CHAPTER 9

FLOW IN CONDUITS AND CANALS

ART. 112. DEFINITIONS

From the earliest times water has been conveyed from place to place in artificial channels, such as troughs, aqueducts, ditches, and canals, there being no head to cause the flow except that due to the slope. The Roman aqueducts were usually rectangular channels about $2\frac{1}{2}$ feet wide and 5 feet deep, lined with cement, sometimes running underground and sometimes supported on arches. The word "conduit" will be used as a general term for a channel of any shape lined with timber, mortar, or masonry, and will also include large metal pipes, troughs, and sewers. Conduits may be either open, as in the case of troughs, or closed, as in sewers and most aqueducts. Ditches and canals are conduits in earth without artificial lining. Most of the principles relating to conduits and canals apply also to streams, and the word "channel" will be used as applicable to all cases.

The wetted perimeter of the cross-section of a channel is that part of its boundary which is in contact with the water. Thus, if a circular sewer of diameter d be half full of water, the wetted perimeter is $\frac{1}{2}\pi d$. In this chapter the letter p will designate the wetted perimeter.

The hydraulic radius of a water cross-section is its area divided by its wetted perimeter, and the letter r will be used to designate it. If a is the area of the cross-section, the hydraulic radius of that section is found by

$$r = a/p$$

The letter r is of frequent occurrence in formulas for the flow in channels; it is a linear quantity which is always expressed in the same unit as p, and hence its numerical value is different in

different systems of measures. It is frequently called the hydraulic depth or hydraulic mean depth, because for a shallow section its value is but little less than the mean depth of the water. Thus, in Fig. 112, if b be the breadth on the water surface, the mean depth is a/b, and the hydraulic radius is a/p; and these are nearly equal, since the length of p is but slightly larger than that of b.

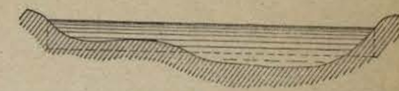

Fig. 112.

The hydraulic radius of a circular cross-section filled with water is one-fourth of the diameter; thus

$$r = a/p = \tfrac{1}{4}\pi d^2/\pi d = \tfrac{1}{4}d$$

The same value is also applicable to a circular section half filled with water, since then both area and wetted perimeter are one-half their former values.

The slope of the water surface in the longitudinal section, designated by the letter s, is the ratio of the fall h to the length l in which that fall occurs, or

$$s = h/l$$

The slope is hence expressed as an abstract number, which is independent of the system of measures employed. To determine its value with precision h must be obtained by referring the water level at each end of the line to a bench-mark by the help of a hook gage or other accurate means, the benches being connected by level lines run with care. The distance l is not measured horizontally but along the inclined channel, and it should be of considerable length in order that the relative error in h may not be large. If $s = o$ there is no slope and no flow; but when there is even the smallest slope the force of gravity furnishes a component acting down the inclined surface, and motion ensues. The velocity of flow evidently increases with the slope.

The flow in a channel is said to be steady when the same quantity of water per second passes through each cross-section. If an empty channel be filled by admitting water at its upper end, the flow is at first non-steady or variable, for more water passes through one of the upper sections per second than is delivered at the lower end. But after sufficient time has elapsed the flow becomes steady; when this occurs the mean velocities in different sections are inversely as their areas (Art. 31).

Uniform flow is that particular case of steady flow where all the water cross-sections are equal, and the slope of the water surface is parallel to that of the bed of the channel. If the sections vary, the flow is said to be non-uniform, although the condition of steady flow is still fulfilled. In this chapter only the case of uniform flow will be discussed.

The velocities of different filaments in a channel are not equal, as those near the wetted perimeter move slower than the central ones, owing to the retarding influence of friction. The mean of all the velocities of all the filaments in a cross-section is called the mean velocity v. Thus if v', v'', etc., be velocities of different filaments,

$$v = \frac{v' + v'' + \text{etc.}}{n}$$

in which n is the number of filaments. Let a be the area of the cross-section and let each filament have the small cross-section of area a'; then $n = a/a'$, and hence,

$$av = a'(v' + v'' + \text{etc.})$$

But the second member is the discharge q; that is, the quantity of water passing the given cross-section in one second. Therefore the mean velocity may be also determined by the relation

$$v = q/a$$

The filaments which are here considered are in part imaginary, for experiments show that there is a constant sinuous motion of particles from one side of the channel to the other. The best definition for mean velocity hence is, that it is a velocity which multiplied by the area of the cross-section gives the discharge, or $v = q/a$.

Prob. 112. Compute the hydraulic radius of a rectangular trough whose width is 5.6 feet and depth 2.8 feet.

Art. 113. Formula for Mean Velocity

When all the wetted cross-sections of a channel are equal, and the water is neither rising nor falling, having attained the condition of steady flow, the flow is said to be uniform. This is the case in a conduit or canal of constant size and slope whose supply does not vary. The same quantity of water per second then passes each cross-section, and consequently the mean velocity in each section is the same. This uniformity of flow is due to the resistances along the interior surface of the channel, for were it perfectly smooth the force of gravity would cause the velocity to be accelerated. The entire energy of the water due to the fall h is hence expended in overcoming resistances caused by surface roughness. A part overcomes friction along the surface, but most of it is expended in eddies of the water, whereby impact results and heat is generated. A complete theoretic analysis of this complex case has not been perfected, but if the velocity be not small, the discussion given for pipes in Art. 90 applies equally well to channels.

Let W be the weight of water passing any cross-section in one second, F the force of friction per square unit along the surface, p the wetted perimeter, and h the fall in the length l. The potential energy of the fall is Wh. The total resisting friction is Fpl, and the energy consumed per second is $Fplv$, if v be the velocity. Accordingly $Fplv$ equals Wh. But the value of W is wav, if w is the weight of a cubic foot of water and a the area of the cross-section in square feet. Therefore $Fpl = wah$, and since a/p is the hydraulic radius r, and h/l is the slope s, this reduces to $F = wrs$, which is an approximate expression for the resisting force of friction on one square unit of the surface of the channel. In order to establish a formula for the mean velocity the value of F must be expressed in terms of v, and this can only be done by studying the results of experiments. These indicate that F is approximately proportional to the square of the mean velocity. Therefore if c is a constant, the mean velocity is

$$v = c\sqrt{rs} \qquad (113)$$

which is the formula first advocated by Chezy in 1775. This is really an empirical expression, since the relation between F and v is derived from experiments. The coefficient c varies with the roughness of the bed and with other circumstances.

Another method of establishing Chezy's formula for channels is to consider that when a pipe on a uniform slope is not under pressure, the hydraulic gradient coincides with the water surface. Then formula (90) may be used by replacing h'' by h and d by its value $4r$. Accordingly

$$h = \tfrac{1}{4} f \frac{l}{r} \frac{v^2}{2g} \quad \text{or} \quad v = \sqrt{8g/f}\,\sqrt{rs}$$

in which the quantity $\sqrt{8g/f}$ is the Chezy coefficient.

This coefficient c is different in different systems of measures since it depends upon g. For the English system it is found that c usually lies between 30 and 160, and that its value varies with the hydraulic radius and the slope, as well as with the roughness of the surface. To determine the value of c for a particular case the quantities v, r, and s are measured, and then c is computed. To find r and s linear measurements and leveling are required. To determine v the flow must be gaged either in a measuring vessel or by an orifice or weir, or, if the channel be large, by floats or other indirect methods described in the next chapter, and then the mean velocity v is computed from $v = q/a$. It being a matter of great importance to establish a satisfactory formula for mean velocity, thousands of such gagings have been made, and from the records of these the values of the coefficients given in the tables in the following articles have been deduced.

Prob. 113. Compute the value of c for a circular masonry conduit 6 feet in diameter which delivers 65 cubic feet per second when running half full, its slope or grade being 1.5 feet in 1000 feet.

Art. 114. Circular Conduits, Full or Half Full

When a circular conduit of diameter d runs either full or half full of water, the hydraulic radius is $\frac{1}{4}d$, and the Chezy formula for mean velocity is

$$v = c\sqrt{rs} = c \cdot \tfrac{1}{2}\sqrt{ds}$$

The velocity can then be computed when c is known, and for this purpose Table 114 gives Hamilton Smith's values of c for pipes and conduits having quite smooth interior surfaces and no sharp bends.* The discharge per second then is

$$q = av = c \cdot \tfrac{1}{2} a\sqrt{ds}$$

in which a is either the area of the circular cross-section or one-half that section, as the case may be.

Table 114. Coefficients c for Circular Conduits

Coeff. 17
Cast I. pipe

Diameter in Feet	Velocity in Feet per Second						
	1	2	3	4	6	10	15
1.	96	104	109	112	116	121	124
1.5	103	111	116	119	123	129	132
2.	109	116	121	124	129	134	138
2.5	113	120	125	128	133	139	143
3.	117	124	128	132	136	143	147
3.5	120	127	131	135	139	146	151
4.	123	130	134	137	142	150	155
5.	128	134	139	142	147	155	
6.	132	138	142	145	150		
7.	135	141	145	149	153		
8.	137	143	148	151			

To use Table 114 a tentative method must be employed since c depends upon the velocity of flow. For this purpose there may be taken roughly

$$\text{mean Chezy coefficient } c = 125$$

and then v may be computed for the given diameter and slope; a new value of c is then taken from the table and a new v computed; and thus, after two or three trials, the probable mean velocity of flow is obtained. The value of the diameter d must be expressed in feet.

For example, let it be required to find the velocity and discharge of a semicircular conduit of 6 feet diameter when laid on a grade of 0.1 feet in 100 feet. First,

$$v = 125 \times \tfrac{1}{2}\sqrt{6 \times 0.001} = 4.8 \text{ feet per second.}$$

* Hydraulics (London and New York, 1886), p. 271.

For this velocity the table gives 147 for c; hence

$$v = 147 \times \tfrac{1}{2}\sqrt{0.006} = 5.7 \text{ feet per second.}$$

Again, from the table c = 150, and

$$v = 150 \times \tfrac{1}{2}\sqrt{0.006} = 5.8 \text{ feet per second.}$$

This shows that 150 is a little too large; for c = 149.5, v is found to be 5.79 feet per second, which is the final result. The discharge per second now is

$$q = 0.7854 \times \tfrac{1}{2} \times 36 \times 5.79 = 81.9 \text{ cubic feet,}$$

which is the probable flow under the given conditions.

To find the diameter of a circular conduit to discharge a given quantity under a given slope, the area a is to be expressed in terms of d in the above equation, which is then to be solved for d; thus,

$$d = \left(\frac{8q}{\pi c\sqrt{s}}\right)^{\frac{2}{5}} \qquad d = \left(\frac{16q}{\pi c\sqrt{s}}\right)^{\frac{2}{5}}$$

the first being for a conduit running full and the second for one running half full. Here c may at first be taken as 125; then d is computed, the approximate velocity found from $v = q/\frac{1}{4}\pi d^2$, and with this value of v a value of c is selected from the table, and the computation for d is repeated. This process may be continued until the corresponding values of c and v are found to be in close agreement.

As an example of the determination of diameter let it be required to find d when $q = 81.9$ cubic feet per second, $s = 0.001$, and the conduit runs full. For c = 125 the formula gives $d = 4.9$ feet, whence $v = 4.37$ feet per second. From the table c may be now taken as 142, and repeating the computation $d = 4.64$ feet, whence $v = 4.84$ feet per second, which requires no further change in the value of c. As the tabular coefficients are based upon quite smooth interior surfaces, such as occur only in new, clean, iron pipes, or with fine cement finish, it might be well to build the conduit 5 feet or 60 inches in diameter. It is seen from the previous example that a semicircular conduit of 6 feet diameter carries the same amount of water as is here carried by one of 4.64 feet diameter which runs entirely full.

Circular conduits running full of water are long pipes and all the formulas and methods of Arts. 94 and 95 can be applied also to their discussion. From Art. 113 it is seen that

$$c = \sqrt{8g/f} \quad \text{or} \quad c = 16.04/\sqrt{f}$$

in which f is to be taken from Table 90*a*. Values of c computed in this manner will not generally agree closely with the coefficients of Smith, partly because the values of f are given only to three decimal places, and partly because Table 90*a* for pipes was constructed from experiments on smoother surfaces than those of conduits. An agreement within 5 per cent in mean velocities deduced by different methods is all that can generally be expected in conduit computations, and if the actual discharge agrees as closely as this with the computed discharge, the designer can be considered a fortunate man.

All of the laws deduced in the last chapter regarding the relation between diameter and discharge, relative discharging capacity, etc., hence apply equally well to circular conduits which run either full or half full. If the conduit be full, it matters not whether it be laid truly to grade or whether it be under pressure, since in either case the slope s is the total fall h divided by the total length. Usually, however, the word "conduit" implies a uniform slope for considerable distances, and in this case the hydraulic gradient coincides with the surface of the flowing water.

Prob. 114. Find the diameter of a circular conduit to deliver when running full 16 500 000 gallons per day, its slope being 0.00016.

Art. 115. Circular Conduits, Partly Full

Let a circular conduit with the slope s be partly full of water, its cross-section being a and hydraulic radius r. Then the mean velocity and the discharge are given by

$$v = c\sqrt{rs} \qquad q = ca\sqrt{rs}$$

The mean velocity is hence proportional to $\sqrt{r}$ and the discharge to $a\sqrt{r}$, provided that c be a constant. Since, however, c varies slightly with r, this law of proportionality is only approximate.

$p = \pi \frac{d}{2} + 2$ (the arc of which the sine is $\frac{d' - \frac{d}{2}}{\frac{d}{2}}$)

$a = p\frac{d}{4} + 2\left[\frac{S(d' - \frac{d}{2})}{2}\right]$ $s^2 = (d - d')d'$

When a circular conduit of diameter d runs either full or half full, its hydraulic radius is $\frac{1}{4}d$ (Art. 112). If it is filled to the depth d' (Fig. 115), the wetted perimeter is

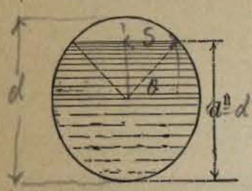

Fig. 115.

$$p = \tfrac{1}{2}\pi d + d \operatorname{arc} \sin \frac{2d' - d}{d}$$

and the sectional area of the water surface is

$$a = \tfrac{1}{4}dp + (d' - \tfrac{1}{2}d)\sqrt{d'(d - d')}$$

From these p and a can be computed, and then r is found by dividing a by p. Table 115 gives values of p, a, and r for a circle of diameter unity for different depths of water. To find from it the hydraulic radius for any other circle it is only necessary to multiply the tabular values of r by the given diameter d. The table shows that the greatest value of the hydraulic radius occurs when $d' = 0.81d$, and that it is but little less when $d' = 0.8d$. In the fifth and sixth columns of the table are given values of $\sqrt{r}$ and $a\sqrt{r}$ for different depths in the circle of diameter unity; these are approximately proportional to the velocity and discharge which occur in a circle of any size. The table shows that the greatest velocity occurs when the depth of the water is about eight-

TABLE 115. CROSS-SECTIONS OF CIRCULAR CONDUITS

Depth d		Wetted Perimeter p	Sectional Area a	Hydraulic Radius r	Velocity $\sqrt{r}$	Discharge $a\sqrt{r}$
Full	1.0	3.142	0.7854	0.25	0.5	0.393
	0.95	2.691	0.7708	0.286	0.535	0.413
	0.9	2.498	0.7445	0.298	0.546	0.406
	0.81	2.240	0.6815	0.3043	0.552	0.376
	0.8	2.214	0.6735	0.3042	0.552	0.372
	0.7	1.983	0.5874	0.296	0.544	0.320
	0.6	1.772	0.4920	0.278	0.527	0.259
Half Full	0.5	1.571	0.3927	0.25	0.5	0.196
	0.4	1.369	0.2934	0.214	0.463	0.136
	0.3	1.159	0.1981	0.171	0.414	0.0820
	0.2	0.927	0.1118	0.121	0.348	0.0389
	0.1	0.643	0.0408	0.0635	0.252	0.0103
Empty	0.0	0.0	0.0	0.0	0.0	0.0

tenths of the diameter, and that the greatest discharge occurs when the depth is about $0.95d$, or $\frac{19}{20}$ of the diameter.

By the help of Table 115 the velocity and discharge may be computed when c is known, but it is not possible on account of the lack of experimental knowledge to state precise values of c for different values of r in circles of different sizes. However, it is known that an increase in r increases c, and that a decrease in r decreases c. The following experiments of Darcy and Bazin show the extent of this variation for a semicircular conduit of 4.1 feet diameter, and they also teach that the nature of the interior surface greatly influences the values of c. Two conduits were built, each with a slope $s = 0.0015$ and $d = 4.1$ feet. One was lined with neat cement, and the other with a mortar made of cement with one-third fine sand. The flow was allowed to occur with different depths, and the discharges per second were gaged by means of orifices; this enabled the velocities to be computed, and from these the values of the coefficient c were found. The following are a portion of the results obtained, d' denoting the depth of water in the conduit, r the hydraulic radius, v the mean velocity, and all linear demensions being in English feet:

For cement lining				For mortar lining			
d'	r	v	c	d'	r	v	c
2.05	1.029	6.06	154	2.04	1.022	5.55	142
1.61	0.867	5.29	147	1.69	0.900	4.94	135
1.03	0.605	4.16	138	1.09	0.635	3.87	125
0.59	0.366	3.02	129	0.61	0.379	2.87	120

It is here seen that c decreases quite uniformly with r, and that the velocities for the mortar lining are 8 or 10 per cent less than those for the neat cement lining.

The value of the coefficient c for these experiments may be roughly expressed for English measures by

$$c = c_1 - 16\left(\tfrac{1}{2}d - d'\right)$$

in which c_1 is the coefficient for the conduit when running half full. How this will apply to different diameters and velocities is not known; when d' is greater than $0.8d$, it will probably prove incorrect. In practice, however, computations on the flow in partly filled conduits are of rare occurrence.

Prob. 115. Compute the hydraulic radius for a circular conduit of 4.1 feet diameter, when it is three-fourths filled with water, and also the mean what relation of x to y will give max x × y ans. x = y

velocity when it is lined with neat cement and laid on a grade of 0.15 feet per 100 feet.

Omit ART. 116. RECTANGULAR CONDUITS

But Outlet

x + y = Constant

In designing an open rectangular trough or conduit to carry water there is a certain ratio of breadth to depth which is most advantageous, because thereby either the discharge is the greatest or the least amount of material is required for its construction. Let b be the breadth and d the depth of the water section, then the area a is bd and the wetted perimeter p is $b + 2d$. If the area a is given, it may be required to find the relation between b and d so that the discharge may be a maximum. If the wetted perimeter p is given, the relation between b and d to produce the same result may be demanded. It is now to be shown that in both cases the breadth is double the depth, or $b = 2d$. This is called the most advantageous proportion for an open rectangular conduit, since there is the least head lost in friction when the velocity and discharge are the greatest possible.

Let r be the hydraulic radius of the cross-section, or

$$r = \frac{a}{p} = \frac{bd}{b + 2d}$$

then, from the Chezy formula (113), the expressions for the velocity and discharge are

$$v = c\sqrt{s}\sqrt{\frac{bd}{b+2d}} \qquad q = c\sqrt{s}\sqrt{\frac{b^3d^3}{b+2d}}$$

In these expressions it is required to find the relation between b and d, which renders both v and q a maximum.

Let the wetted perimeter p be given, as might be the case when a definite amount of lumber is assigned for the construction of a trough; then $b + 2d = p$, or $d = \frac{1}{2}(p - b)$, and

$$v = c\sqrt{s}\sqrt{\frac{b(p-b)}{2p}} \qquad q = c\sqrt{s}\sqrt{\frac{b^3(p-b)^3}{8p}}$$

in which p is a constant. Differentiating either of these expressions with respect to b and equating the derivative to zero, there

is found $b = \frac{1}{2}p$, and hence $d = \frac{1}{4}p$. Accordingly $b = 2d$, or the breadth is double the depth.

Again, let the area a be given, as might be the case when a definite amount of rock excavation is to be made; then $bd = a$, or $d = a/b$, and

$$v = c\sqrt{s}\sqrt{\frac{ab}{b^2 + 2a}} \qquad q = c\sqrt{s}\sqrt{\frac{a^3b}{b^2 + 2a}}$$

in which a is constant. By equating the first derivative to zero, there is found $b^2 = 2a$, and hence $d^2 = \frac{1}{2}a$. Accordingly $b = 2d$, or the breadth is double the depth, as before.

It is seen in the above cases that the maximum of both v and q occur when r is a maximum, or when $r = \frac{1}{2}d$. It is indeed a general rule that r should be a maximum in order to secure the least loss of head in friction. The circle has a greater hydraulic radius than any other figure of equal area.

In these investigations c has been regarded as constant, although strictly it varies somewhat for different ratios of b to d. The rule deduced is, however, sufficiently close for all practical purposes. It frequently happens that it is not desirable to adopt the relation $b = 2d$, either because the water pressure on the sides of the conduit becomes too great or because it is advisable to limit the velocity so as to avoid scouring the bed of the channel. Whenever these considerations are more important than that of securing the greatest discharge, the depth is made less than one-half the breadth.

The velocity and discharge through a rectangular conduit are expressed by the general equations

$$v = c\sqrt{rs} \qquad q = av = ca\sqrt{rs}$$

and are computed without difficulty for any given case when the coefficient c is known. To determine this, however, is not easy, for it is only from recorded experiments that its value can be ascertained. When the depth of the water in the conduit is one-half of its width, thus giving the most advantageous section, the values of c for smooth interior surfaces may be estimated by the use of Table 114 for circular conduits, although c is probably smaller for rectangles than for circles of equal area. When the depth of the water is less or greater than $\frac{1}{2}d$, it must be remembered that c increases with r. The value of c also is subject to slight variations with the slope s, and to great variations with the degree of roughness of the surface.

Table 116, derived from Smith's discussion of the experiments of Darcy and Bazin, gives values of c for a number of wooden and masonry conduits of rectangular sections, all of which were laid on the grade of 0.49 per cent or $s = 0.0049$. The great influence

Table 116. Coefficients c for Rectangular Conduits

Unplaned Plank $b = 3.93$ Feet		Unplaned Plank $b = 6.53$ Feet		Neat Cement $b = 5.94$ Feet		Brick $b = 6.27$ Feet	
d	c	d	c	d	c	d	c
0.27	99	0.20	89	0.18	116	0.20	89
.41	108	.30	101	.28	125	.31	98
.67	112	.46	109	.43	132	.49	104
.89	114	.60	113	.56	135	.57	105
1.00	114	.72	116	.63	136	.65	105
1.19	116	.78	116	.69	136	.71	106
1.29	117	.89	118	.80	137	.85	107
1.46	118	.94	120	.91	138	.97	110

of roughness of surface in diminishing the coefficient is here plainly seen. For masonry conduits with hammer-dressed surfaces c may be as low as 60 or 50, particularly when covered with moss and slime.

Prob. 116. Find the size of a trough, whose width is double its depth, which will deliver 125 cubic feet per minute when its slope is 0.002, taking the coefficient c as 100.

Art. 117. Trapezoidal Sections

Ditches and conduits are often built with a bottom nearly flat and with side slopes, thus forming a trapezoidal section. The side slope is fixed by the nature of the soil or by other circumstances, the grade is given, and it may be then required to

ascertain the relation between the bottom width and the depth of water, in order that the section shall be the most advantageous. This can be done by the same reasoning as used for the rectangle in the last article, but it may be well to employ a different method, and thus be able to consider the subject in a new light.

Let the trapezoidal channel have the bottom width b, the depth d, and let θ be the angle made by the side slopes with the horizontal. Let it be required to discharge q cubic units of water per second. Now $q = \text{c}a\sqrt{rs}$, and the most advantageous proportions may be said to be those that will render the cross-section a a minimum for a given discharge, for thus the least excavation, will be required. From Fig. 117,

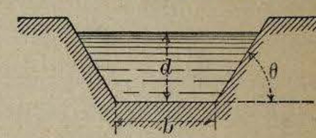

Fig. 117.

$$a = d(b + d\cot\theta) \qquad p = b + 2d/\sin\theta$$

and from these the value of r may be expressed in terms of a, d, and θ; inserting this in the formula for q, it reduces to

$$\frac{\text{c}^2sa^3}{d} - \frac{q^2a}{d^2} = q^2\left(\frac{2}{\sin\theta} - \cot\theta\right)$$

in which the second member is a constant. Obtaining the first derivative of a with respect to d, and then replacing q^2 by its value c^2a^2rs, there results

$$d = 2q^2/\text{c}^2a^2s \qquad d = 2r$$

which is the relation that renders the area a a minimum; that is, the advantageous depth is double the hydraulic radius. Now since $a/p = r$, it is easy to show that

$$b + 2d\cot\theta = 2d/\sin\theta$$

or, the top width of the water surface should equal the sum of the two side slopes in order to give the most advantageous section. Since c has been regarded constant, the conclusion is not a rigorous one, although it may safely be followed in practice. As in all cases of an algebraic minimum, a considerable variation in the value of the ratio d/b may occur without materially effecting the value of the area a. In many cases it is not possible to