DEAD LOADS AND EARTH PRESSURE

With open spandrels having columns or transverse walls, the dead loads act vertically upon the arch ring and can be more accurately found than with filled spandrels.

With spandrels filled with earth the dead load carried by the arch ring is that due to the weight of the roadway, of the filling, and of the arch ring itself. The earth filling is usually assumed to act vertically, in which case the forces acting on the arch are easily computed. For arches in which the ratio of rise to span is small, such an assumption is sufficiently correct. A common assumption for weight of earth fill where the actual value is unknown is 100 pounds per cubic foot.

Since the pressure produced by the earth filling against the extradosal surface of the ring is really inclined, being nearly vertical near the crown and considerably inclined near the springings, it is sometimes advisable in an arch of large rise to take account of the horizontal component of the pressure near the springings. The earth pressure acting against an inclined plane may be found either algebraically or graphically. The algebraic solution is given under the subject of retaining walls, page 665, and in the example of arch design the inclined pressure is taken into account for illustration, although it is really unnecessary in the case selected. (See p. 576.)

OUTLINE OF DISCUSSION ON ARCH DESIGN

The method of designing an arch by the elastic theory is illustrated by the example on pages 574 to 582. The steps to be taken are there stated in full.

In the following pages the reactions at the supports, which in an arch are not simple vertical forces, and the relations between the outer loads and the internal stresses, are first treated briefly so as to understand the theory in a general way. Next (p. 553), the working formulas are given for finding the thrust, shear and bending moment at the crown, and at intermediate points in the arch ring. From these, the force polygon and the line of pressure, which is an equilibrium polygon drawn for a pole distance equal to the horizontal thrust, may be drawn (p. 555). The method of determining the stresses due to temperature and rib shortening is given (p. 556). Since the lines of pressure do not ordinarily pass through the center line of the arch ring, the pressures on the various sections are eccentric, and the distribution of stress in an arch under different conditions is discussed at length, the same analyses applying also to any other member like a column, subjected to eccentric pressure (p. 558 to 574). Diagrams are presented to aid in the determinations. Following the example, the design of arch abutments is given (p. 583), and beyond this are general directions with reference to construction details. Several typical arches are illustrated (p. 589).

RELATION BETWEEN OUTER LOADS AND REACTIONS AT SUPPORTS

An arch differs from a beam in that under vertical loads the reactions at the supports of the arch are inclined, while for a beam the reactions are vertical. The loads acting on the arch, together with the reactions caused by the loads, constitute the entire system of forces acting, and for a complete analysis of the arch the relation between these forces should be determined. This relation is more simply deduced if for each reaction there are substituted its horizontal and vertical components.

For arches symmetrical about the center line of span the following analysis is applicable. For unsymmetrical arches, methods similar to those presented in the following pages are to be employed although the necessary formulas are too long to be given here.

NOTATION

H_1 and V_1 = horizontal and vertical components of the left reaction.
H_2 and V_2 = horizontal and vertical components of the right reaction.
M_1 and M_2 = moments at left and right supports respectively.
M = moment at any point on arch axis having coördinates x and y.
M_c, H_c, V_c = moment, thrust and shear at the crown.
M_L = moment at any point on left half of arch axis of all loads between the point and the crown.
M_R = moment at any point on right half of arch axis of all outer loads between the point and crown.
m = number of divisions into which the half length of arch axis is divided.
s = short length of arch axis.
I = moment of inertia of cross section about the gravity axis.
L = horizontal span of arch axis.
r = rise of arch.
E_c = modulus of elasticity of concrete.
n = ratio of moduli of elasticity of steel to concrete.
R = resultant force acting on any section of the arch ring.
N = thrust = normal component of resultant R.
V = shear = radial component of resultant R.

H = horizontal component of resultant R.
P = any concentrated load.
Δ_L = change in span length due to any cause, + for an increase, − for a decrease.
t = rise or fall in temperature of the arch ring from the mean in degrees Fahrenheit.
c = coefficient of linear expansion or contraction.
f = average unit compression in concrete of arch ring due to thrust.
ϕ = central angle subtended by the axis of the arch.
x,y = coördinates of any point on the axis of the arch ring.

Three-Hinged Arch. The use of the three-hinged arch is discussed on page 539. Since its analysis is simplest and at the same time illustrates important principles of arch design, it is considered first.

Referring to Fig. 157, it is seen that there are two unknown components

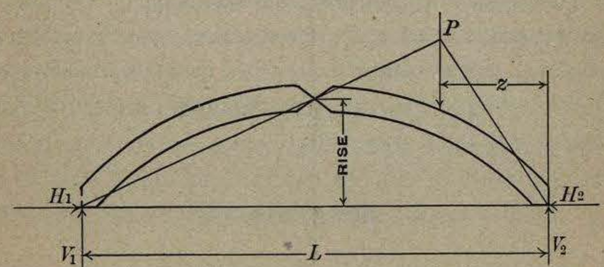

FIG. 157.—Arch with Three Hinges. *(See p. 546).*

of each reaction, making four unknown quantities, H_1, V_1, H_2, V_2, which require four equations to solve them. From statics we have the three equations of equilibrium.:

Algebraic sum of vertical components = zero.
Algebraic sum of horizontal components = zero.
Algebraic sum of moments of all forces about any point = zero.

We have here an additional equation from the fact that the bending moment at the crown hinge = o. Therefore the four components of the reactions can easily be found. Suppose there is only one load, P, on the span. Then

$$V_1 = \frac{Pz}{L} \text{ and } V_2 = \frac{P\ (L - z)}{L} \qquad (3)$$

Since, for equilibrium, the moment at the crown hinge must be o, the resultant reaction on the left must pass through the left hinge, or

$$V_1\left(\frac{L}{2}\right) - H_1 r = 0. \quad \text{Hence } H_1 = \frac{V_1 L}{2r} \qquad (4)$$

V_1, H_1 = components of left reaction. V_2, H_2 = components of right reaction. L = span. r = rise.

When all loads are vertical, or in any case when the loads are symmetrical about the center, $H_1 = H_2$.

When the loads are not symmetrical and also not vertical, H_2 can be easily found, after H_1 has been determined as above, from the relation that the algebraic sum of all the outer horizontal forces = o. In a three-hinged arch, then, the reactions having been found by means of simple statics as above described, the thrust, shear and bending moment on any section of the arch can be computed and sections designed.*

Two-Hinged Arch. Under the action of the loads on this arch there are produced two components of the reaction at each support, making in all four unknowns, H_1, V_1, H_2, V_2. From statics we have the three fundamental equations of equilibrium, as given above. We must find an additional equation from the theory of elasticity. This additional equation is obtained from the fact that the span does not change its length under the

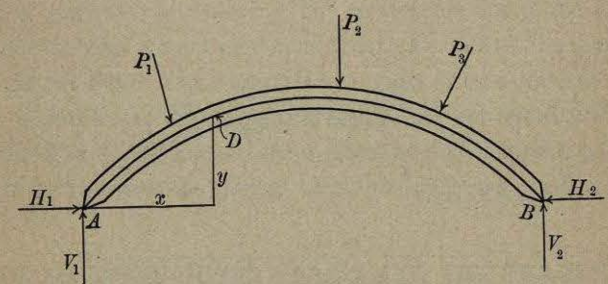

FIG. 158.—Two-Hinged Arch. *(See p. 547).*

action of the loads. From mechanics† we know that if the arch were fixed at B and free at A, the horizontal motion of A (the origin of coördinates) is given by $\Sigma My \frac{s}{EI}$, where Σ denotes the summation of the products of $My \frac{s}{EI}$ for each section of the arch. Now, since the arch is really prevented by the support from moving horizontally at point A, the above deformation can be placed equal to o, and we have then the fourth equation $\Sigma My \frac{s}{EI} = 0$, which, in addition to the three from statics, enables us to find the reactions H_1, V_1, H_2, V_2. As soon as the reactions are known, the thrust, shear and bending moment at any section of the arch can be found.

*Three Hinged Masonry Arches; Long Spans Especially Considered, by David A. Molitor, Transactions American Society of Civil Engineers, Vol. XL, p. 31.
†"Mechanics of Engineering", by Irving P. Church, 1908, p. 449.

In a similar manner the conditions of equilibrium can be obtained for an arch with only one hinge (at the crown).

"Fixed" or "Continuous" Arches. A method frequently followed with the hingeless arch is to consider the reactions at the ends in the same way as in hinged arches, but the simpler method is to take the forces at a section through the crown. However, in order to better understand the theory and the relation of the external to the internal forces, the arch reactions at the supports will be discussed first and afterward the analysis will consider the forces at the crown.

Let Fig. 159 represent a hingeless arch. The loads having been determined, there are at each support three unknown quantities, namely, the vertical and the horizontal components and the point of application of the reaction. Or, instead of saying that the point of application of the reaction

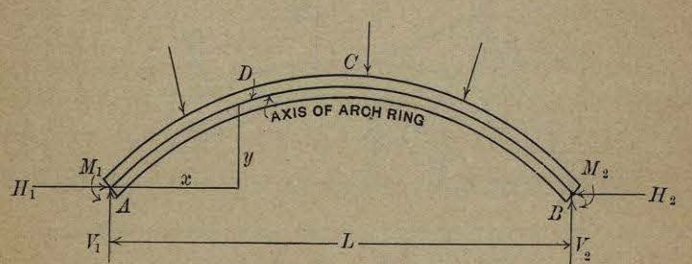

FIG. 159.—Continuous Arch. (*See p.* 548).

is unknown, we can say that there is a bending moment at each support, and that this moment, together with the horizontal and vertical components of the reaction, makes three unknown quantities at each support to be found. There are then six unknown quantities to be determined, namely, H_1, V_1, M_1, H_2, V_2, M_2.

Statics provides the three fundamental equations of equilibrium (see page 546), hence three additional equations must be determined from the theory of elasticity. These three additional equations are given from the three following conditions:

The change in span of the arch $= \Delta x = O$

The vertical deflection at A (the origin of coördinates) $= \Delta y = O$

The change in direction of the tangent at the arch axis at A $= \Delta\phi = O$

These three conditions must be true since the arch is fixed at A and at B, the abutments being assumed immovable

From mechanics,*

$$\Delta x = \Sigma_A^B My \frac{s}{EI} = O \qquad (5)$$

$$\Delta y = \Sigma_A^B Mx \frac{s}{EI} = O \qquad (6)$$

$$\Delta \phi = \Sigma_A^B M \frac{s}{EI} = O \qquad (7)$$

These three equations are general formulas. They are not used directly in arch computations but are necessary in the theoretical derivation of the working formulas given in paragraphs which follow.

These three equations express the conditions that the horizontal, vertical and rotary movements of the left end of the arch ring each equal zero, so far as these motions are caused by the *bending moments only*, acting on the different sections from B to A. The movements due to the *thrust* and *shear* within the ring are not here considered. By means of equations (5), (6), (7) and the three from statics (see p. 549) we can solve for the six unknown quantities at the supports, namely, the horizontal and vertical components of each reaction and the moment at each support, and having thus found the reactions, the stresses within the arch ring can then be computed.

RELATION BETWEEN OUTER FORCES AND THE THRUST, SHEAR AND BENDING MOMENT FOR THE FIXED ARCH

In Fig. 160 let the arch A B be fixed at the two supports. If the loads are known, the horizontal and vertical components of the reactions and also the moment at each support of the arch may be found, as it has been shown above. Having these three quantities for each support, the *point of application* of each reaction may then be determined.

Thus in Fig. 160 the point of application at the left support is at *a*, distant y_1 vertically from A, where $y_1 = \frac{M_1}{H}$. Similarly at B, $y_2 = \frac{M_2}{H}$. Having computed y_1 and y_2, thus locating the points of application of the reactions, the force polygon and its equilibrium polygon, *a b c d*, can be drawn, as described more fully on page 579, and the latter will be the true line of pressure for the loading shown. The stresses on any section such as D may

M = moment. s = short length of arch axis. E = modulus of elasticity. I = moment inertia. Δx = change of span length. xy = coördinates of a point.

*See "Mechanics of Engineering," by Irving P. Church, 1908, p. 449, or any general treatise on mechanics.

be then studied. The resultant of all outer forces on the left of D is a force acting along the line $a\,b$ of the equilibrium polygon and having a magnitude equal to the force O_0 of the force polygon. This resultant outer force O_0

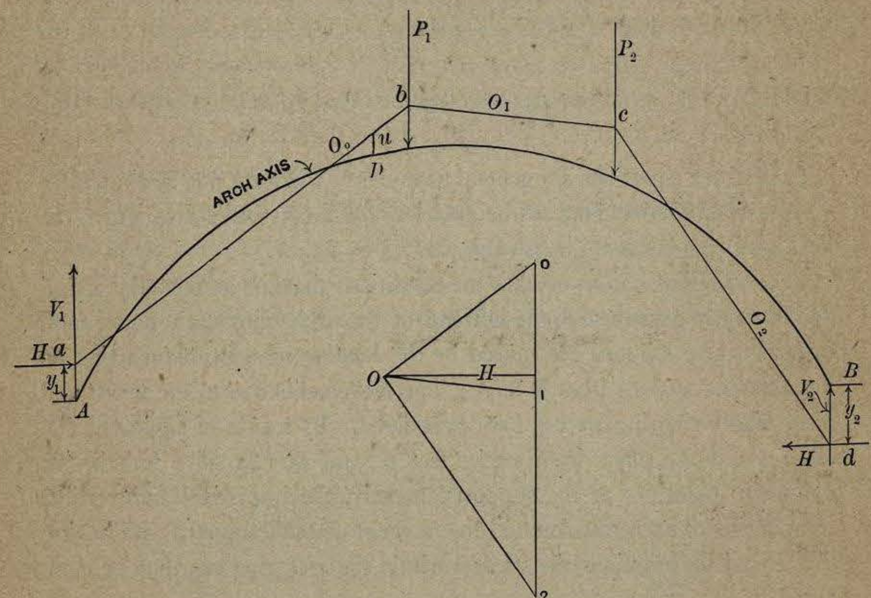

FIG. 160.—Line of Pressure in an Arch. (*See p.* 549).

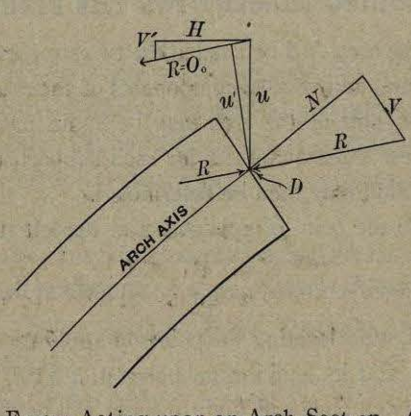

FIG. 161.—Forces Acting upon an Arch Section. (*See p.* 550.)

acting along *ab* is resisted by inner forces, i. e., stresses, on the section *D* which is redrawn in Fig. 161.

The force *R* is the force opposing the resultant O_0. This force is equiva-

lent to a force *R* acting at the arch axis and a bending moment $= Ru' = Hu$, where *H* is the *horizontal* component of *R* and *u* is the *vertical* distance from point D on the arch axis to the equilibrium polygon; u' is the *perpendicular* distance from point D to the force $R = O_0$. For vertical loads *H* is constant throughout the length of the arch ring.

The resultant force *R* acting at D can be resolved into two components one of which, *N*, is *tangential* to the axis at D and therefore normal to the section of the arch ring; the other component, *V*, is *perpendicular* to the axis and parallel to the section.

N is the *thrust*, that is, the tangential component of the resultant force on the section.

V is the *shear*, that is, the radial component of the resultant force on the section.

Hu or *Ru'* is the *bending moment* about the gravity axis of the section.

Evidently there are sections of the arch where the equilibrium polygon intersects the arch axis. At these sections the bending moment is zero. Furthermore, if the equilibrium polygon is normal to any section there will be no shear on that section. It is possible then to find sections where there is no moment, or no shear, or possibly where there is neither moment nor shear. There is always a *thrust* on every section.

THRUST, SHEAR AND MOMENT AT THE CROWN

Instead of actually finding the components of the reactions and the moments at the supports by the plan indicated on page 549, it is simpler to find the thrust, shear and moment at the crown. Having these, the equilibrium polygon may be drawn and the thrust, shear and moment at any point may be found. The thrust, shear and moment at the crown can be found by use of equations (5), (6), (7), page 549, in which *M* is the moment of any point D of Fig. 160, page 550, expressed in terms of the values at the crown. Instead, however, of determining these quantities by means of these equations, shorter expressions for the thrust, shear and moment at the crown may be obtained by *taking the origin of coördinates at the crown and studying the motion at that point.*

In Fig. 162, *CD* represents the vertical section at crown, upon which acts the resultant pressure along the line *AB*. In the lower part of the figure, for this resultant force is substituted the horizontal thrust, H_c, the shear, V_c, acting at the center of the section *CD*, and the moment M_c.

Referring to Fig. 163, page 552, and accepting *C* as origin of coördinates, Let

x, y, = coördinates of any point D,

M_L = moment at any point D on left half of arch axis of all loads between the point and the crown.

M_R = moment at any point D on right half of arch of all loads between the point and the crown.

m = number of divisions of half of the arch axis.

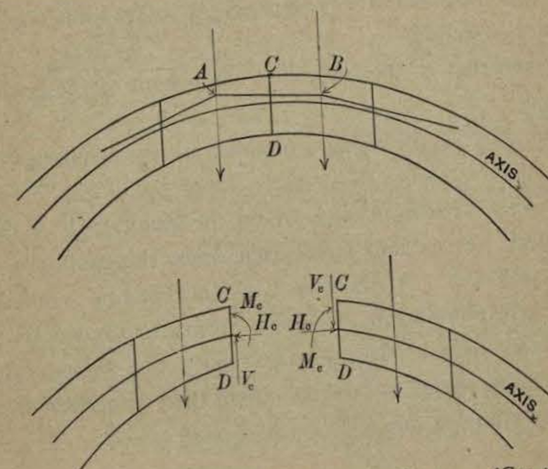

FIG. 162.—Moment and Thrust at the Crown. (See p. 551)

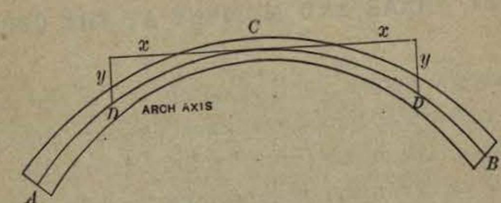

FIG. 163.—Coördinates of Any Point in Arch Axis. (See p. 551.)

The greater the number of divisions the more accurate the results.

The formulas given below require that the arch be divided so that the ratio of length of any divisiom to its average moment of inertia is constant. Because of this requirement the end divisions with large moments of inertia may be long, even with comparatively short divisions at the crown. This may cause an inaccuracy which can be eliminated by subdividing the load on the end divisions.

For an arch divided in such a way that the ratio of the length of any division to its average moment of inertia is constant (see page 554) the three unknown quantities, V_c, H_c, and M_c may be found from formulas*

$$H_c = \frac{m\,\Sigma M_R y + m\,\Sigma M_L y - \Sigma M_R \Sigma y - \Sigma M_L \Sigma y}{2\,[m\Sigma y^2 - (\Sigma y)^2]} \tag{16}$$

$$V_c = \frac{\Sigma M_L x - \Sigma M_R x}{2\,\Sigma x^2} \tag{17}$$

$$M_c = \frac{\Sigma M_R + \Sigma M_L - 2\,H_c\,\Sigma y}{2m} \tag{18}$$

*The horizontal motion of C, Fig. 163, as inpreceding analysis, due to bending moments on sections between B and C, is $\Sigma_C^B My\frac{s}{EI}$. The horizontal motion of C due to the bending moments on sections between A and C, is $\Sigma_C^A My\frac{s}{EI}$. These two motions are equal but opposite in direction, hence,

$$\Sigma_C^B My\frac{s}{EI} = -\Sigma_C^A My\frac{s}{EI} \tag{8}$$

Similarly the vertical motions at C are equal,

$$\Sigma_C^B Mx\frac{s}{EI} = \Sigma_C^A Mx\frac{s}{EI} \tag{9}$$

Also the changes in direction of the tangent to the axis at C are equal, but opposite in direction, hence,

$$\Sigma_C^B M\frac{s}{EI} = -\Sigma_C^A M\frac{s}{EI} \tag{10}$$

If each half of the arch axis be divided into m divisions in such a way as to make $\frac{s}{I}$ constant for all the divisions (See p. 554) the factor $\frac{s}{I}$ and also E may be cancelled. In the equations (8), (9) (10), M, I, x, y, denote respectively the bending moment, moment of inertia of the cross-section, and coördinates at the center point of each division of the arch axis.

At center of any division between B and C the bending moment is

$$M = M_C - V_C x + H_C y - M_R \tag{11}$$

At center of any division between B and C the bending moment is

$$M = M_C + V_C x + H_C y - M_L \tag{12}$$

Placing these values of M in equations (8), (9) and (10) and collecting terms, we have

$$2\,M_C\Sigma y + 2\,H_C\Sigma y^2 - \Sigma M_R y - \Sigma M_L y = 0 \tag{13}$$

$$2\,V_C\Sigma x^2 - \Sigma M_L x + \Sigma M_R x = 0 \tag{14}$$

$$2\,mM_C + 2\,H_C\Sigma y - \Sigma M_R - \Sigma M_L = 0 \tag{15}$$

Combining (13) and (15),

$$H_C = \frac{m\Sigma M_R y + m\Sigma M_L y - \Sigma M_R \Sigma y - \Sigma M_L \Sigma y}{2\,[m\Sigma y^2 - (\Sigma y)^2]} \tag{16}$$

From (14) $$V_C = \frac{\Sigma M_L x - \Sigma M_R x}{2\,\Sigma x^2} \tag{17}$$

From (15) $$M_C = \frac{\Sigma M_R + \Sigma M_L - 2\,H_C\Sigma y}{2\,m} \tag{18}$$

M = moment. H_C = crown thrust. V_C = crown shear. m = number divisions of half axis. x, y = coördinates of a point.

These are fundamental equations in arch analysis. The method of application is illustrated in the example, page 574.

All Σ signs denote summations for *one-half* of the arch axis.

All numerical values of M_L, M_R, x, y, are positive.

A positive value of V_c indicates that the line of pressure at the crown slopes upward toward the left; a negative value, upward towards the right.

A positive value of M_c indicates a positive moment at the crown; a negative value, a negative moment.

The moment at any point between B and C is

$$M = M_c - V_c x + H_c y - M_R \qquad (19)$$

while at any point between A and C

$$M = M_c + V_c x + H_c y - M_L \qquad (20)$$

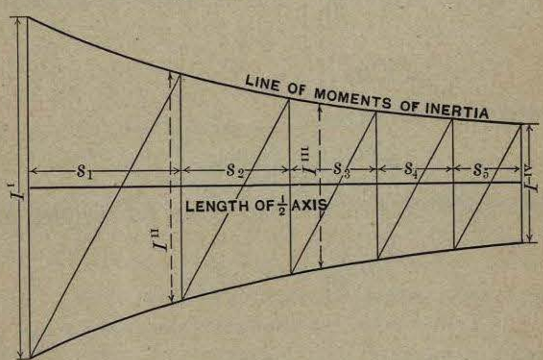

FIG. 164.—Diagram for finding Constant $\frac{s}{I}$. (*See p.* 554)

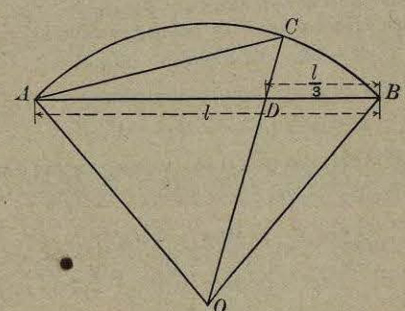

FIG. 165.—Diagram for finding Length of Arc of a Circle. (*See p.* 554.)

GRAPHICAL METHOD FOR FINDING CONSTANT $\frac{s}{I}$

Fig. 164 and Fig. 165 give a graphical method of determining the length

M = moment. H_c = crown thrust. V_c = crown shear. x, y = coördinates of a point. s = length of division of axis. I = moment of inertia.

of divisions for a constant $\frac{s}{I}$. If the arch axis is made up of arcs of circles, the length of any arc ACB is equal to three halves of the straight line AC.* The point C is found in Fig. 165 by dividing the chord AB into thirds and drawing a radius through the one-third point. If the arc is an ellipse, a simple method of drawing which is given on page 202, the length may be measured from the drawing. Having found the length of the half axis and drawn it as a horizontal line, the constant $\frac{s}{I}$ is found as shown in Fig. 164 by computing four or more values of I, the moment of inertia, at different points and plotting these to locate the curves as shown. Beginning at the lower left corner of the diagram, trial diagonals (parallel to each other) and vertical lines are drawn, so that the number of spaces between the verticals will represent the number of divisions into which the half arch must be divided. If at the first trial the final diagonal does not come out exactly at the upper right corner which represents the crown of the arch, a new slope is tried for the parallel diagonals.

LINE OF PRESSURE

Having determined the thrust and moment at the crown, the line of pressure may be drawn as shown in folding Fig. 181, opposite page 581, from which the compression and tension at different sections may be found after determining the thrust and eccentricity from the formulas which follow.

It is well to draw the line of pressure before considering the temperature and the effect of the rib shortening, and then afterwards study these, adding or deducting the stresses for the most unfavorable conditions.

EFFECT OF TEMPERATURE AND THRUST

The thrust acting throughout the ring tends to shorten the span. A change of temperature of the ring tends to shorten the span when the temperature falls or to lengthen the span when the temperature rises. The tendency for the span to change its length by a distance Δ_L due to any cause is resisted by a horizontal component H_1 and a moment M_1 acting at each support, and by a thrust and moment in the arch ring. Δ_L is positive for an increase and negative for a decrease in span length.

*Method given in Nouvelles Annales de Mathematiques, Jan. 1907. The error for 40 degrees is less than $\frac{1}{10000}$, for 70 degrees is less than $\frac{1}{1000}$, for 90 degrees is less than $\frac{5}{1000}$.

The thrust and moment at the crown may be found from formulas*

$$H_c = \frac{I}{s}\,\frac{mE\,\Delta_L}{2[m\Sigma y^2 - (\Sigma y)^2]} \tag{23}$$

and

$$M_c = -\frac{H_c\,\Sigma y}{m} \tag{24}$$

Rise in Temperature. Under a rise of temperature of the arch ring of t degrees Fahr. the span L would tend to increase in length an amount of ctL, c being the coefficient of linear expansion. Substituting for Δ_L in (23) the value of ctL, the thrust at crown is

$$H_c = \frac{I}{s}\,\frac{ctLmE}{2[m\,\Sigma y^2 - (\Sigma y)^2]}, \tag{25}$$

The value of the temperature coefficient, c, in equation (25) may be taken for concrete as 0.0000055. Dimensions must all be in same units; if in feet, *E must be in pounds per square foot.* Using a value of E_c of 2,000,000, E is therefore 2,000,000 × 144= 288,000,000 pounds per square foot.

Moment at crown is

$$M_c = -\frac{H_c\,\Sigma y}{m} \tag{26}$$

*The change in total span length, the two halves of the arch being equal, is

$$2\Sigma_C^A My\frac{s}{EI} = \Delta_L \tag{21}$$

The change in inclination of tangent to axis at crown is

$$2\Sigma_C^A M\frac{s}{EI} = 0 \tag{22}$$

Replacing the M of equations (20) and (22) by $M_c + H_c y$, which is the moment at any point D, Fig. 166, in terms of moment and thrust at the crown, and making $\frac{s}{I}$ constant, there results

$$2\frac{s}{EI}M_c\Sigma y + 2\frac{s}{EI}H_c\Sigma y^2 = \Delta_L$$
$$mM_c + H_c\Sigma y = 0$$

From which

$$H_c = \frac{I}{s}\,\frac{mE\Delta_L}{2[m\,\Sigma y^2 - (\Sigma y)^2]} \tag{23}$$

and

$$M_c = -\frac{H_c\,\Sigma y}{m} \tag{24}$$

M = moment. H_c = crown thrust. m = number divisions of half axis. s = length of division of axis. I = moment inertia. L = span. E = modulus of elasticity. Δ_L=change of span length. t = rise or fall of temperature. c = coefficient of expansion. x, y = coördinates of a point.

The moment at any point D may be found as soon as the values of H_c and M_c have been determined by means of the relation

$$M = M_c + H_c y \tag{27}$$

or we can say that the moment at any point equals the thrust H_c* multiplied by the distance from the point in question to the line OO, Fig. 166.

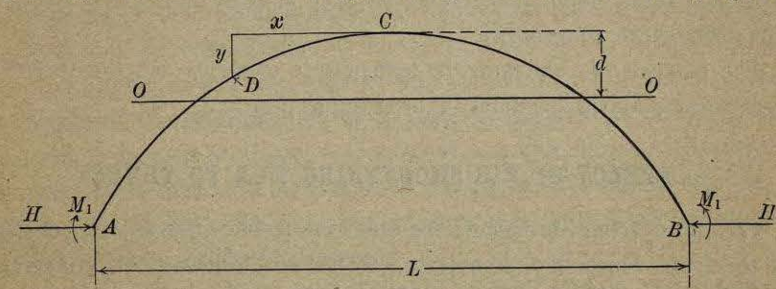

FIG. 166.—Moments and Thrusts due to Changes of Temperature. (*See p.* 556.)

Above the line OO, Fig. 166, the moments are all negative, being a maximum at the crown, and below OO they are all positive, being maximum at A and B. The line OO is below the crown a distance $d = \frac{\Sigma y.}{m}$ At the two points where OO intersect the arch axis the moments are zero, as is evident from equations (24) and (26).

Fall in Temperature. Here the thrust at crown is

$$H_c = -\frac{I}{s}\,\frac{ctLmE}{2[m\,\Sigma y^2 - (\Sigma y)^2]} \tag{28}$$

where c is 0.0000055, and moment at crown is

$$M_c = -\frac{H_c\,\Sigma y}{m} \tag{29}$$

and, as above,

$$M = M_c + H_c y \tag{30}$$

In placing a numerical value for H_c in the last two equations, it should be observed that it is a negative quantity. If in the equations the values of L and y are in feet, *E is in pounds per square foot.* Above OO the moments are all positive, below they are all negative. The thrust at the crown is really a tension in this case.

M = moment. H_c = crown thrust. m = number divisions of half axis. s = length of division of axis. I = moment inertia. L = span. E = modulus of elasticity. t =rise or fall of temperature from mean. c = coefficient of expansion. x, y = coördinates of a point.

*The horizontal thrust is constant throughout the arch, hence H_c at the crown equals H at the support.