c. Since total shear should not exceed 120 pounds per square inch, bd should not be less than $\frac{V}{105}$.

For T-beams the same rules apply except that only the web of the beam is effective; hence, b' must be substituted for b.

Vertical and Inclined Reinforcement. When the allowable working strength of the concrete in shear, as indicated in the preceding paragraphs, is exceeded, web reinforcement must be introduced. This may consist of the bent-up portion of the horizontal bars or of inclined or vertical members attached to or looped about the main reinforcement. Where inclined members are used, their connection with the horizontal reinforcement must be such as to insure against slipping.

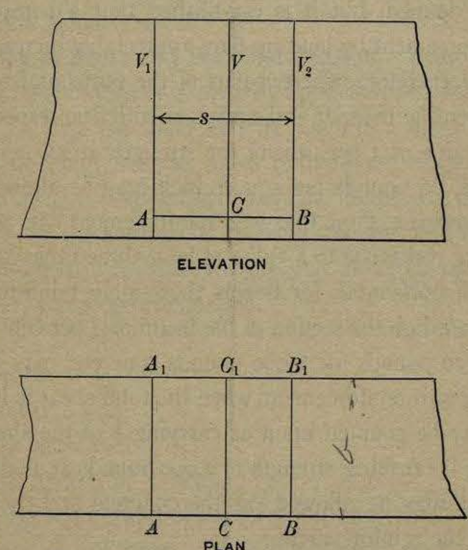

FIG. 143.—Illustration of Shearing Stresses. (*See p.* 448.)

Let

A_s = area of section of stirrup or bnet bar.
V = total shear.
s = distance between stirrups.
d = depth from top of beam to center of steel.
jd = distance from center of compression to center of tension in the beam (approximately $\frac{7}{8}d$.)
f_s = unit stress in steel.

The shear, V, of any section, $A_1 A$, Fig. 143, as determined by the loads

(see p 435.) is divided by bdj to obtain the vertical unit shear, v, as indicated in formulas (29) and (30).

The horizontal unit shear of a section, $A_1 A$, is equal to the vertical unit shear, v, at the same section. Multiplying this shear by b, the breadth of the beam, gives the shear per unit of length of the beam. Consider now the plane $AA_1B'_1B$. The intensity of the horizontal unit shear on this plane varies proportionally with the total shear V, and is equal at $A_1 A$ to $\frac{V_1}{jd}$ and at $B_1 B$ to $\frac{V_2}{jd}$. Hence the total amount of shear on this plane is equal to $\frac{V_1 + V_2}{2jd} \times s$, or when V at C is the average shear, to $\frac{V}{jd} \times s$.

When a stirrup is placed at C and the distance between stirrups is s, the diagonal tension tributary to this stirrup, the measure of which is the horizontal shear (see page 446), is equivalent to two-thirds of the total shear produced on the plane AB, it being assumed, when the allowable shear of the concrete is exceeded, that steel is provided to take two-thirds of the total shear, leaving the concrete to take the remaining one-third.

The stress in a single vertical stirrup may be assumed as $A_s f_s$, hence the area of vertical steel required in a length of s may be taken as

$$A_s = \tfrac{2}{3}\frac{sV}{f_s jd} \qquad (31)$$

The spacing of the stirrups is treated under a separate heading which follows.

For bars inclined at 45 degrees, it may be assumed that the stress in any single reinforcing member is $\frac{A_s f_s}{0.7}$ and the required area of bar, assuming the steel to take two-thirds of the shear, is

$$A_s = \tfrac{2}{3}\frac{0.7\ sV}{f_s jd} \qquad (32)$$

For other angles of inclination, it may be assumed as approximately correct for the present to use the formula

$$A_s = \tfrac{2}{3}\frac{\sin \alpha\ sV}{f_s jd} \qquad (33)$$

The use of these formulas is illustrated in the example 6, page 473.

Formerly, as already stated, stirrups were figured to take direct shear across the rod, but this has been proved by tests to be incorrect.

Stirrups in a Continuous Beam. In a continuous beam in the part near the support subjected to negative bending moment, the diagonal tension acts in the opposite direction to that in the part subjected to positive bending moment. The maximum stress then is in the upper end of the stirrups, so that they should be inverted near the supports. In any case the stirrups should be anchored in the tensile portion of the beam with their free ends (straight or preferably hooked) extending into the compressive part of the beam.

Spacing of Stirrups. The spacing of stirrups must be less than the effective depth of the beam, and a practical limit for spacing is suggested as three-fourths the depth of the beam. Closer spacing than this, however, may be required in order to make the rods small enough to have sufficient bond, as given in the following paragraphs.

Let

x = distance in feet from left support to point at which required spacing is desired.

x_1 = distance in feet from left support to point beyond which stirrups are unnecessary.

l = span of beam in feet.

w = uniform load in pounds per foot.

V = total vertical shear at section x feet from left support in pounds.

v = total unit shear at section in pounds per square inch.

v' = allowable unit shear (or diagonal tension) on concrete alone.

A_s = cross-sectional area of vertical stirrup in square inches. (In a *U*-stirrup this is the sum of the area of the two legs).

f_s = allowable unit stress in stirrups in pounds per square inch.

jd = depth of beam in inches from center of compression to center of horizontal reinforcement. (In a T-beam this may be taken as distance between center of slab and steel; in a rectangular beam as 0.87 of the total depth to steel.)

b = breadth of beam in inches.

s = spacing of stirrups in inches at point x feet from left support.

Then the allowable spacing of a given size stirrup in a certain part of the beam assuming the total shear not to exceed 3 times the working shear in the concrete is obtained by solving formula (31) for s.

$$s = \frac{3 A_s f_s jd}{2V} \qquad (34)$$

This equation becomes for a uniformly loaded beam.*

$$s = \frac{3 A_s f_s jd}{w(l - 2x)} \qquad (35)$$

For f_s = 16 000 pounds per square inch, formula (34) becomes

$$s = \frac{24000 A_s jd}{V} \qquad (36)$$

and formula (35) changes to

$$s = \frac{48000 A_s jd}{w(l - 2x)} \qquad (37)$$

The above formulas, while applying strictly to supported beams, may be used for continuous beams with safety.

Stirrups should thus be spaced by equation (34) or (35) up to a section where the unit shear equals the working shearing strength of concrete, bear-

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NOTE—Formulas (31) to (37) assume, according to the authors' suggestion, that the shearing unit stress in the beam does not exceed three times the allowable shearing strength of concrete. If, however, the above suggestion is not followed, and the shear does exceed three times the allowable strength of concrete, the following formulas should be used in place of (31), (32) and (33):

$$A_s = \frac{(V - v'\,bjd)\,s}{f_s\,j\,d} \quad (31a) \qquad A_s = \frac{0.7\,(V - v'\,bjd)\,s}{f_s\,j\,d} \quad (32a)$$

$$A_s = \frac{\sin\alpha \quad (V - v'\,bjd)\,s}{f_s\,jd} \quad (33a)$$

In place of (34) and (35), use

$$s = \frac{A_s f_s\,j\,d}{V - v'\,b\,j\,d} \quad (34a) \qquad s = \frac{2\,A_s f_s\,j\,d}{w\,(l - 2x) - 2\,v'\,b\,j\,d} \quad (35a)$$

and for f_s = 16 000 pounds per square inch, instead of (36) and (37),

$$s = \frac{16\,000\,A_s\,j\,d}{V - v'\,b\,j\,d} \quad (36a) \qquad s = \frac{32\,000\,A_s\,j\,d}{w\,(l - 2x) - 2\,v'\,b\,j\,d} \quad (37a)$$

$s =$ $2V$ (34)

This equation becomes for a uniformly loaded beam.*

$$s = \frac{3\,A_s\,f_s\,jd}{w(l - 2x)} \qquad (35)$$

For f_s = 16 000 pounds per square inch, formula (34) becomes

$$s = \frac{24000\,A_s\,jd}{V} \qquad (36)$$

and formula (35) changes to

$$s = \frac{48000\,A_s\,jd}{w\,(l - 2\,x)} \qquad (37)$$

The above formulas, while applying strictly to supported beams, may be used for continuous beams with safety.

Stirrups should thus be spaced by equation (34) or (35) up to a section where the unit shear equals the working shearing strength of concrete, bearing in mind, however, that the maximum spacing should not exceed three-fourths the depth of the beam. The distance from the support to the point where no stirrups are required, for uniform loading is†

$$x_1 = \frac{l}{2} - \frac{v'bjd}{w} \qquad (38)$$

From formulas (34) and (36) it is evident that the necessary spacing of stirrups is inversely proportional to the total shear V at any point and therefore is the smallest at the end of the beam and increases toward its middle.

Many constructors advise the insertion of occasional stirrups throughout the entire length of the beam even if they are not theoretically necessary.

For a small beam where the stirrups are spaced uniformly, for convenience, only the minimum value of s needs to be figured by substituting for V in equation (34) and (36) the total shear at an arbitrary distance $\frac{1}{2}\,d$ from the support, or in equation (35) and (37) substituting $\frac{1}{2}\,d$ for x.

* By substituting in equation (34), $V = \frac{wl}{2} - wx$ as the total shear at any point in a uniformly loaded beam.

† The unit shear $v = \frac{V}{bjd}$. Stirrups are unnecessary at section where $v = v'$ or less, or $v' = \frac{V'}{bjd}$. For the case of uniform load $V' = \frac{wl}{2} - wx_1$ Substituting this for V' and solving for x_1, we have $x_1 = \frac{l}{2} - \frac{v'\,bjd}{w}$

Graphical Method for Spacing Stirrups. In a large and important beam, the spacing should vary with the shear. The following graphical method will be of use in such cases:

Lay out half of the span $\frac{l}{2}$ to any convenient scale as shown in Fig. 144.

Compute the values of s at three or four points (point 1, 2 and 3 in the figure) and lay them out on the perpendiculars erected at the respective points to the same scale as the span. Draw a smooth curve located by the points on the perpendiculars. From point a on the perpendicular at the

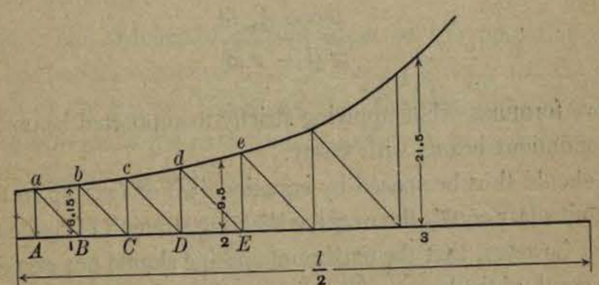

FIG. 144.—Graphical Method for Spacing Stirrups. (*See p.* 452.)

the point where the first stirrup will be placed, draw a line at 45 degrees to intersect with the line representing the span and erect at the point of intersection, B, a perpendicular to cut the curve in point b. A line drawn from b at 45 degrees will intersect the span in point C, where the above process is repeated. The points, A, B, C, D, E, thus obtained are the points in which stirrups are required.

For uniformly loaded beams it is only necessary to compute the minimum spacing of stirrups, that is, at the support. The spacing at two other points may be obtained from the fact that the spacing for $x = \frac{l}{8}$ is four-thirds the minimum and for $x = \frac{l}{4}$ is twice the minimum. For $x = \frac{l}{2}$ the spacing is infinity.

Types of Shear Reinforcement. Fig. 145 illustrates different types of diagonal tension reinforcement, showing beams reinforced with stirrups alone, with bent bars, and with a combination of bent bars and stirrups. The method of providing for the negative bending moment over the support is also indicated.

Fig. 146, page 455 shows different types of stirrups.

Diameter of Stirrups. The diameter to select for stirrups is governed by the limiting spacing of the stirrups as given in the preceding paragraphs, by the bond of the stirrup prongs, and by convenience in selecting and placing the reinforcement. The effective length of the stirrup should be taken less than the total length because of the slight change in the intensity of shear below the neutral axis where a lower bond strength may be expected.

Tests by Prof. Talbot indicate that it is safe to use up to at least $\frac{6}{10}$ of the total length of the stirrup in figuring the bond.

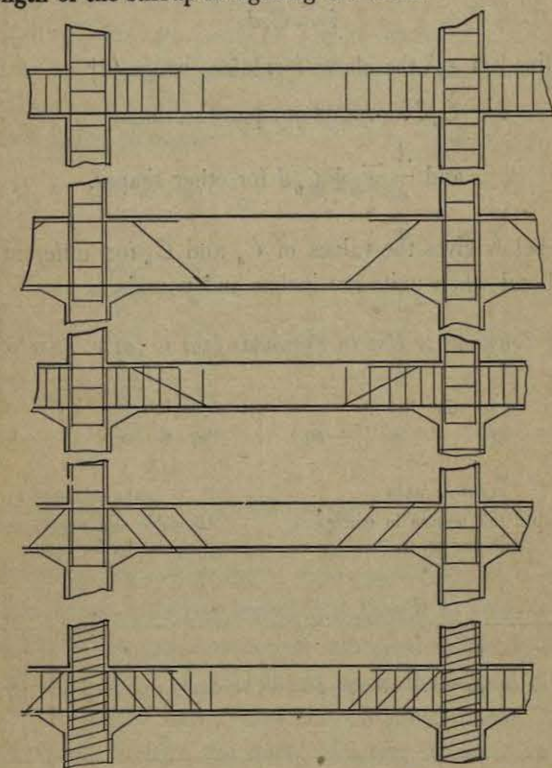

FIG. 145. Reinforcement of Continuous Beam. (*See p.* 452.)

The maximum diameter of stirrups which can be used by these assumptions without danger of slipping is determined by the bond and can be figured by the formulas given below.

Let

i = diameter of stirrup bar.

A = area of stirrup bar.

o = circumference of stirrup bar.

d = depth from surface of beam to center of tension steel.
u = allowable bond stress per unit of surface of bar.
C_s and C_b = constants to use in formulas (39) to (42).

Then for vertical stirrup with straight upper end.*

$$\frac{A}{o} \overline{<} 0.6 \frac{u}{f_s} d \text{ or } \frac{A}{o} \overline{<} \tfrac{1}{4} C_s d \qquad (39)$$

For round or square stirrups $\frac{A}{o} = \frac{1}{4} i$,

Hence

$$i \overline{<} C_s d \qquad (40)$$

For rods inclined at 45° the above formulas change to†

$$i \overline{<} C_b d \text{ for round or square sections} \qquad (41)$$

$$\text{and } \frac{A}{o} \overline{<} \tfrac{1}{4} C_b d \text{ for other shapes.} \qquad (42)$$

The table below gives the values of C_s and C_b for different values of tension and bond when units are inches and pounds.

Values of Constants to Use in Formulas (39) *to* (42.) (*See p.* 454.)

Allowable unit bond stress.	C_s					C_b				
	VERTICAL BARS Allowable unit tension in stirrups in lb. per sq. in.					BARS INCLINED 45° Allowable unit tension in bars in lb. per sq. in.				
Lb. per sq. in.	12 000	14 000	16 000	18 000	20 000	12 000	14 000	16 000	18 000	20 000
80	0.016	0.014	0.012	0.011	0.010	0.022	0.019	0.017	0.015	0.014
100	0.020	0.017	0.015	0.013	0.012	0.028	0.024	0.021	0.019	0.017
120	0.024	0.020	0.018	0.016	0.014	0.033	0.028	0.025	0.022	0.020
150	0.030	0.026	0.023	0.020	0.018	0.042	0.036	0.031	0.028	0.025

* $f_s A \overline{<} 0.6\, dou$, hence $\frac{A}{o} \overline{<} 0.6 \frac{u}{f_s} d$ Call $0.6 \frac{u}{f_s} = \frac{1}{4} C_s$ and obtain $\frac{A}{o} = \frac{1}{4} C_s d$

† For rods inclined at 45° substitute for d in the above equation $\sqrt{2}\, d$.

Hence $\frac{1}{4} C_b = 0.6 \sqrt{2} \frac{u}{f_s}$ and $\frac{A}{o} \overline{<} \frac{1}{4} C_b d$

The above formulas and table apply directly only to straight rods.

The bond stress between concrete and plain reinforcing bars may be assumed at $\frac{1}{25}$ of the compressive strength at 28 days, or 80 pounds for 2 000 pound concrete (see p. 528), which for the allowable tension in steel, f_s = 16 000 pounds per square inch gives a diameter of i = 0.012 d. For deformed bars the bond may be increased to 100 or 150 pounds per square inch, varying with the character of the bar. Using the highest figure and 16 000 pounds per square inch as the allowable tension in steel, a beam 20 inches deep to center of steel, making no allowance for the value of a bent end, would require stirrups not to exceed 0.5 inch or ½-inch diameter if deformed bars are used, or ¼-inch diameter plain stirrups. Deformed bars are therefore useful for stirrups to permit larger diameters, although the total quantity of stirrup steel required with a given allowable tensile stress is not changed.

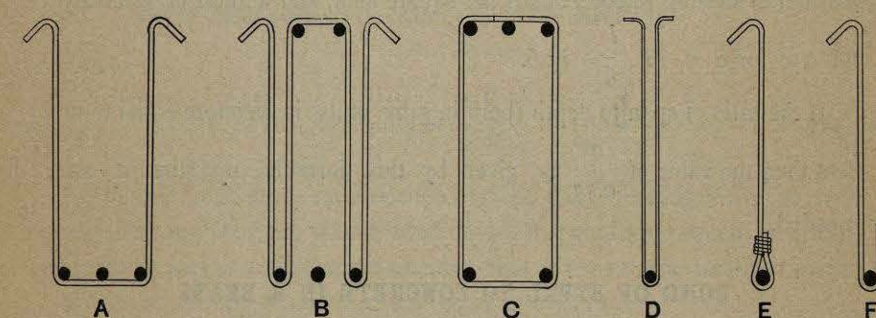

FIG. 146. Types of Stirrups. (*See p.* 453.)

Recent tests (p. 467) show that a right-angle bend of 5 diameters or a semi-circular bend of similar length is sufficient to stress the steel to its elastic limit provided the hook is well imbedded in the concrete so that it cannot kick out. With an imbedment in concrete in all directions equal to 8 diameters of the bar, a hook of 5 diameters may be assumed to develop the elastic limit of the steel and larger stirrups can be used than the table indicates,

Ratio of Span to Depth in Rectangular Beam which Renders Stirrups Unnecessary. In a T-beam stirrups are almost always needed and every case must be computed by rules already given. For a beam which is rectangular throughout its length stirrups are unnecessary if at the section of maximum shear the intensity of diagonal tension does not exceed the allowable stress in the concrete. For a rectangular beam uniformly loaded we may deduce the following expression for the ratio of span to depth which will render stirrups unnecessary.

Let

l = span of beam in inches.
d = depth from surface of beam to steel in inches.
v = maximum unit shear at end of beam in lb. per sq. in.
C = a constant from Table 00 on page 000.
a = denominator in formula $M = \frac{wl^2}{a}$

Then it may be shown* that, when $M = \frac{wl^2}{a}$

$$\frac{l}{d} \geqq \frac{a}{1.74\, C^2 v} \tag{43}$$

Adopting values of working compression in concrete of 650 pounds per square inch, working unit shear of 40 pounds per square inch, working tension in steel of 16,000 pounds per square inch, and a ratio of elasticity of 15; for $a = 12$, $\frac{l}{d} = 15.6$.

If the ratio of span to depth (both in same units) is therefore equal to or less than the value of $\frac{\alpha}{0.77}$ as given by this formula, no stirrups are needed.

BOND OF STEEL TO CONCRETE IN A BEAM

The bonding of the steel to the concrete is discussed on page 461, the values being based on the resistance to withdrawal of a steel rod imbedded in concrete. **In a reinforced concrete beam the bond of the tension steel per unit of length must not exceed its safe working value.** The concrete surrounding the steel acts as a web between its tensile and compressive parts,

* In addition to above notation let, w = load per linear inch of span, b = breadth of beam in inches, M = bending moment.

With uniform load the shear is a maximum at the support and is equal to $\frac{wl}{2}$. Taking 0.87 d as the approximate depth from the center of compression to the center of tension, the maximum intensity of shear (and consequently of diagonal tension) in the concrete is therefore (See formula (29) p. 447) $v = \frac{wl}{2 \times 0.87\, bd}$. From page 754, formula (11), it is evident that for $M = \frac{wl^2}{\alpha}$, $wl = \frac{\alpha\, bd^2}{C^2 l}$ Substituting this value of wl in the above expression for v and solving for $\frac{l}{d}$, bearing in mind that the maximum unit shear must not be exceeded, we obtain $\frac{l}{d} \geqq \frac{\alpha}{1.74\, C^2 v}$, Similarly when $M = \frac{wl^2}{12}$, $\frac{l}{d} \geqq \frac{6.8}{C^2 v}$. For a cantilever where $M = \frac{wl^2}{2}$, $\frac{l}{d} = \frac{1.15}{C^2 v}$

and the pull in the rods as it becomes less and less, because of the reducing bending moment, passes into the beam, thus producing a bond stress between the steel and the concrete. If the bond is insufficient the rod will slip.

Care must be taken, therefore, to see that the size of horizontal bars in a beam is not too large to give sufficient bond surface between the steel and the concrete. Using the formula suggested by Prof. Talbot.*

Let

V = total shear.
v = unit shear in pounds per square inch.
u = unit bond in pounds per square inch of surface area.
o = perimeter of bar in inches.
Σ_0 = sum of perimeters of all bars.
m = number of bars in tension.
jd = distance between centers of tension and compression.
d = depth from surface to center of tension steel.

Then

$$u = \frac{V}{jd\, \Sigma_0} \tag{44}$$

The unit bond stress recommended by the Joint Committee for concrete whose strength is 2 000 pounds at 28 days is 80 pounds per square inch, and assuming also as a close approximation that $jd = \frac{7}{8} d$, the total perimeter of bars which are required at any point of a beam is

$$\Sigma o = \frac{V}{70\, d} \tag{45}$$

In a continuous beam this formula applies to the steel which is in tension whether it is located in the top or the bottom of the beam. Since the negative bending moment decreases quite rapidly, the bond stress at the support of a continuous beam is more apt to exceed the safe working limit than in the middle of the beam, thus requiring more attention and frequently limiting the diameter of the bars.

The above formula does not apply to the compression steel and therefore has no relation to the steel in the bottom of a continuous beam at the support.

* Bulletin No. 4, University of Illinois, 1906, p. 19.
The formula may be derived from the relation of the bond to the shear.
The tendency to slip, or the bond stress, is equal to the shear because the measure of both of them is the increment of the moment. Hence $M = vb$, from which, since

$$v = \frac{V}{bjd} \text{ then } u = \frac{V}{jd\, \Sigma o}$$

Tests by Prof. M. O. Withey* indicate that the bond of tension bars in a beam is much less than shown by tests in which bars are pulled out from blocks of concrete, probably because of the compression on the head of the block in the latter case. For 1 : 2 : 4 concrete, the ultimate bond strength at the age of 60 days averaged 276 pounds per square inch.

POINTS TO BEND HORIZONTAL REINFORCEMENT

The bending moment in a reinforced concrete beam decreases toward the ends, reducing in the same ratio the pull in the tension bars. Since these must be designed to take the maximum moment at the center of the beam, the steel at the ends, when the bars are carried horizontally through the whole length of the beam, is stressed away below its working strength. By bending up a part of the bars not required for tension, the inclined portion assists in providing for the diagonal tension, and by carrying the ends horizontally over the top of the supports the tension due to negative bending moment may be resisted there.

If part of the rods are bent up at a certain point, those remaining must have sufficient sectional area to carry the tension beyond this point, and must also have sufficient length imbedded to prevent slipping. The limiting locations for bending the rods may therefore be found as follows:

Let

m = number of bars at the center.

m_1 = number of bars to be bent.

M = maximum moment $\frac{wl^2}{a}$ in which

a = denominator in the expression for bending moment.

x_1 = distance from support to point where m_1 bars may be bent up leaving sufficient steel to carry the pull.

Then it may be proved that the distance in feet from support to point where m_1 bars may be bent up and still leave sufficient steel to take the pull is†

*Proceedings American Society for Testing Materials, 1909.

† The ratio of pull at the middle to that at the point under consideration equals the ratio of moments in these points. Thus if the steel is stressed equally at both points,

$$M_{x_1} : M = (m - m_1)\frac{t^2\pi}{4} : \frac{m\,t^2\pi}{4}$$

Substituting

$$M = \frac{wl^2}{\alpha} \text{ and } M_{x_1} = \frac{wl^2}{\alpha} - w\frac{\left(\frac{l}{2} - x_1\right)^2}{2}$$

and solving for x_2

$$x_1 \overline{<} \frac{l}{2}\left(1 - \sqrt{\frac{8\,m_1}{\alpha m}}\right)$$

$$x_1 \overline{<} \frac{l}{2}\left(1 - \sqrt{\frac{8m_1}{ma}}\right) \qquad (47)$$

The last equation may be used for beams designed for any bending moment $M = \frac{wl^2}{a}$. For $a = 8$, the formula changes to

$$x_1 \overline{<} \frac{l}{2}\left(1 - \sqrt{\frac{m_1}{m}}\right) \qquad (48)$$

Having thus determined x_1 see that the remaining horizontal bars are secure against slipping by the use of formula (44), page 457.

The use of the formulas are illustrated in the Example 6, page 472.

SPACING OF TENSION BARS IN A BEAM

The tension bars in a beam must be a sufficient distance apart to properly transmit the pull to the concrete in the beam and prevent cleaving the concrete between them.* At most points in a beam, with bars of ordinary size, the bond stress as determined from formula (44) page 457, is low, and there is therefore but little tendency to slip and the bars may be placed as close together as proper placing of the concrete between them will permit. At points where a part of the rods are bent up, and especially in the top of the beam over the supports, the bond stress may be high, and it is advisable to make a rule that the rods shall not be spaced nearer together in the clear than 1½ times their diameter. To permit the concrete to be readily placed between them and to give sufficient concrete on the sides of the beam for fire protection, it is advisable further to make the minimum spacing between the rods 1 inch and the minimum distance of the rods from the sides of the beam 1½ inches in the clear.

There is less danger of vertical splitting, and where two layers of rods are used the rods in a vertical plane may be placed directly over each other, and with sufficient space simply to permit the mortar to run between them. The Joint Committee specify a limiting clear space of ½ inch.

Prof. McKibben has suggested a mathematical demonstration for determining the width of concrete required between the rods in order to make the resistance in shear equivalent to the adhesion of the concrete to the steel.

*The relation of bond to shear is discussed on the following page.