The moment of resistance may be found by taking moments about the center of compression in the concrete, thus,

$$M = f_s pbd^2 \left(1 - \tfrac{3}{8} k\right) \qquad (68)$$

or by taking moments about the center of pull in the steel,

$$M = \tfrac{2}{3} f_c kbd^2 \left(1 - \tfrac{3}{8} k\right) \qquad (69)$$

Eliminating k from these equations by substituting its value from equation (63), and also substituting the value of p from equation (66), we have

$$M = \tfrac{2}{3} f_s bd^2 \frac{1}{\frac{f_s}{f_c}\left(1 + \frac{f_s}{nf_c}\right)} \left[1 - \frac{3}{8\left(1 + \frac{f_s}{nf_c}\right)}\right] \qquad (70)$$

or

$$M = \tfrac{2}{3} f_c bd^2 \frac{1}{1 + \frac{f_s}{nf_c}} \left[1 - \frac{3}{8\left(1 + \frac{f_s}{nf_c}\right)}\right] \qquad (71)$$

APPENDIX III

FORMULAS FOR REINFORCED CONCRETE CHIMNEY AND HOLLOW CIRCULAR BEAM DESIGN

The analysis which follows is based upon the several fundamental assumptions adopted in reinforced concrete beam design with the additional assumption that, since the concrete is usually thin as compared to the diameter of the chimney, no appreciable error is involved in assuming all material as concentrated on the mean circumference of the shell. An analysis for shear is also given together with an example of chimney design and review.

The principles involved in the demonstration of the thickness of steel and concrete are taken by permission from the analysis by Messrs. C. Percy Taylor, Charles Glenday, and Oscar Faber.*

The principal formulas given below are quoted in the text, where the general subject of concrete chimneys is discussed, and tables are presented there with the values of constants for use in design.

NOTATION

W = weight in pounds of the chimney above the section under consideration.
M = moment in inch pounds of the wind about that section.
P = total compression in concrete.
T = total tension in steel.
$n = \frac{E_s}{E_c}$ = ratio of modulus of elasticity of steel to that of concrete
f_c = maximum compression in concrete in pounds per square inch (measured at the mean circumference).
f_s = maximum tension in the steel in pounds per square inch.
D = mean diameter of shell in inches.
r = mean radius of shell in inches.
t = total thickness of shell in inches.
t_c = thickness in inches of concrete only.

* Engineering (London), Mar. 13, 1908.

t_s = thickness in inches of an imaginary steel shell of mean radius r. and having a cross-sectional area equivalent to the total area of reinforcing bars.

A_s = total cross-sectional area, in square inches, of reinforcing bars in the section under consideration.

k = ratio of distance of neutral axis, from mean circumference on compression side, to diameter D.

j, z, C_P and C_T = constants for any given value of k. (Tables 1 and 2, pp. 635 and 636.)

jD = distance between center of compression and centre of tension.

zD = distance from center of compression to center of force due to weight.

Referring to Fig. 243, if f_c is the maximum intensity of stress in the concrete at the mean circumference on the compression side, then the intensity of compression in the steel at that point is nf_c. Since f_s is the maximum intensity of stress in the steel at the mean circumference on the tension side, then the variation of the stress in the steel, across the section cd, is represented by the straight line ab which cuts the line cd at e, thus locating the neutral axis or the line of zero stress. Having assumed a constant value for the modulus of elasticity of the concrete in compression, it therefore follows that, at any point of a given section, the stress in either the concrete or the steel is directly proportional to the distance of that point from the neutral axis.

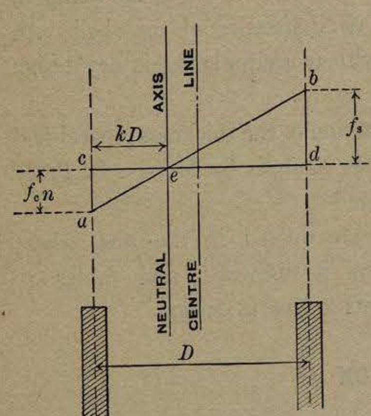

FIG. 243.—Resisting Forces in a Reinforced Chimney. (*See p. 766.*)

Calling kD the distance of the neutral axis from the mean circumference on compression side as shown in Fig. 243, we have by similar triangles

$$\frac{kD}{D} = \frac{nf_c}{f_s + nf_c}$$

whence

$$k = \frac{1}{1 + \dfrac{f_s}{nf_c}} \qquad (1)$$

By this formula the position of the neutral axis may be determined for any combinations of f_c, f_s, and n.

If now, as shown in Fig. 244, α represents half the angle subtended at the center by the portion in compression, we have

$$\cos \alpha = (1 - 2k)$$

from which, for any given value of z, $\cos \alpha$ becomes known as well as α and $\sin \alpha$. Thus having located the neutral axis for any given combinations of f_c, f_s and n and bearing in mind that the stress at any point of the shell is proportional to the distance of that point from the neutral axis, it is now possible to determine the total force on the compression side, the total force on the tension side, and also the location of the center of compression and the center of tension.

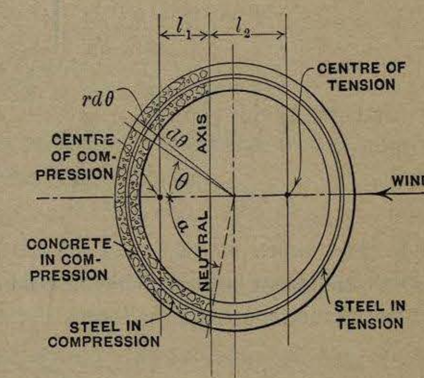

FIG. 244.—Distribution of Stresses in the Steel of a Reinforced Chimney. (*See p. 767.*)

Considering a small radial element subtending an angle $d\theta$, as shown in Fig. 244, we have in this element, since the length of an arc is its radius times the angle,

$$\text{area of concrete} = t_c r d\theta$$

$$\text{area of steel} = t_s r d\theta$$

The distance of the element from the neutral axis is $r(\cos \theta - \cos \alpha)$, while the distance from the neutral axis to the point of extreme stress f_c is $r(1 - \cos \alpha)$. Therefore the intensity of stress on this elemental area is

$$f_c \frac{r(\cos \theta - \cos \alpha)}{r(1 - \cos \alpha)} \text{ in the concrete}$$

and

$$f_c\, n \frac{r(\cos \theta - \cos \alpha)}{r(1 - \cos \alpha)} \text{ in the steel.}$$

Assuming these intensities at the mean circumference to represent the average for the entire element, we have the total force on the elemental area (concrete and steel)

$$d\,P = (t_c + n\,t_s)\,r\,d\,\theta\,\frac{f_c\,r\,(\cos\theta - \cos\alpha)}{r\,(1 - \cos\alpha)}$$

The total force P on the compression side of the section is therefore

$$P = (t_c + n\,t_s)\,2\int_0^\alpha \frac{f_c\,r\,(\cos\theta - \cos\alpha)}{(1 - \cos\alpha)}\,d\,\theta$$

Integrating this expression, gives

$$P = f_c\,r\,(t_c + n\,t_s)\,\frac{2}{(1 - \cos\alpha)}\,(\sin\alpha - \alpha\cos\alpha)$$

Since any given position of the neutral axis determines α, as shown above, this equation may take the form

$$P = C_P f_c r\,(t_c + n\,t_s) \qquad (2)$$

in which C_P is a constant for a given position of the neutral axis. (See Table 1, page 635.)

Having determined the magnitude of P, its location, with respect to the neutral axis, may best be found by taking its moment about that axis and dividing by P, thus giving the distance from the neutral axis to the center of compression l_1, as shown in Fig. 244.

As before, the compressive force on an elemental area is

$$d\,P = (t_c + n\,t_s)\,r\,d\,\theta\,\frac{f_c r\,(\cos\theta - \cos\alpha)}{r\,(1 - \cos\alpha)}$$

The distance of this force from the neutral axis being $r(\cos\theta - \cos\alpha)$, we have as its moment about that axis

$$d\,M_c = (t_c + n\,t_s)\,r\,d\,\theta\,\frac{f_c r^2\,(\cos\theta - \cos\alpha)^2}{r\,(1 - \cos\alpha)}$$

while the moment of the total compressive force P is

$$M_c = (t_c + n\,t_s)\,2\int_0^\alpha r\,\frac{f_c\,r\,(\cos\theta - \cos\alpha)^2}{(1 - \cos\alpha)}\,d\,\theta$$

$$= (t_c + n\,t_s)\,\frac{2\,f_c r^2}{(1 - \cos\alpha)}\left[\int_0^\alpha \cos^2\theta\,d\,\theta - 2\cos\alpha\int_0^\alpha \cos\theta\,d\,\theta + \cos^2\alpha\int_0^\alpha d\,\theta\right]$$

Integrating, we have

$$M_c = (t_c + n\,t_s)\,f_c r^2\,\frac{2}{(1 - \cos\alpha)}\left[(\alpha\cos^2\alpha. - \tfrac{3}{2}\sin\alpha\cos\alpha + \tfrac{1}{2}\alpha)\right]$$

Dividing M_c by P we have

$$l_1 = \frac{M_c}{P} = \frac{(\alpha\cos^2\alpha - \frac{3}{2}\sin\alpha\cos\alpha + \frac{1}{2}\alpha)}{(\sin\alpha - \alpha\cos\alpha)}\,r \qquad (3)$$

Following a similar method of procedure it is possible to determine the total tension and the location of the center of tension.

In accordance with our assumption that the concrete is to take no tensile stress it is evident that in considering the forces on the tension side of the section we are concerned merely with the steel. On the tension side a small element therefore has an area $= t_s\,r\,d\,\theta$

The intensity of stress on this element, being proportional to its distance from the neutral axis, is

$$f_s\,\frac{r\,(\cos\theta + \cos\alpha)}{r\,(1 + \cos\alpha)}$$

while the total tension on the small element is

$$d\,T = t_s\,r\,d\,\theta\,f_s\,\frac{(\cos\theta + \cos\alpha)}{(1 + \cos\alpha)}$$

The total force T on the tension side of the section is therefore

$$T = 2\int_0^{(\pi - \alpha)} t_s\,r\,f_s\,\frac{(\cos\theta + \cos\alpha)}{(1 + \cos\alpha)}\,d\,\theta$$

Integrating, we have

$$T = f_s\,r\,t_s\,\frac{2}{(1 + \cos\alpha)}\,(\sin\alpha + (\pi - \alpha)\cos\alpha)$$

Since, as before, any given position of the neutral axis determines α, this equation may take the form

$$T = C_T f_s\,r\,t_s \qquad (4)$$

in which C_T is a constant for a given position of the neutral axis (see Table 1, page 635). By a method similar to that used in considering the force on the compression side we may write the moment, about the neutral axis, of the force on a small element on the tension side as

$$d\,M_T = t_s\, r\, d\,\theta\, f_s \frac{r\,(\cos\theta + \cos\alpha)^2}{(1 + \cos\alpha)}$$

while the moment of the total tensile force T about this axis is

$$M_T = 2\int_0^{(\pi-\alpha)} t_s\, r\, f_s \frac{r\,(\cos\theta + \cos\alpha)^2}{(1 + \cos\alpha)}\, d\,\theta$$

Integrating, we have

$$M_T = t_s\, r^2 f_s \frac{2}{(1 + \cos\alpha)}\left[(\pi - \alpha)\cos^2\alpha + \tfrac{3}{2}\sin\alpha\cos\alpha + \tfrac{1}{2}(\pi - \alpha)\right]$$

Dividing M_T by T we have as the distance of the center of tension from the neutral axis

$$l_2 = \frac{\left((\pi - \alpha)\cos^2\alpha + \frac{3}{2}\sin\pi\cos\alpha + \frac{1}{2}(\pi - \alpha)\right)}{(\sin\alpha + (\pi - \alpha)\cos\alpha)}\, r \qquad (5)$$

From formulas (3) and (5) it is evident that the distance between the total force in compression and the total force in tension (i. e., $l_1 + l_2$) may, for any given position of the neutral axis, be expressed as a constant times the diameter D. Thus $l_1 + l_2 = jD$ as shown in Fig. 245. Likewise, as shown in Fig. 245, zD may represent the distance of the center of compression from the center of the chimney, z also being a constant for any given position of the neutral axis.

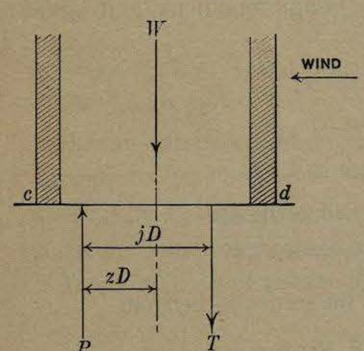

FIG. 245.—External and Internal Forces Acting upon a Chimney. (See p. 770.)

In a chimney the tensile and compressive stresses which we have been considering are produced by a combination of wind pressure and the weight of the chimney. Thus, on any horizontal section cd, as shown in Fig. 245, the forces external to that section are: the horizontal pressure of the wind, causing a moment M about the section, and a central vertical load W representing the weight of that portion of the chimney above the section under consideration. These forces are resisted, and held in equilibrium, by the forces P and T which represent the compressive and tensile stresses in the concrete and steel.

The system of forces as shown in Fig. 245 must be in equilibrium. Hence, taking moments about the force P, we may write

$$TjD = M - WzD$$

But

$$T = C_T f_s r t_s$$

Therefore

$$C_T f_s r t_s jD = M - WzD$$

Whence

$$r t_s = \frac{M - WzD}{C_T f_s jD}$$

The total area of steel $A_s = 2\pi r t_s$

Therefore

$$A_s = \frac{2\pi(M - WzD)}{C_T f_s jD} \qquad (6)$$

From Table I, page 635, it may be seen that the constant j changes but slightly for a considerable variation in the position of the neutral axis. Taking $\frac{2\pi}{j} = 8$ for all cases, equation (6) may be

$$A_s = \frac{8\,(M - WzD)}{C_T f_s D} \qquad (7)$$

While this formula is not exact, the error involved is inappreciable for almost any case so that formula (7) may always be used instead of formula (6).

Applying now the condition that the summation of all vertical forces must be zero, we have

$$P - T = W$$

Substituting values of P and T as previously found, the equation becomes

$$C_P f_c r\,(t_c + n t_s) - C_T f_s r t_s = W$$

Transposing and solving for t_c we obtain

$$t_c = \frac{W + (C_T f_s - C_P f_c n)\, r t_s}{C_P f_c r}$$

The total thickness of the shell is

$$t = t_c + t_s$$

whence

$$t = \frac{W + (C_T f_s - C_P f_c n)\, r t_s}{C_P f_c r} + t_s$$

For convenience in use, after having determined A_s by the formula given above, by substituting $r = \frac{D}{z}$ and $t_s = \frac{A_s}{\pi D}$, this formula for t may best be written

$$t = \frac{2W + (C_T f_s - C_P f_c n)\frac{A_s}{\pi}}{C_P f_c D} + \frac{A_s}{\pi D} \tag{8}$$

In view of the fact that formulas (6), (7) and (8) contain the constants z, j, C_T and C_P, which, as has been shown, are dependent for their value solely upon the location of the neutral axis, it is evident that, for any specific values of f_c, f_s, and n, which in turn will determine the position of the neutral axis, the expressions for A_s and t will admit of a further simplification. For given values of f_c, f_s and n, the necessary thickness of shell and area of reinforcement may be expressed merely in terms of the moment of the wind M, the weight W, and the mean diameter D. The expressions, as given, however, seem best adapted to general use, and when supplemented by the tables given on pages 635, 636, are rendered quite simple of solution for specific values.

In Table 2, page 636, is given values of k, the location of the neutral axis, for various combinations of f_c, f_s and n; while Table 1, page 635, gives the corresponding values of the constants C_P, C_T, z and j for various positions of the neutral axis.

Shear or Diagonal Tension. Having determined the necessary thickness of shell and vertical reinforcement, the size and spacing of the circular steel hoops must be considered. The external forces produce shear and diagonal tension which may be analyzed similarly to like stresses in rectangular beams, and the reinforcement necessary to resist the diagonal tension, which is a function of the vertical tension, may be determined. Usually this reinforcement is not so great as that which it is advisable to insert for the proper distribution of temperature stresses, but nevertheless it should be determined to be sure that it is sufficient in quantity.

The concrete should never be relied upon to carry any tension or vertical shear because the expansion from the heat may cause vertical cracks in the concrete. These need not be considered dangerous if sufficient horizontal reinforcement is provided any more than the vertical cracks in a brick or tile chimney. Considering the stresses due to vertical shear, it may be easily shown that at any horizontal section of a chimney the vertical shear per inch of height is the total horizontal shear on that section divided by the distance between centers of tension and compression, jD. With this as a

basis there may be developed a formula for practical use in determining the necessary area and spacing of horizontal steel hoops at any given section.

Thus let

h_l = height, in feet, of chimney above section under consideration.
F = effective wind pressure against chimney in pounds per square foot.
f_s = allowable tensile stress in pounds per square inch in steel hoops.
D = mean diameter of shell in inches.
P_0 = ratio of area of steel hoop to area of concrete.

At any horizontal section of a chimney the total shear on that section is equal to

$$\frac{D}{12} h_l F$$

while the maximum shear per inch of height is therefore

$$\frac{D}{12}\frac{h_l F}{jD}$$

Having seen that for all positions of the neutral axis j remains practically constant, and giving j an average value of, say, 0.783, the expression for the maximum vertical shear per inch of height becomes

$$0.106\ hF$$

while the shear or diagonal tension in one foot of height is $12 \times 0.106\ h_l F$.

The area of steel in one foot of height of chimney will be $12\ bp_0$ and the stress the hoops in this height are capable of sustaining on their two sections is

$$2 \times 12\ tp_0 f_s$$

Equating these we have

$$12 \times .106\ h_l F = 2 \times 12\ tp_0 f_s$$

whence

$$p_0 = \frac{h_l F}{18.8 f_s t}$$

This ratio of steel is for shear or diagonal tension only. To provide for temperature stresses or rather to distribute the strains so as to prevent the localization of cracks an additional amount of horizontal steel is needed. This may be provided for arbitrarily by assuming 0.25% steel or rather

0.0025 for temperature stress in addition to the steel for shear. Expressing this as a formula for ratio of steel gives

$$p_0 = \frac{h_1 F}{18.8 f_s} + 0.0025 \qquad (9)$$

Small rods spaced 6 to 10 inches apart except in the upper part of the stack where the spacing may be greater are advised.

The spacing of hoops in many of the chimneys already built has been 18 inches to 36 inches, but as such chimneys have frequently cracked quite seriously, more recent designs have called for 8 or 9 inch spacing through the entire stack.

Design of Hollow Circular Beams. The analysis of a hollow circular reinforced concrete beam whose thickness, compared relatively with its diameter, is small, is similar in principle to that of a chimney. In this case the weight of the member acts in the same direction as the external forces, so that in formulas (7) and (8) W the weight in the axial direction, is zero. The forces of compression, P, and tension, T, are equal. The area of steel and the thickness of shell are therefore obtained from formulas (7) and (8), pages 771 and 772, by making $W = O$.

APPENDIX IV

METHOD OF COMBINING MECHANICAL ANALYSIS CURVES

In Chapter XI the method of forming mechanical analysis curves is discussed, and approximate rules are given for combining individual curves to form the curve of the mixture. More exact methods, which also illustrate the principles, are given in the following pages, taking up first simple cases and then the more complicated ones.

Case I. Curves which meet, but do not overlap. In Fig. 246 are shown three curves, No. 1, No. 2, and No. 3, representing ideal grades of sand and stone, which may be combined in such proportions that the curve of the mixture will be of the ideal form required. The problem requires the determination of the percentages of each of the three materials which when combined will form a mixture whose curve is nearly the ideal. In order to prove that the percentages found will produce the resultant curve, and also to illustrate the theory of the mixture, the resultant curve will be first plotted and described in a very elementary manner, and afterwards by the method of ratios which would be employed in practice.

Curve No. 3 represents a material all of whose particles will pass through a sieve having holes 2.00 inches diameter and all of whose particles will be retained on a sieve having holes 0.75 inch diameter. Stone represented by curve No. 2 lies between diameters 0.75 and 0.25 inch, while the material of curve No. 1 is all finer than 0.25 inch, that is, is all under ¼ inch. Curves No. 3_1 and No. 3_2 are referred to later.

The curve *OebA* is first plotted* as a parabola. Although the latest tests indicate that the best curve is a combination of an ellipse and a straight line,† the parabola will illustrate the principle of combination as well as any other, and so for this problem we may assume now that the required theoretical mix of materials lies in this parabolic curve. This is equivalent to saying that the desired theoretical mixture of materials is such, that at any ordinate

* CONSTRUCTION OF THE PARABOLA.

D = largest diameter of stone
d = any given diameter
P = per cent. of mixture smaller than any given diameter

The equation of the parabola is

$$d = \frac{P^2 D}{10000}$$

The parabola can be constructed in any of the numerous ways given in text-books, the writer finding it easiest to use a slide rule. Set D on the B scale of the rule opposite 100 on D scale, read any value of d on the B scale opposite any corresponding value of P on the D scale.

† "Laws of Proportioning Concrete," by William B. Fuller and Sanford E. Thompson, Transactions American Society of Civil Engineers.