CHAPTER 3

THEORETICAL HYDRAULICS

Art. 21. Laws of Falling Bodies

Theoretical Hydraulics treats of the flow of water when unretarded by opposing forces of friction. In a perfectly smooth inclined trough water would flow with accelerated velocity and be governed by the same laws as those for a body sliding down a frictionless inclined plane. Such a flow is, however, never found in practice, for all surfaces over which water moves are more or less rough. Friction retards the motions caused by gravity so that the theoretic velocities deduced in this chapter constitute limits which cannot be exceeded by the actual velocities. Many of the laws governing the free fall of bodies in a vacuum are similar to those of both theoretical and practical hydraulics, and hence they will here be briefly discussed.

A body at rest above the surface of the earth immediately falls when its support is removed. When the fall occurs in a vacuum, its velocity at the end of one second is g feet, the mean value of g being 32.16 feet per second per second, and at the end of t seconds its velocity is $V = gt$. The distance passed through in the time t is the product of the mean velocity $\frac{1}{2}V$ by the number of seconds, or $h = \frac{1}{2}gt^2$. Eliminating t from these two equations gives

$$V = \sqrt{2gh} \qquad \text{or} \qquad h = V^2/2g \qquad (21)_1$$

which show that the velocity varies with the square root of the height and that the height varies as the square of the velocity.

When a falling body has the initial velocity u at the beginning of the time t, its velocity at the end of this time is $V = u + gt$ and the distance passed over in that time is $h = ut + \frac{1}{2}gt^2$. Eliminating t from these equations gives

$$V = \sqrt{2gh + u^2} \qquad \text{or} \qquad h = (V^2 - u^2)/2g \qquad (21)_2$$

as the relations between V and h for this case. These formulas are also true whatever be the direction of the initial velocity u.

When a body of weight W is at the height h above a given horizontal plane, its potential energy with respect to this plane is Wh. When it falls from rest to this plane, the potential energy is changed into the kinetic energy $WV^2/2g$ if no work has been done against frictional resistance, and therefore $V^2 = 2gh$. When it has a velocity u in any direction at the height h above the plane, its energy there is partly potential and partly kinetic, the sum of these being $Wh + W \cdot u^2/2g$; on reaching the plane it has the kinetic energy $WV^2/2g$. Placing these equal, there results $V^2 = 2gh + u^2$, as found above by another method. In general, reasoning from the standpoint of energy is more satisfactory than that in which the element of time is employed.

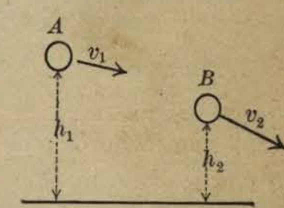

Fig. 21.

The general case of a body moving toward the earth is represented in Fig. 21. When the body is at A, it is at a height h_1 above a certain horizontal plane and has the velocity v_1. When it has arrived at B, its height above the plane is h_2 and its velocity is v_2. In the first position the sum of its potential and kinetic energy with respect to the given horizontal plane is

$$W\left(h_1 + \frac{v_1^2}{2g}\right)$$

and in the second position the sum of these energies is

$$W\left(h_2 + \frac{v_2^2}{2g}\right)$$

If no energy has been lost between the two positions, these two expressions are equal, and hence

$$h_1 + \frac{v_1^2}{2g} = h_2 + \frac{v_2^2}{2g} \qquad (21)_3$$

This equation is the simplest form of Bernouilli's theorem (Art. 31). It contains two heights and two velocities, and when three of these quantities are given, the fourth can be found; thus, if v_1, h_1, and h_2 are given, the value of v_2 is

$$v_2 = \sqrt{2g(h_1 - h_2) + v_1^2}$$

where $h_1 - h_2$ is the vertical height of A above B. With proper changes in notation this expression reduces to $(21)_2$, which is for the case where the horizontal plane passes through B, and to $(21)_1$, which is the case where there is no initial velocity.

Prob. 21. A body enters a room through the ceiling with a velocity of 47 feet per second, and in a direction making an angle of 17° with the vertical. If the height of the room is 16 feet, find the velocity of the body as it strikes the floor, resistances of the air being neglected.

Art. 22. Velocity of Flow from Orifices

When an orifice is opened, either in the base or side of a vessel containing water, the water flows out with a velocity which is greater for high heads than for low heads. The theoretic velocity of flow is given by the theorem established by Torricelli in 1644:

> The theoretic velocity of flow from the orifice is the same as that acquired by a body after having fallen from rest in a vacuum through a height equal to the head of water on the orifice.

One proof of this theorem is by experience. When a vessel is arranged, as in the first diagram of Fig. 22, so that a jet of water from an orifice is directed vertically upward, it is known that it never attains to the height of the level of the water in the vessel, although under favorable conditions it nearly reaches that level. It may hence be inferred that the jet would actually rise to that height were it not for the resistance of the air and the friction of the edges of the orifice. Now, since the velocity required to raise a body vertically to a certain height is the same as that acquired by it in falling from rest through that height, it is re-

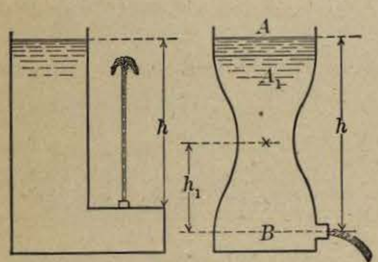

Fig. 22.

garded as established that the velocity at the orifice is that stated in the theorem.

The following proof rests on the law of conservation of energy. Let, as in the second diagram of Fig. 22, the water surface in a vessel be at A and let the flow through the orifice occur for a very short interval of time during which the water surface descends to A_1. Let W be the weight of water between the planes A and A_1, which is evidently the same as that which flows from the orifice during the short time considered. Let W_1 be the weight of water between the planes A_1 and B, and h_1 the height of its center of gravity above the orifice. Let h be the height of A above the orifice, and δh the small distance between A and A_1. At the beginning of the flow the water in the vessel has the potential energy $W_1H_1 + W(h - \frac{1}{2}\delta h)$ with respect to B. V being the velocity at the orifice, the same water at the end of the short interval of time has the energy $W_1h_1 + W \cdot V^2/2g$. By the law of conservation these are equal if no energy has been expended in overcoming frictional resistances; thus $h - \frac{1}{2}\delta h = V^2/2g$. Here δh is very small if the area A is large compared with the area of the orifice, and thus $V^2 = 2gh$, which is the same as for a body falling from rest through the height h. Or $h - \frac{1}{2}\delta h$ may be regarded as an average head corresponding to an average velocity V, so that in general $V^2/2g$ is equal to the average head on the orifice.

For any orifice, therefore, whether its plane is horizontal, vertical, or inclined, provided the head h is so large that it has practically the same value for all parts of the orifice, the relation between V and h is

$$V = \sqrt{2gh} \qquad \text{or} \qquad h = V^2/2g \qquad (22)_1$$

the first of which gives the theoretic velocity of flow due to a given head, while the second gives the theoretic head that will produce a given velocity. The term "velocity-head" will generally be used to designate the expression $V^2/2g$, this being the height to which the jet would rise if it were directed vertically upward and there were no frictional resistances. Using for g the mean value 32.16 feet per second per second (Art. 7), these formulas become

$$V = 8.020\sqrt{h} \qquad h = 0.01555\,V^2 \qquad (22)_2$$

in which h must be in feet and V in feet per second. The following table gives values of the velocity V corresponding to a given head h and also values of the velocity-head h corresponding to a given velocity V. It is seen that small heads produce high theoretic velocities. The relation between h and V is the same as that between the ordinate and abscissa of the common parabola when the origin is at the vertex. It may also be noted that the discussion here given applies not only to water but to any liquid; thus $V^2 = 2gh$ is theoretically true for alcohol and mercury as well as for water.

TABLE 22. VELOCITIES AND VELOCITY-HEADS

$V = \sqrt{2gh} = 8.020\sqrt{h}$				$h = V^2/2g = 0.01555\, V^2$			
Head in Feet	Velocity in Feet per Second	Head in Feet	Velocity in Feet per Second	Velocity in Feet per Second	Head in Feet	Velocity in Feet per Second	Head in Feet
0.1	2.537	1	8.02	1	0.016	10	1.56
0.2	3.587	2	11.33	2	0.062	20	6.22
0.3	4.393	3	13.89	3	0.140	30	13.99
0.4	5.072	4	16.04	4	0.249	40	24.88
0.5	5.671	5	17.93	5	0.389	50	38.87
0.6	6.212	6	19.64	6	0.560	60	55.97
0.7	6.710	7	21.22	7	0.762	70	76.19
0.8	7.171	8	22.68	8	0.995	80	99.51
0.9	7.608	9	24.06	9	1.260	90	125.95
1.0	8.020	10	25.36	10	1.555	100	155.50

When a Pitot tube (Art. 41) is placed with its mouth in the plane of the horizontal orifice in Fig. 22, and at the contracted section of the jet (Art. 45), it will be found that the water in it stands practically at the level of the water in the vessel.* In this manner the frictional resistance of the air is eliminated, and a valuable experimental demonstration of the theorem which connects the velocity and the velocity-head is obtained.

Prob. 22. Find from Table 22 the velocity due to a head of 0.085 feet, and the velocity-head corresponding to a velocity of 65.5 feet per second.

* Engineering Record, Feb. 15, 1902.

ART. 23. FLOW UNDER PRESSURE

The level of water in the reservoir and the orifice of outflow have been thus far regarded as subjected to no pressure, or at least only to the pressure of the atmosphere which acts upon both with the same mean force of 14.7 pounds per square inch, since the head h is rarely or never so great that a sensible variation in atmospheric pressure can be detected between the orifice and the water level. But the upper level of the water may be subject to the pressure of steam or to the pressure due to a heavy weight or to a piston. The orifice may also be under a pressure greater or less than that of the atmosphere. It is required to determine the velocity of flow from the orifice under these conditions.

First, suppose that the surface of the water in the vessel or reservoir is subjected to the uniform pressure of p_0 pounds per square unit above the atmospheric pressure, while the pressure at the orifice is the same as that of the atmosphere. Let h be the depth of water on the orifice. The velocity of flow V is greater than $\sqrt{2gh}$ on account of the pressure p_0, and it is evidently the same as that from a column of water whose height is such as to produce the same pressure at the orifice. If w is the weight of a cubic unit of water, the unit-pressure at the orifice due to the head is wh, and the total unit-pressure at the depth of the orifice is $p = wh + p_0$, and from formula $(11)_1$ the head of water which would produce this total unit-pressure is

$$\frac{p}{w} = h + \frac{p_0}{w}$$

Accordingly the theoretic velocity of flow from the orifice is

$$V = \sqrt{2g(h + p_0/w)}$$

or, if h_0 denote the head corresponding to the pressure p_0,

$$V = \sqrt{2g(h + h_0)}$$

The general formula $(22)_1$ thus applies to any small orifice if H be the head corresponding to the static pressure at the orifice.

Secondly, suppose that the surface of the water in the vessel is subjected to the unit-pressure p_0, while the orifice is under the external unit-pressure p_1. Let h be the head of actual water on the orifice, h_0 the head of water which will produce the pressure p_0, and h_1 the head which will produce p_1. The theoretic velocity of flow at the orifice is then the same as if the orifice were under a head $h + h_0 - h_1$, or

$$V = \sqrt{2g(h+h_0-h_1)} \qquad (23)_1$$

in which the values of h_0 and h_1 are

$$h_0 = p_0/w \quad \text{and} \quad h_1 = p_1/w$$

Usually p_0 and p_1 are given in pounds per square inch, while h_0 and h_1 are required in feet; then (Art. 11)

$$h_0 = 2.304 p_0 \qquad h_1 = 2.304 p_1$$

The values of p_0 and p_1 may be absolute pressures, or merely pressures above the atmosphere. In the latter case p_1 may sometimes be negative, as in the discharge of water into a condenser.

As an illustration of these principles let the cylindrical tank in Fig. 23 be 2 feet in diameter, and upon the surface of the water let there be a tightly fitting piston which with the load W weighs 3000 pounds. At the depth 8 feet below the water level are three small orifices: one at A, upon which there is an exterior head of water of 3 feet; one not shown in the figure, which discharges directly into the atmosphere; and one at C, where the discharge is into a vessel in which the air pressure is only 10 pounds per square inch. It is required to determine the velocity of efflux from each orifice. The head h_0 corresponding to the pressure on the upper water surface is

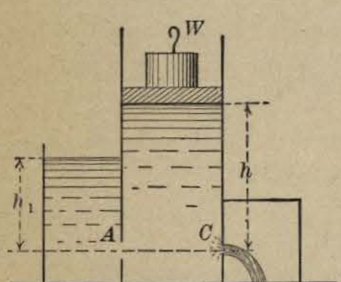

Fig. 23.

$$h_0 = \frac{p_0}{w} = \frac{3000}{3.142 \times 62.5} = 15.28 \text{ feet}$$

The head h_1 is 3 feet for the first orifice, 0 for the second, and $-2.304(14.7-10) = -10.83$ feet for the third. The three theoretic velocities of outflow then are:

$$V = 8.02\sqrt{8+15.28-3} = 36.1 \text{ feet per second,}$$
$$V = 8.02\sqrt{8+15.28+0} = 38.7 \text{ feet per second,}$$
$$V = 8.02\sqrt{8+15.28+10.83} = 46.8 \text{ feet per second.}$$

In the case of discharge from an orifice under water, as at A in Fig. 23, the value of $h - h_1$ is the same wherever the orifice be placed below the lower level, and hence the velocity depends upon the difference of level of the two water surfaces, and not upon the depth of the orifice.

The velocity of flow of oil or mercury under pressure is to be determined in the same manner as water by finding the heads which will produce the given pressure. Thus in the preceding numerical example, if the liquid is mercury whose weight per cubic foot is 850 pounds the head of mercury corresponding to the pressure of the piston is

$$h_0 = \frac{3000}{3.142 \times 850} = 1.12 \text{ feet,}$$

and, accordingly, for discharge into the atmosphere at the depth $h = 8$ feet the velocity is

$$V = 8.02\sqrt{8+1.12} = 24.2 \text{ feet per second,}$$

while for water the velocity was 38.7 feet per second. The general formula $(22)_1$ is applicable to all cases of the flow of liquids from a small orifice if for h its value p/w be substituted where p is the resultant unit-pressure at the depth of the orifice and w the weight of a cubic unit of the liquid. Thus for any liquid

$$V = \sqrt{2gp/w} \qquad (23)_2$$

is the theoretic velocity of flow from the orifice. Accordingly for the same unit-pressure p the velocities are inversely proportional to the square roots of the densities of the liquids.

Prob. 23. What is the theoretic velocity of flow from a small orifice in a boiler 1 foot below the water level when the steam-gage reads 60 pounds per square inch? What is the theoretic velocity when the gage reads 0?

Art. 24. Influence of Velocity of Approach

Thus far in the determination of the theoretic velocity and discharge from an orifice, the head upon it has been regarded as constant. But if the cross-section of the vessel is not large, the head can only be kept constant by an inflow of water, and this will modify the previous formulas. In this case the water approaches the orifice with an initial velocity. Let a be the area of the orifice and A the area of the horizontal cross-section of the vessel. Let V be the velocity of flow through a and v be the vertical velocity of inflow through A. Let W be the weight of water flowing from the orifice in one second; then an equal weight must enter at A in one second in order to maintain a constant head h. The kinetic energy of the outflowing water is $W \cdot V^2/2g$, and this is equal, if there be no loss of energy, to the potential energy Wh of the inflowing water plus its kinetic energy $W \cdot v^2/2g$,

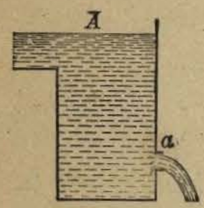

Fig. 24a.

or

$$W\frac{V^2}{2g} = Wh + W\frac{v^2}{2g}$$

Now since the same quantity of water Q passes through the two areas in one second, $Q = aV = Av$, whence $v = V \cdot a/A$. Inserting this value of v in the equation of energy, there is found

$$V = \sqrt{\frac{2gh}{1-(a/A)^2}} \qquad (24)_1$$

which is always greater than the value $\sqrt{2gh}$.

The influence of the velocity of approach on the velocity of flow at the orifice can now be ascertained by assigning values to the ratio a/A. Thus, if $a = A$, the velocity V must be infinite in order that the water may fill the entire section of the vessel and orifice. Further,

for	$a = \frac{2}{3}A$	$V = 1.342\sqrt{2gh}$
for	$a = \frac{1}{2}A$	$V = 1.154\sqrt{2gh}$
for	$a = \frac{1}{3}A$	$V = 1.061\sqrt{2gh}$
for	$a = \frac{1}{5}A$	$V = 1.021\sqrt{2gh}$
for	$a = \frac{1}{10}A$	$V = 1.005\sqrt{2gh}$

It is here seen that the common formula $(22)_1$ is in error 2.1 percent when $a = \frac{1}{5}A$, if the head be maintained constant by a uniform vertical inflow at the water surface, and 0.5 percent when $a = \frac{1}{10}A$. Practically, if the area of the orifice be less than one-twentieth of the cross-section of the vessel, the error in using the formula $V = \sqrt{2gh}$ is too small to be noticed, even in the most precise experiments, and fortunately most orifices are smaller in relative size than this.

A more common case is that where the reservoir is of large horizontal and small vertical cross-section, and where the water approaches the orifice with velocity in a horizontal direction, as in Fig. 24*b*. Here let A be the area of the vertical cross-section of the trough or pipe, a the area of the orifice, and h the head on its center. Then if h be large compared with the depth of the

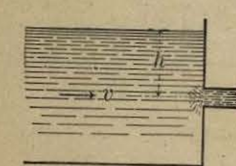

Fig. 24*b*.

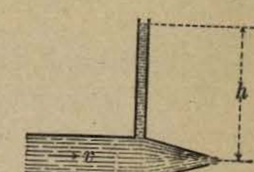

Fig. 24*c*.

orifice, exactly the same reasoning applies as before, and the theoretic velocity at the orifice is given by the above formula $(25)_1$. The same is also true for the case shown in Fig. 24*c*, where water is forced through a hose with the velocity v and issues from a nozzle with the velocity V, the head h being that due to the pressure at the entrance of the nozzle.

The "effective head" on an orifice is the head that will produce the theoretic velocity V. If H is this effective head, then $H = V^2/2g$, and from the first equation of this article

$$H = h + \frac{v^2}{2g} \qquad (24)_2$$

The effective head on an orifice is, therefore, the sum of the pressure and velocity heads which exist behind it. Another expression for the effective head can be obtained from $(24)_1$, or

$$H = \frac{h}{1-(a/A)^2}$$

When H has been found from either of these formulas, the theoretic velocity and discharge are given by

$$V = \sqrt{2gH} \quad \text{and} \quad Q = aV = a\sqrt{2gH}$$

for all instances where h is sufficiently large so that its value is sensibly constant for all parts of the orifice. But if this is not the case, the value of Q is to be found by the methods of Arts. 47 and 48.

Prob. 24. In Fig. 24*c* let the head h be 50 feet, the diameter of the nozzle $1\frac{1}{2}$ inches, and the diameter of the hose 3 inches. Compute the effective head H, and also the discharge Q in cubic feet per second.

Art. 25. The Path of a Jet

When a jet of water issues from a small orifice in the vertical side of a vessel or reservoir, its direction at first is horizontal, but the force of gravity immediately causes the jet to move in a curve which will be shown to be the common parabola. Let x be the abscissa and y the ordinate of any point of the curve, measured from the orifice as an origin, as seen in Fig. 25*a*. The effect of the impulse at the orifice is to cause the space x to be described uniformly in a certain time t, or, if v be the velocity of flow, $x = vt$. The effect of the force of gravity is to cause the space y to be described in accordance with the laws of falling bodies (Art. 21), or $y = \frac{1}{2}gt^2$. Eliminating t from these two equations, and replacing v^2 by its theoretic value $2gh$, gives

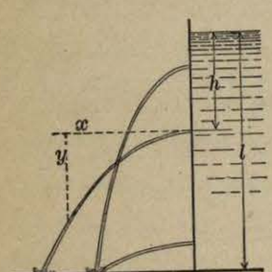

Fig. 25*a*.

$$y = gx^2/2v^2 = x^2/4h$$

which is the equation of a parabola whose axis is vertical and whose vertex is at the orifice.

The horizontal range of the jet for any given ordinate y is found from the equation $x^2 = 4hy$. If the height of the vessel be l, the horizontal range on the plane of the base is

$$x = 2\sqrt{h(l-h)}$$

This value is 0 when $h = 0$ and also when $h = l$, and it is a maximum when $h = \frac{1}{2}l$. Hence the greatest range is from an orifice at the mid-height of the vessel.

A more general case is that where the side of the vessel is inclined to the vertical at the angle θ, as in Fig. 25*b*. Here the jet at first issues perpendicularly to the side with a velocity v, having the theoretic value $\sqrt{2gh}$, and under the action of the impulsive force a particle of water would describe the distance AB in a certain time t with the uniform velocity v. But in that same time the force of gravity causes it to descend through the distance BC. Now let x be the horizontal abscissa and y the vertical ordinate of the point C measured from the origin A. Then $AB = x\sec\theta$, and $BC = x\tan\theta - y$. Hence

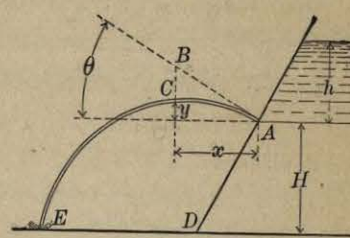

Fig. 25*b*.

$$x\sec\theta = vt \qquad x\tan\theta - y = \tfrac{1}{2}gt^2$$

The elimination of t from these expressions gives, after replacing v^2 by its value $2gh$,

$$y = x\tan\theta - x^2\sec^2\theta/4h \tag{25}$$

which is also the equation of a common parabola.

To find the horizontal range in the level of the orifice take $y = 0$ in the last equation; then

$$x = 4h\tan\theta/\sec^2\theta = 2h\sin 2\theta$$

This is 0 when $\theta = 0°$ or $\theta = 90°$; it is a maximum and equal to $2h$ when $\theta = 45°$. To find the highest point of the jet the first derivative of y with reference to x is to be equated to zero in order to obtain the maximum ordinate, and there results

$$x = h\sin 2\theta \qquad y = h\sin^2\theta$$

which are the coordinates of the highest point with respect to the origin A. In these if $\theta = 90°$, x is 0 and y is h; that is, if a jet be directed vertically upward, it will, theoretically, rise to the height of the water level in the reservoir.

As a numerical example let a vessel whose height is 16 feet stand upon a horizontal plane DE, Fig. 25b, the side of the vessel being inclined to the vertical at the angle $\theta = 30°$. Let a jet issue from a small orifice at A under a head of 10 feet. The jet rises to its maximum height, $y = \frac{1}{4} \times 10 = 2.5$ feet, at the distance $x = \frac{1}{2}\sqrt{3} \times 10 = 8.66$ feet from A. At $x = 17.32$ feet the jet crosses the horizontal plane through the orifice. To locate the point where it strikes the plane DE, the value of y is made -6 feet; then, from the equation of the curve, x is found to be 24.6 feet, whence the distance DE is 21.2 feet.

In practice the above equations are modified by the frictional resistance of the edges of the orifice which renders v less than the theoretic value $\sqrt{2gh}$, and also by the resistance of the air. They are, indeed, extreme limits which may be approached but not reached by equations that take these resistances into account.

Prob. 25. A jet issues from a vessel under a head of 6 feet, one side of the vessel being inclined to the vertical at an angle of 45° and its depth being 10 feet. Find the maximum height to which the jet rises, the point where it strikes the horizontal plane of the base, and its theoretic velocity as it strikes that plane.

Art. 26. The Energy of a Jet

Let a jet or stream of water have the velocity v, and let W be the weight of water per second passing any given cross-section. The kinetic energy of this moving water is the same as that stored up by a body of weight W falling freely under the action of gravity through a height h and thereby acquiring the velocity v. Thus, if K represents kinetic energy per second,

$$K = Wh = W \cdot v^2/2g \tag{26$_1$}$$

Now if a be the area of the cross-section and w the weight of a cubic unit of water, W is the weight of a prism of water of length v and cross-section a, or $W = wav$, whence

$$K = wav^3/2g \tag{26$_2$}$$

and accordingly the energy which a jet can yield in one second is directly proportional to its cross-section and to the cube of its velocity. The term "power" is often used to express energy

per second, and when K is in foot-pounds per second, the horse-power that a jet can yield is ascertained by dividing K by 550. Hence the horse-powers of jets of the same cross-section vary as the cubes of their velocities. For example, if the velocity of a jet be doubled, the cross-section remaining the same, the horse-power is made eight times as great. The term "energy of a jet" is often used in hydraulics for brevity, but it always means energy per second of the jet; that is, the power of the jet.

The expressions just deduced give the theoretic energy of the jet, that is, the maximum work which can be obtained from it in one second, but this, in practice, can never be fully utilized. The actual work realized when a jet strikes a moving surface, like the vane of a water-motor, depends upon a number of circumstances which will be explained in a later chapter, and it is the constant aim of inventors so to arrange the conditions that the work realized may be as near the theoretic energy as possible. The "efficiency" of an apparatus for utilizing the power of moving water is the ratio of the work k actually utilized to the theoretic energy, or the efficiency e is

$$e = k/K \tag{26$_3$}$$

The greatest possible value of e is unity, but this can never be attained, owing to the imperfections of the apparatus and the frictional resistances. Values greater than 0.90 have, however, been obtained; that is, 90 percent or more of the theoretic power of the water has been utilized in some of the best forms of hydraulic motors.

For example, let water issue from a pipe 2 inches in diameter with a velocity of 10 feet per second. The cross-section in square feet is 3.142/144, and the kinetic energy of the jet in foot-pounds per second is

$$K = 0.01555 \times 62.5 \times 0.0218 \times 10^3 = 21.2$$

which is 0.0385 horse-power. If the velocity is 100 feet per second, the theoretic horse-power will be 38.5; if this jet operates a motor yielding 27.7 effective horse-powers, the efficiency of the apparatus is $27.7/38.5 = 0.72$, or 72 percent of the theoretic energy is utilized.