}$$

in which $c_1^2, c_2^2, \cdots c_n^2$ are the Chezy coefficients for the different lengths. Now let q be the discharge per second; then, since the flow is steady, the mean velocities are

$$v_1 = q/a_1 \qquad v_2 = q/a_2 \cdots v_n = q/a_n$$

and, inserting these in the equation, it reduces to

$$h = \frac{q^2}{2g}\left(\frac{1}{a_n^2} - \frac{1}{a_1^2}\right) + q^2\left(\frac{l_1}{c_1^2 a_1^2 r_1} + \frac{l_2}{c_2^2 a_2^2 r_2} + \cdots + \frac{l_n}{c_n^2 a_n^2 r_n}\right)$$

which is a fundamental formula for the discussion of steady flow through non-uniform channels. This formula shows that the discharge q is a consequence not only of the total fall h in the entire length of the channel, but also of the dimensions of the various cross-sections. The assumption has been made that a and r are constant in each of the parts considered; this can be realized by taking the lengths $l_1, l_2, \cdots l_n$ sufficiently short. If only one part be considered in which a and r are constant, a_n and a_1 are equal, all the terms but one in the second member disappear, and the last equation reduces to $q = ca\sqrt{rh/l}$, which is the Chezy formula for the discharge in a channel of uniform cross-section.

An important practical problem is that where the steady flow is non-uniform in a channel having a bed with constant slope, a condition which may be caused by an obstruction below the part considered or by a sudden fall below it. Let a_1 and a_2 be the areas of the two sections, l their distances apart, and v_1 and v_2 the mean velocities. Then, if a and r be average values of the areas and hydraulic radii of the cross-sections throughout the length l, the last formula becomes

$$h = \frac{q^2}{2g}\left(\frac{1}{a_2^2} - \frac{1}{a_1^2} + \frac{2gl}{c^2a^2r}\right)$$

Now the important problem is to discuss the change in depth between the two sections. For this purpose let A_1A_2 in Fig. 137 be the longitudinal profile of the water surface, let A_1D be horizontal, and A_1C be drawn parallel to the bed B_1B_2. The depths A_1B_1 and A_2B_2 are represented by d_1 and d_2, the latter being taken as the larger. Let i be the constant slope of the bed B_1B_2; then $DC = il$, and since $DA_2 = h$ and $A_2C = d_2 - d_1$, there is found for the fall in the length l,

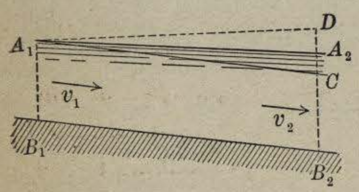

Fig. 137.

$$h = il - (d_2 - d_1)$$

Inserting this value of h in the preceding equation and solving for l, there is obtained the important formula

$$l = \frac{(d_2 - d_1) - \frac{q^2}{2g}\left(\frac{1}{a_1^2} - \frac{1}{a_2^2}\right)}{i - q^2/c^2a^2r} \qquad (137)_1$$

from which the length l corresponding to a change in depth $d_2 - d_1$ can be approximately computed. This formula is the more accurate the shorter the length l, since then the mean quantities a and r can be obtained with greater precision, and c is subject to less variation.

The inverse problem, to find the change in depth when l is given, cannot be directly solved by this formula, because the areas are functions of the depths. When $d_2 - d_1$ is small compared with either d_1 or d_2, it is allowable to regard d_2 as equal to d_1 when they are to be added or multiplied together. Hence

$$\frac{1}{a_1^2} - \frac{1}{a_2^2} = \frac{a_2^2 - a_1^2}{a_1^2a_2^2} = \frac{d_2^2 - d_1^2}{b^2d_1^2d_2^2} = \frac{(d_2 + d_1)(d_2 - d_1)}{b^2d_1^4} = \frac{2(d_2 - d_1)}{b^2d_1^3}$$

also making a equal to a_1 and r equal to d_1 in the last formula, and solving for $d_2 - d_1$, there is found

$$\frac{d_2 - d_1}{l} = \frac{i - q^2/c^2b^2d_1^3}{1 - q^2/gb^2d_1^3} \qquad (137)_2$$

from which the change in depth can be computed when all the other quantities are given.

Fig. 137 is drawn for the case of depth increasing downstream, but the reasoning is general and the formulas apply equally well when the depth decreases with the fall of the stream. In the latter case the point A_2 is below C, and $d_2 - d_1$ will be negative. As an example, let it be required to determine the decrease in depth in a rectangular conduit 5 feet wide and 333 feet long, which is laid with its bottom level, the depth of water at the entrance being maintained at 2 feet, and the quantity supplied being 20 cubic feet per second. Here $l = 333$, $b = 5$, $d_1 = 2$, $q = 20$, and $i = 0$. Taking $c = 89$, and substituting all values in the formula, there is found $d_2 - d_1 = -0.09$ feet; whence $d_2 = 1.91$ feet, which is to be regarded as an approximate probable value. It is likely that values of $d_2 - d_1$ computed in this manner are liable to an uncertainty of 15 or 20 percent, the longer the distance l the greater being the error of the formula. In strictness also c varies with depth, but errors from this cause are small when compared to those arising in ascertaining its value from the tables.

Prob. 137. Explain why formula $(137)_2$ cannot be used for the above example when the slope i is 0.01.

Art. 138. The Surface Curve

In the case of steady uniform flow, in the channel where the bed has a constant grade, the slope of the water surface is parallel to that of the bed, and the longitudinal profile of the water surface is a straight line. In steady non-uniform flow, however, the slope of the water surface continually varies, and the longitudinal profile is a curve whose nature is now to be investigated. As in the last article, the width of the channel will be taken as constant, its cross-section will be regarded as rectangular, and it will be assumed that the stream is wide compared to its depth, so that the wetted perimeter may be taken as equal to the width and the hydraulic radius equal to the mean depth (Art. 112). These assumptions are closely fulfilled in many canals and rivers.

The last formula of the preceding article is rigidly exact if the sections a_1 and a_2 are consecutive, so that l becomes δl and $d_2 - d_1$ becomes δd. Making these changes,

$$\frac{\delta d}{\delta l} = \frac{i - q^2/c^2b^2d^3}{1 - q^2/gb^2d^3} \qquad (138)_1$$

in which d is the depth of the water at the place considered. This is the general differential equation of the surface curve, l being measured parallel to the bed BB, and d upward, while the angle whose tangent is the derivative $\delta d/\delta l$ is also measured from BB.

To discuss this curve, let CC be the water surface if the slope were uniform, and let D be the depth of the water in the wide

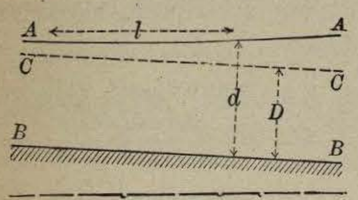

Fig. 138a.

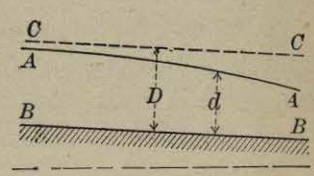

Fig. 138b.

rectangular channel. The slope s of the water surface is here equal to the slope i of the bed of the channel, and from the Chezy formula (113),

$$q = av = cbD\sqrt{ri} = cbD\sqrt{Di}$$

This value of q, inserted in the differential equation of the surface curve, reduces it to the form,

$$\frac{\delta d}{\delta l} = i\,\frac{1 - (D/d)^3}{1 - \frac{c^2 i}{g}(D/d)^3} \qquad (138)_2$$

in which d and l are the only variables, the former being the ordinate and the latter the abscissa, measured parallel to the bed BB, of any point of the surface curve. The derivative $\delta d/\delta l$ is the tangent of the angle which the tangent at any point of the surface curve makes with the bed BB or the surface CC.

First, suppose that D is less than d, as in Fig. 138*a*, where AA is the surface curve under the non-uniform flow, and CC is the line which the surface would take in case of uniform flow. The numerator of $(138)_2$ is then positive, and the denominator is also positive, since i is very small. Hence δd is positive, and it increases with d in the direction of the flow; going up-stream it decreases with d, and the surface curve becomes tangent to CC when $d = D$. This form of curve is that usually produced above a dam; it is called the "backwater curve," and will be discussed in detail in Art. 140.

Second, let d be less than D, as in Fig. 138*b*. The numerator is then negative and the denominator positive; δd is accordingly negative and AA is concave to the bed BB, whereas in the former case it was convex. This form of surface curve is produced when a sudden fall occurs in the stream below the point considered; it is called the "drop-down curve" and is discussed in Art. 141.

Formula $(138)_1$ may also be put into another form by substituting for q its value bdv, where v is the mean velocity in the cross-section whose depth is d. It thus becomes

$$\frac{\delta d}{\delta l} = \frac{g}{c^2} \cdot \frac{v^2 - c^2 di}{v^2 - gd} \qquad (138)_3$$

and by its discussion the same conclusions are derived as before. When v is equal to $c\sqrt{di}$, the inclination $\delta d/\delta l$ becomes zero, and the slope of the water surface is parallel to the bed of the stream. When v is less than $c\sqrt{di}$, the numerator is negative, and if the denominator is also negative, the case of Fig. 138*a* results. When v is greater than $c\sqrt{di}$ and the denominator is negative, the case of Fig. 138*b* obtains. When v equals $\sqrt{gd}$, the value of $\delta d/\delta l$ is infinity and the water surface stands normal to the bed of the stream; this remarkable case can actually occur in two ways, and they will be discussed in Art. 139.

Prob. 138. Let the velocity of the stream be 20 feet per second, the value of c be 80, and the slope be 1 on 2000. Compute values of $\delta d/\delta l$ for depth of 12.2, 12.3, 12.4, 12.5, and 12.6 feet; then draw the surface curve.

Art. 139. The Jump and the Bore

A very curious phenomenon which sometimes occurs in shallow channels is that of the so-called "jump," as shown in Fig. 139*a*. This happens when the denominator in $(138)_3$ is zero; then $\delta d/\delta l$ is infinite, and the water surface stands normal to the bed. Placing that denominator equal to zero, there is found $v^2 = gd$. Now

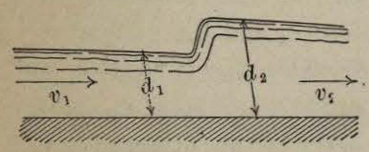

Fig. 139*a*.

by further consideration it will appear that the varying denominator in passing through zero changes its sign. Above the jump where the depth is d_1 the velocity is slightly greater than $\sqrt{gd_1}$, and below it is less than $\sqrt{gd_2}$. The conditions for the occurrence of the jump are that an obstruction should be in the stream below, that the slope i should not be small, and that the velocity v_1 should be greater than $\sqrt{gd_1}$. To find the necessary slope, the algebraic conditions are

$$v_1 = c\sqrt{d_1 i} \quad \text{and} \quad v_1 > \sqrt{gd_1} \quad \text{whence} \quad i > g/c^2$$

and accordingly the jump cannot occur when i is less than g/c^2. For an unplaned planked trough c may be taken at about 100; hence the slope for this must be equal to or greater than 0.00322.

To determine the height of the jump, let $d_2 - d_1$ be represented by j. It is then to be observed that the lost velocity-head is $(v_1^2 - v_2^2)/2g$, and that this is lost in two ways, first by the impact due to the expansion of section (Art. 76), and second by the uplifting of the whole quantity of water through the height

$\frac{1}{2}(d_2 - d_1)$, loss in friction between d_1 and d_2 being neglected. Hence

$$\frac{v_1^2 - v_2^2}{2g} = \frac{(v_1 - v_2)^2}{2g} + \frac{j}{2}$$

Inserting in this the value of v_2, found from the relation $v_2(d_1 + j) = v_1 d_1$, and solving for j, gives

$$j = -d_1 + 2\sqrt{d_1 \frac{v_1^2}{2g}} \qquad (139)$$

The following is a comparison of heights of the jump computed by this formula and the observed values in four experiments made by Bidone, the depths being in feet:

Depth d_1	Velocity v_1	Observed j	Computed j
0.149	4.59	0.274	0.290
0.154	4.47	0.267	0.283
0.208	5.59	0.305	0.428
0.246	6.28	0.493	0.531

The agreement is very fair, the computed values being all slightly greater than the observed, which should be the case, because the reasoning omits the frictional resistances between the points where d_1 and d_2 are measured. Experiments made at Lehigh University, under velocities ranging from 2.2 to 6.2 feet per second, show also a good agreement between computed and observed value.* The depths in these experiments were less than in those of Bidone, but higher relative jumps were obtained. For instance, for v_1 = 4.33 feet per second and d_1 = 0.039 feet, the observed value of j was 0.166 feet, whereas the value computed from the above formula is 0.173 feet; here the jump is more than four times the depth d_1, while it is usually less than twice d_1 in the above records from Bidone.

Another remarkable phenomenon is that of the so-called "bore," where a tidal wave moves up a river with a vertical front. It is also seen when a large body of water moves down a cañon after a heavy rainfall, or when a reservoir bursts and allows a large discharge to suddenly escape down a narrow valley. In the great flood of 1889 at Johnstown, Pa., such a vertical wall of water,

* Engineering News, 1895, vol. 34, p. 28.

variously estimated at from 10 to 30 feet in height, was seen to move down the valley, carrying on its front brush and logs mingled with spray and foam.* In 41 minutes it traveled a distance of 13 miles down the descent of 380 feet. The velocity was hence about 28 feet per second.

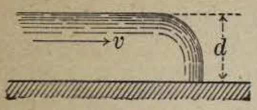

Fig. 139*b*.

Fig. 139*b* shows the form of surface curve for this case, and by reference to $(138)_3$ it is seen that $\delta d/\delta l$ must be negative and that it has the value ∞ at the vertical front. The conditions for the occurrence of the bore then are

$$v = \sqrt{gd} \quad \text{and} \quad v > c\sqrt{di} \quad \text{whence } i < g/c^2$$

For the Johnstown flood, taking v as 28 feet per second, the value of d found from this equation is 24 feet; it was probably greater than this in the upper part of the valley and less in the lower part. Since the value of i is about 1/180, it follows that c must have been less than 76. The conditions here established show that the flood bore will occur when the velocity becomes equal to $\sqrt{gd}$, provided c is less than $\sqrt{g/i}$. It appears, therefore, that roughness of surface is an essential condition for the formation of the bore in a steep valley.

The bore can also occur in a canal with horizontal bed when a lock breaks above an empty level reach, provided v becomes equal to $\sqrt{gd}$. No case of this kind appears to be on record, and there seems to be no way of ascertaining whether the actual velocity will reach the limit $\sqrt{gd}$. If the bore occurs and the depth of the vertical wall be d_2, its distance from a point where the depth is d_1 is found from $(139)_2$ by inserting in it the value of g corresponding to the critical velocity v. Thus may be shown that for c = 80 and $d_1 = \frac{1}{2}d_1$ the length l is $275d_1$.

The tidal bore, which occurs in many large rivers when the tide flows in at their mouths, obeys similar laws. Here the slope i may be taken as zero, while c is probably very large, so that roughness of surface is not an essential condition. The great bore at Hangchow, China, which occurs twice a year, is said to travel up the river at a rate of from 10 to 13 miles per hour, the height of the vertical front being

* Transactions American Society of Civil Engineers, 1889, vol. 21, p. 564.

from 10 to 20 feet.* From $v = \sqrt{gh}$, the velocity corresponding to a depth of 10 feet is 12.6 miles per hour, while that corresponding to a depth of 20 feet is 17 miles per hour, so that the statements have a fair agreement with the theoretical law. This investigation indicates that the velocity of the tidal bore depends mainly upon the depth of the tidal wave above the river surface, but it may be noted that other discussions† regard the depth of the river itself as an element of importance, and Art. 191 considers this with respect to common waves.

Prob. 139. When the height of the jump is three times the depth d_1, show that the velocity v_1 must be $2\sqrt{2gd_1}$. Also show that $0.414d_1$ is the minimum height of a jump.

Art. 140. The Backwater Curve

When a dam is built across a channel the water surface is raised for a long distance up-stream. This is a fruitful source of contention, and accordingly many attempts have been made to discuss it theoretically, in order to be able to compute the probable increase in depth at various distances back from a proposed dam. None of these can be said to have been successful except for the simple case where the slope of the bed of the channel is constant and its cross-section such that the width may be regarded as uniform and the hydraulic radius be taken as equal to the depth. These conditions are closely fulfilled for some streams, and an approximate solution may be made by the formula $(137)_2$. It is desirable, however, to obtain an exact equation of the surface curve.

For this purpose take the differential equation of the surface curve given in $(138)_2$, and let the independent variable d/D be represented by x. Then it may be put into the more convenient form

$$\frac{\delta l}{\delta x} = \frac{D}{i}\left(1 + \frac{1 - c^2 i/g}{x^3 - 1}\right) \qquad (140)_1$$

in which l is the abscissa and Dx the ordinate of any point of the curve. The general integral of this is

$$l = \frac{Dx}{i} - D\left(\frac{1}{i} - \frac{c^2}{g}\right)\left(\frac{1}{6}\log_e \frac{x^2 + x + 1}{(x-1)^2} - \frac{1}{\sqrt{3}}\,\text{arc cot}\,\frac{2x+1}{\sqrt{3}}\right) + C$$

* Skidmore's China, the Long-lived Empire (New York), 1900, p. 294.
† G. H. Darwin, The Tides, p. 65; Century Magazine, vol. 34, p. 903.