The determination of reactions and moments of continuous beams is referred to on page 439.

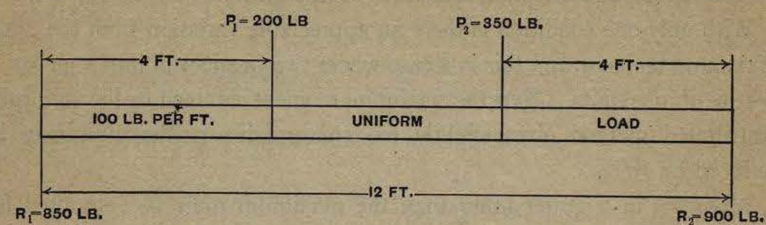


FIG. 135. Beam Loaded with Distributed and Concentrated Loads. (*See pp.* 438 and 439.)

Rule to Find Bending Moment at Any Point in a Beam. Consider either side of the vertical section passing through the point and disregard the other side. Multiply each load and reaction by its average distance from the section and add the products, taking loads acting downward as negative and those acting upward as positive.

This sum is the bending moment at the section.

Moments in English measure are usually taken in inch pounds. Hence, the distance must be in inches and the weights in pounds.

Example 4: In Fig. 135, what is the bending moment in inch pounds at the middle of the span?

Solution: $M = R_1 (6 \times 12) - P_1 (2 \times 12) - (100 \times 6) \times 3 \times 12 = 34\,800$ inch pounds.

Rule to Find Shear at any Point in Beam. Consider either side of the section passing through the points and disregard the other side. Add the loads and reactions, taking the loads acting downward as negative and those acting upward, such as a reaction, as positive. The sum is the shear at the section.

Example 5, In Fig. 135 what is the shear at the left support and at the center?

Solution: $R_1 = 850$ pounds at left support and at the center the shear is $R_1 - P_1 - (100 \times 6) = 50$ pounds,

Table of Common Bending Moments and Shearing Forces. The following table for convenient reference gives values of the shearing forces and bending moments for common cases. The values for external forces are independent of the structure of the beam.

*Bending Moments and Shearing Forces.**

Description.	Loading.	Section Considered.	Shearing Force. At distance x from support.	Shearing Force. Greatest.	Bending Moment. At distance x from support.	Bending Moment. Greatest.
Beam fixed at one end, unsupported at other.	At end		W	W	$W(l-x)$	Wl
	Uniform.		$\frac{W}{l}\left(l-x\right)$	W	$\frac{W}{2l}\left(l-x\right)^2$	$\frac{Wl}{2}$
Beam supported at both ends.	At middle.	Between support and middle.	$\frac{W}{2}$	$\frac{W}{2}$	$\frac{W}{2}x$	$\frac{Wl}{4}$
		Beyond middle.	$-\frac{W}{2}$		$\frac{W}{2}\left(l-x\right)$	
	Uniform.		$\frac{W}{l}\left(\frac{l}{2}-x\right)$	$\frac{W}{2}$	$\frac{W}{2l}\left(lx-x^2\right)$	$\frac{Wl}{8}$
	At distance *a* from support.	Between support and load.	$\frac{W(l-a)}{l}$	$\frac{W(l-a)}{l}$	$\frac{W(l-a)}{l}x$	$\frac{Wa(l-a)}{l}$
		Beyond load.	$-\frac{Wa}{l}$	$\frac{Wa}{l}$	$\frac{Wa}{l}\left(l-x\right)$	

* W = total load; l = span; x = distance of section considered from support. If moment is in inch pounds, l and x must be in inches and W in pounds. If load is distributed so as to be in terms of weight per unit length, substitute wl for W in the formulas.

Table of Moments of Inertia. The table on page 438 gives the moment of inertia for beams of a few sections which might be used in concrete construction. The reinforcement, if any, may be considered as replaced by an area of concrete which is the area of the steel times the ratio of elasticity, *n*, and is located at the same distance from the neutral axis.

SHEAR AND BENDING MOMENT DIAGRAMS

The diagrams in Figs. 136 and 137, pages 436 and 437, give bending moments and shears for beams continuous over four spans. In diagram 136, various distributions of uniform loading are given; in the first place, at the top of the page, with all the spans loaded and ends fixed; next, all spans loaded and the ends supported; and below these curves, different spans loaded in such a way as to produce maximum and minimum bending

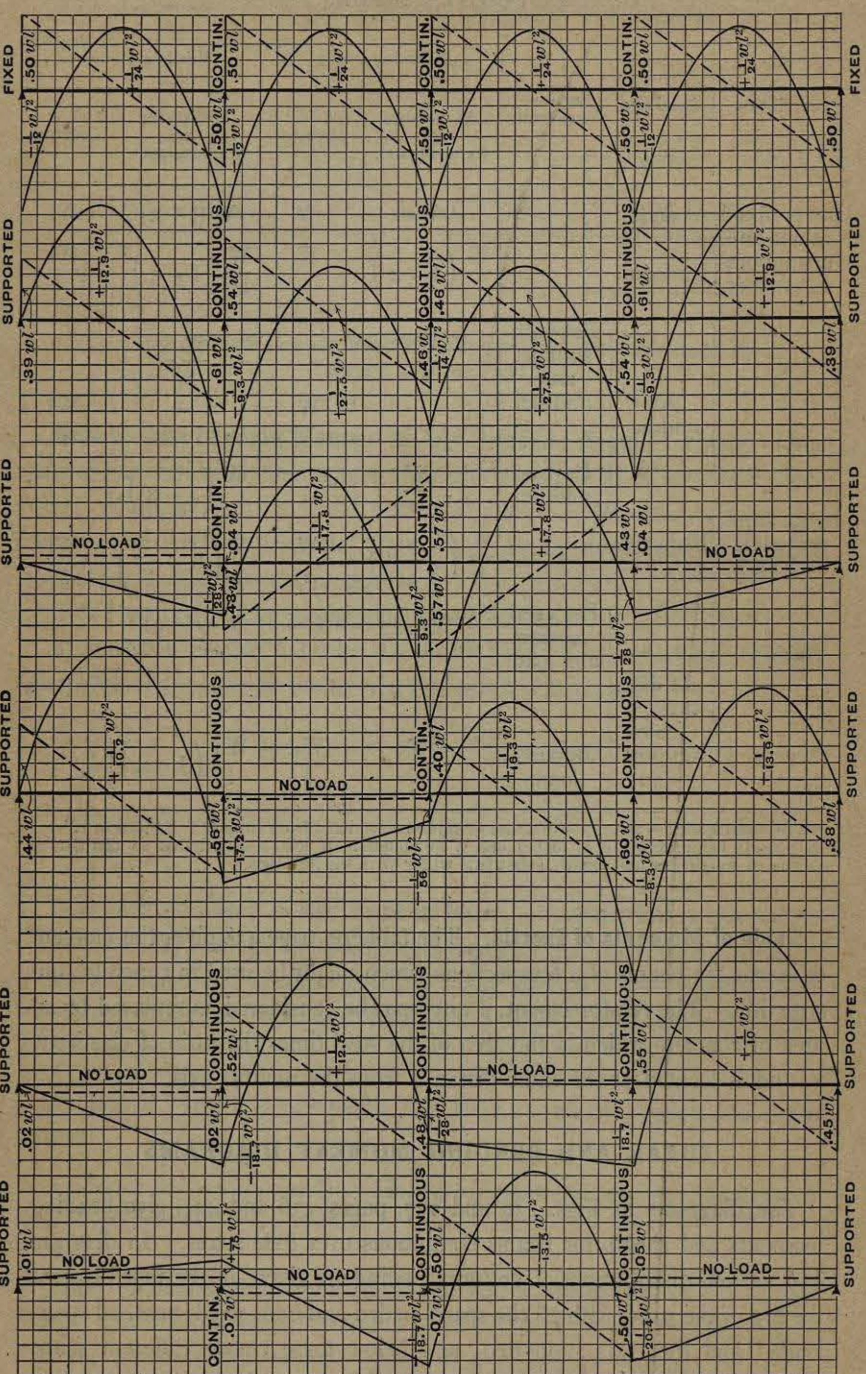


FIG. 136.—Bending Moments and Shears for Continuous Beams, Distributed Loads. (*See p.* 435.)

moments and shears. The cases chosen are sufficiently representative to be used without appreciable error as maximum and minimum values for beams of any number of spans and any distribution of uniform loading.

As stated with the diagrams, the curves are all drawn to scale on cross-section ruling so that proportionate values may be read. The loads are given in terms of *w*, the load per unit of length. The horizontal scale has twelve divisions per span, so that the moments and shears can be readily

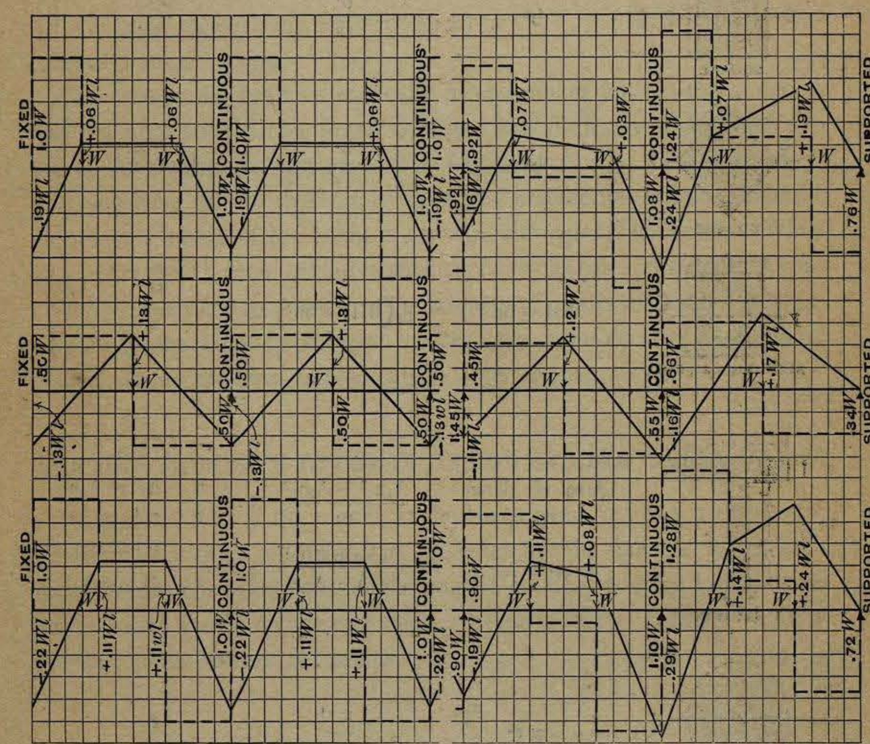
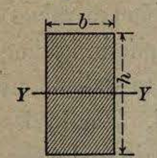
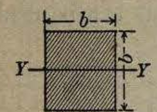
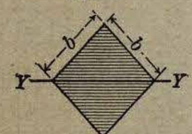
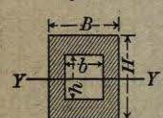
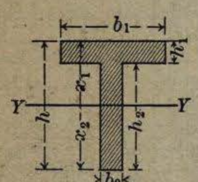
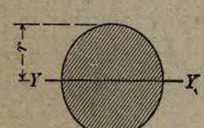
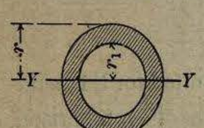
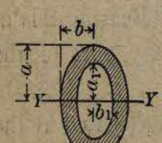


FIG. 137.—Bending Moments and Shears for Continuous Beams, Concentrated Loads. (*See p.* 437.)

estimated at $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{1}{2}$ points. The values which are printed for the bending moments are in common fractions for convenience of comparison, although in order to scale them they must be changed from the common fraction to a decimal. Each vertical division for the bending moment scale represents 0.01 and for the shear scale represents 0.1. Bearing this in mind, the bending moment and shear can be scaled at any part of the span.

Concentrated loads are treated in Fig. 137, the loads being located at the

Moments of Inertia. (*See p.* 435.)

Figure.	Area A	Moment of Inertia. I	Distance of Neutral Axis from most Strained Fiber.* Y
 RECTANGLE	bh	$\frac{bh^3}{12}$	$\frac{h}{2}$
 SQUARE	b^2	$\frac{b^4}{12}$	$\frac{b}{2}$
 SQUARE	b^2	$\frac{b^4}{12}$	$\frac{1}{2}b\sqrt{2}$
 RECTANGULAR CELL	$BH - bh$	$\frac{1}{12}(BH^3 - bh^3)$	$\frac{H}{2}$
 T SECTION	Area of flange + area of web $= A_1 + A_2$	$\frac{A_1h_1^2 + A_2h_2^2}{12} + \frac{A_1A_2(h_1+h_2)^2}{4(A_1+A_2)}$	$x_1 = \frac{h}{2} - \frac{A_1h_2 - A_2h_1}{2(A_1 + A_2)}$ $x_2 = \frac{h}{2} + \frac{A_1h_2 - A_2h_1}{2(A_1 + A_2)}$
 CIRCLE	πr^2	$\frac{\pi r^4}{4}$	r
 HOLLOW CIRCLE	$\pi(r^2 - r_1^2)$	$\frac{\pi(r^4 - r_1^4)}{4}$	r
 HOLLOW ELLIPSE	$\pi(ab - a_1b_1)$	$\frac{\pi a^3b}{4} - \frac{\pi a_1^3b_1}{4}$	a

*Applicable only to homogeneous (not to reinforced) beams.

quarter points, middle points, and third points respectively. This diagram is of special use in studying girders supporting cross beams. The stresses are computed for a beam of four spans and as the curves are symmetrical at each end, the diagram is broken in two, one-half being shown with fixed end and the other half with end supported. The results with a larger number of spans will not be appreciably different.

The vertical scale for concentrated loading is 0.05 per division for bending moments and 0.2 per division for shears.

The concentrated loads are given in terms of W, the load which is concentrated at each point.

The continuous beam is statically indeterminate, so that the moments and reactions have to be found by the theory of flexure, using the formula of three moments first evolved by Clayperon.*

In applying this to the various cases, the assumption is made that the moment of inertia of the beam is constant throughout its length. While this is not strictly true, extensive studies of various cases in reinforced concrete show that a large change in the moment of inertia makes a very small change in the bending moment, so that the relations are substantially correct until a member enters a much larger member.

BENDING MOMENTS TO USE IN DESIGN OF REINFORCED BEAMS

An examination of the curves in the diagram of bending moments for different loads, Fig. 136, page 436, indicates that in concrete beams built continuously it is safe to use for the positive bending moment in the center of the beam, except for the end spans, and also for the negative bending moment at the ends of the beams,

$$M = \frac{wl^2}{12}$$

and for end spans, for the center and also for the adjoining support

$$M = \frac{wl^2}{10}$$

the customary American and English units being adopted, viz:

M = bending moment in inch pounds.
w = load uniformly distributed in pounds per inch of length.
l = length of beam in inches.

* See Lanza's "Applied Mechanics."

In case the load is in pounds per foot of length and l is in feet, the moment in inch pounds to satisfy the former equation is simply the product of the load per foot times the square of the length in feet.

The value of $\frac{wl^2}{12}$ for the bending moment has been widely adopted in Continental Europe, is being used in general practice in Germany, and is recommended in the 1909 recommendations of the American Joint Committee and in the 1907 French rules. However, it is absolutely necessary, when designing by this formula, that the beam be really continuous both in design and construction; that the stresses due to negative bending moment at the support be provided for; that the steel be accurately located; and that, to obtain the best workmanship, the concrete be laid by a responsible builder and superintended by a man experienced in concrete construction.

An examination of the diagrams referred to will show that under these conditions the value is conservative, since a uniformly distributed load except in the end spans does not exceed $\frac{wl^2}{24}$ and the worst panel loading shown for the middle of a span gives $\frac{wl^2}{12.5}$.

Many of the building laws in the United States, to provide for the possibility of poor construction or unforeseen conditions, give the more conservative figure,

$$M = \frac{wl^2}{10} \qquad (26)$$

and for this reason and also because other assumptions may be made by multiplying by a decimal, this value is used in many of the tables in this book, and in fact it is advised for constructors who are not thoroughly familiar with reinforced concrete.

The same diagram, Fig. 136, shows that the negative bending moments are usually greater than the moments at the middle of spans. However, partial floor loading greatly reduces the negative moment, and as a live load is scarcely ever uniform over two full panels, it is considered safe to use the same value for negative moment as is used for the positive moment in the center of the beam, that is

$$M = \frac{wl^2}{12} \qquad (27)$$

At the end support, the beam, if it runs into a column or heavy wall girder,

may be practically "fixed" and thus require top reinforcement for the negative moment, $-\frac{wl^2}{12}$, and the rods running into the support must be bent or otherwise anchored. Sometimes, if the slab has cross reinforcement running into the wall girder, it may be assumed to assist in connecting the beam and girder.

Some designers, in making more exact computations, separate the dead and live load, considering the dead load extended over all panels and finding the most unfavorable position for the live load. Unless the live load is a very exact quantity this is needless refinement.

Bending Moments for Independent Concentrated Loads. If the principal live loads on a beam are concentrated, as they often are upon a girder bridge, the moments and shears at all points must be specially computed. For occasional concentrated loads in connection with uniform live and dead loads, and for loads produced by beams running into girders, it is suggested that the maximum moment under the load be computed as if the beam or girder was supported, and this be reduced by the same ratio used in the distributed loading. Thus, since the maximum moment for a concentrated load at the middle of a supported beam is $\frac{1}{4} Wl$, if $\frac{1}{12} wl^2$ is used in distributed loading instead of the $\frac{1}{8} wl^2$ required for a supported beam, $\frac{8}{12}$ of $\frac{1}{4} Wl$, or $\frac{1}{6} Wl$, may be used for concentrated center loads. The negative bending moment with concentrated loading usually may be taken the same as the maximum positive moment due to concentrated loading, reduced as indicated, except that with loads at $\frac{1}{3}$ or $\frac{1}{4}$ points, this gives for the support next to the end a negative moment which is slightly low (see diagram, Fig. 136, page 436), and in some cases it should be separately figured or else estimated from this diagram.

SHEARING FORCES IN A BEAM OR SLAB

The bending of a beam produces a tendency of the particles to slide upon each other or shear. It is therefore necessary to study

(1) Vertical shear.
(2) Horizontal shear.
(3) Diagonal tension.

Most important of all is the resultant of the shearing forces with the tension which produces the pull in a diagonal direction termed diagonal tension.

Vertical Shear in a Beam. Concrete is strong in direct shear (see p. 382) and capable of standing a working shearing stress of at least 200 pounds per square inch, so that a concrete girder or beam or slab, unless perhaps of hollow or tapered construction, always has sufficient area of section to withstand this direct shear. However, since the direct shear is a measure of the diagonal tension (see p. 446), which is excessive when the direct shear is comparatively low, it must always be computed in a beam or girder for use in the computation of diagonal stresses, as described on page 447.

The vertical shear is a maximum at the support, where it is equal to the reaction. Maximum shears for various loads are given in the diagram (Fig. 136, page 436), in terms of the loads. While with uniform or symmetrical loading the reaction, and therefore the maximum vertical shear, is one-half the total load upon the beam, it will be noticed from the diagram that where the end beams in a series of continuous beams are supported, which is very nearly the case when a beam runs into a light wall girder, the shear at the first support away from the end may be 25 per cent greater than normal, and should be specially provided for in cases like a warehouse where the full live load is liable to be constantly maintained. A further study of the two diagrams (Figs. 136 and 137, pp. 436 and 437) will illustrate the cases where allowances should be made.

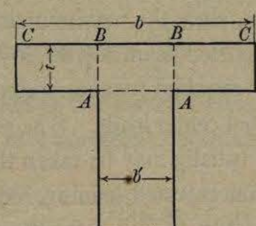


FIG. 138.—Section of a T-Beam. (*See p.* 442.)

In case the concrete in a beam or slab has cracked vertically next to the support because of accident or poor design, the bearing value of the horizontal rods may have to be estimated.

Longitudinal Vertical Shear in Flange of T-Beam. Vertical shear in a longitudinal direction is present in the wings of a T-beam due to the load upon a beam being maximum next to the flange, as shown by lies *BA* in Fig. 138.

The area of concrete in a solid horizontal floor slab is generally sufficient to take care of this shear, but the following method may be used for computing it if desired:

Let

v_h = unit horizontal shear at *AA*.
v_v = unit vertical shear at *BA*
b' = breadth of stem.
b = breadth of flange.
t = thickness of flange.

The shear along the two planes *BA* may be considered as caused by the external forces acting not on the whole breadth, but only on the projecting flanges of the T-beam *BC*.

Then it is readily shown* that

$$v_v = \frac{v_h b' (b - b')}{2tb} \qquad (28)$$

Although this vertical shear through the flanges is readily borne by the concrete, it is advisable, as stated on page 422, to place horizontal rods across the top of the beam, even if the bearing rods in the slab run parallel to the beam, in order to resist unequal bending moment which is liable to occur and to assure T-beam action.

Fillets at the angles between the flange and the beam, that is, between the slab and the beam, are not theoretically necessary, but they may be used for appearance sake and as an additional security in a deep beam with relatively shallow flanges or slabs. Small fillets are also advisable to aid in the removal of forms.

Horizontal Shear. The concrete in a solid rectangular beam or in a T-beam is nearly always sufficient to sustain the direct horizontal shear which at any part is equal to the direct vertical shear. In a skeleton beam the horizontal shear may be excessive, but the reinforcement for diagonal tension will also take care of this, so that the direct horizontal shear as such need never be considered. Formerly, before tests of direct shear proved the high strength of concrete in shear, horizontal shearing stress was determined when designing a beam and the vertical stirrups or bent-up rods were spaced to act in shear, using a value of 10 000 pounds per square inch in the steel to resist it. More recent tests have proved the stirrups and bent up rods to be in tension instead of shear.

Diagonal Tension. Not only does the high strength of concrete in direct shear indicate that cracks which form in the web of a beam are not caused by this, but tests of beams themselves show that such cracks are diagonal and in the direction which would be expected from the theory of diagonal tension. A typical crack due to diagonal tension is shown in Fig. 139, page 444.

Such cracks as these can be due only to a combination of the shearing stress with the horizontal tensile stress, whose resultant forms diagonal

* The above principle may be expressed by the equation $v_v \, 2t = v_h b' \frac{b-b'}{b}$, which solved for v_v will give formula (28).

tension. It is this diagonal tension which must be sustained in a reinforced concrete beam by the area of concrete or by bent up rods and stirrups, as indicated in paragraphs which follow.

Tests by Prof. Talbot* and Prof. Withey† indicate that for 1 : 2 : 4 concrete the first diagonal crack in a beam without stirrups or inclined reinforcement is apt to occur when the maximum shear is from 100 to 200 pounds per square inch. Since failure by diagonal tension is sudden, it is advisable to provide a high factor of safety. In a beam with diagonal

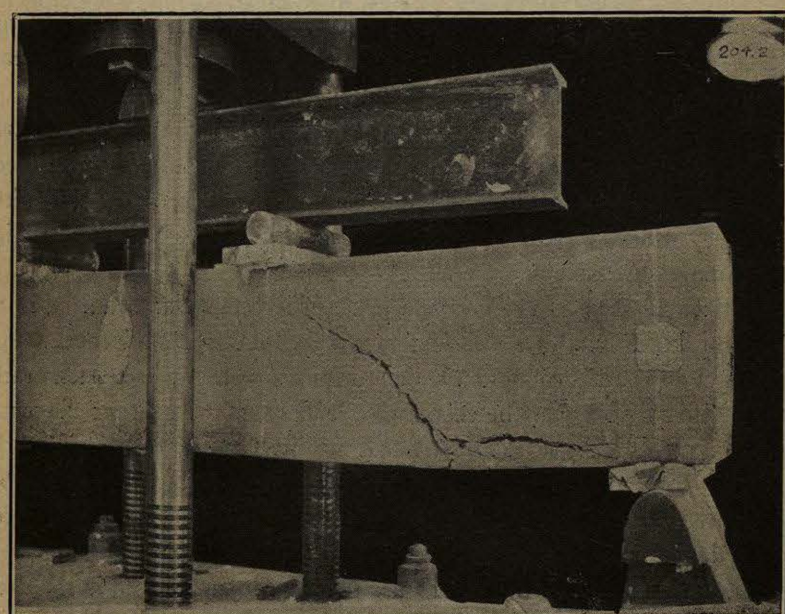


FIG. 139.—Beam under Load at University of Illinois Cracked by Diagonal Tension. (*See p.* 443.)

tension reinforcement, the first diagonal crack occurs at a period but slightly later than in a beam with horizontal rods alone, but in this case it is very small and not dangerous if the steel is designed to take the stress. However, it is desirable that there should be always a sufficient area of concrete, even when reinforced, to prevent the diagonal tension from exceeding the cracking point in the concrete.

* Bulletin No. 12, University of Illinois, 1907.
† Bulletin of University of Wisconsin, Vol. 4, No. 2, 1907.

REINFORCEMENT TO PREVENT DIAGONAL CRACKS IN BEAMS

The failure of a beam from diagonal tension is more sudden than from ordinary tension or compression, and therefore must be guarded against even more carefully. Formerly when beams were designed with full rect-

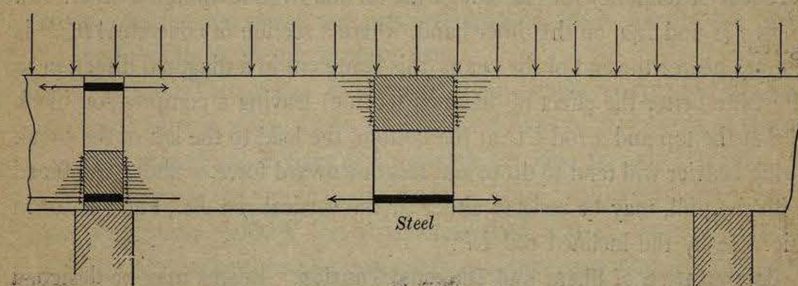


FIG. 140.—Beam with Break in Center illustrating no Shear. (*See p.* 446.)

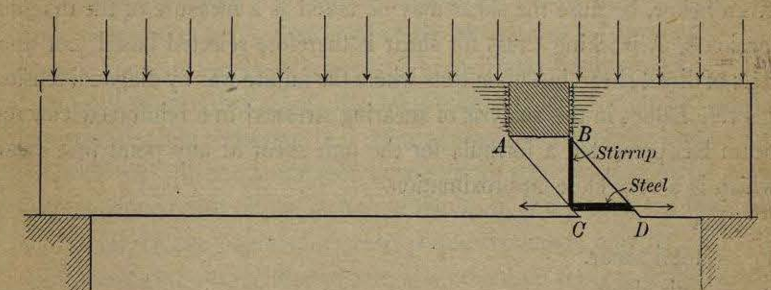


FIG. 141.—Beam with Break near End, illustrating Action of Vertical Stirrup. (*See p.* 446.)

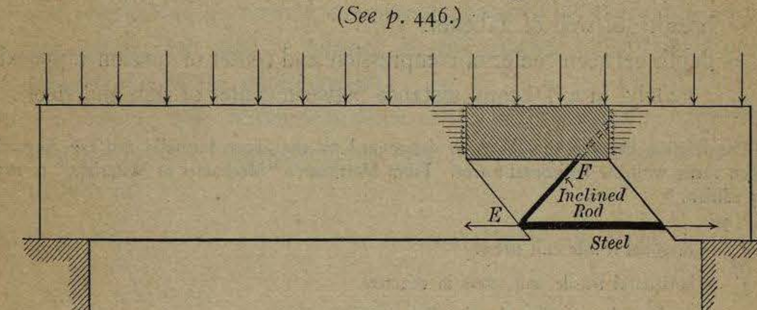


FIG. 142.—Beam with Break near End, illustrating Action of Inclined Rod. (*See p.* 446.)

angular section the concrete often had sufficient area to resist the diagonal tension without assistance from the reinforcement. With the advent of the T-section and the consequent reduction in the width of the stem, it nearly always becomes necessary to introduce stirrups or inclined rods to take the diagonal tension.

An elementary illustration of the action of the stirrups or inclined rods is shown in Figs. 140 to 142, page 445. In Fig. 140 the uniformly loaded beam is cut through in the middle, leaving simply a compression block at the top and tension rod at the bottom. There is pull in the rod at the bottom but no shear or tendency for one side of the section to slide upon the other. In Figs. 141 and 142, on the other hand, where a section of concrete *ABCD* is cut out nearer the end of the beam (this being cut in a diagonal direction to illustrate better the effect of diagonal tension) leaving a compression block *AB* at the top and a rod *CD* at the bottom, the load to the left of the break being heavier will tend to drop, and this downward force or shear, combined with the pull, may be resisted either by the vertical rod *BC*, Fig. 141, or in Fig. 142 by the inclined rod *EF*.

Computation of Shear and Diagonal Tension. Beams may be designed safe against diagonal tension failure by application of the formula for shear given below, because the shear may be taken as a measure of the diagonal tension.* A working stress for shear is therefore selected based, not upon tests of direct shear, but upon tests where the failure was by diagonal tension.

Prof. Talbot in the analysis of shearing stresses† in a reinforced concrete beam has presented a formula for the unit shear at any point in a beam which is a very close approximation.

Let

V = total shear.

v = unit shear.

b = breadth of beam.

b' = breadth of web of T-beam.

jd = depth between center of compression and center of tension (approximately, in a T-beam, distance between center of slab and steel).

* The relation between the shear, as determined by the above formula, and the diagonal tension varies with the horizontal forces. From Merriman's "Mechanics of Materials," p. 265, 1905 edition,

If

f_d = diagonal tensile unit stress.

f'_c = horizontal tensile unit stress in concrete.

v = horizontal or vertical shearing unit stress.

Then

$$f_d = \frac{1}{2} f'_c + \sqrt{\frac{1}{4} f'^2_c + v^2}$$

The direction of this maximum diagonal tension, as Prof. Talbot points out, makes an angle with the horizontal equal to one-half the angle whose co-tangent is $\frac{1}{2}\frac{f'_c}{v}$. If there is no tension in the concrete the last formula reduces to $f_d = v$. The maximum diagonal tension makes an angle of 45 degrees with the horizontal and is equal in intensity to the vertical shearing stress.

† Bulletin No. 14, University of Illinois, 1906, p. 20.

Then

$$v = \frac{V}{bjd} \quad (29) \qquad \text{or for a T-beam,} \quad v = \frac{V}{b'jd} \quad (30)$$

That is, at any section of a beam, the unit shear, either vertical or horizontal, is the total shear at the section produced by the loads divided by the product of the breadth times the moment arm.

The Joint Committee recommend that beams be reinforced against diagonal tension when the shear exceeds a limit of 2 per cent of the compressive strength at 28 days or 40 pounds for 2000 pounds concrete. The laws governing the internal stresses in a beam with a reinforced web are not yet clearly defined, but it is established that a comparatively small amount of reinforcement by bent-up bars appreciably increases the strength of the beam and, therefore, where a part of the horizontal reinforcement is bent up in a scientific manner and arranged with due respect to the shearing stresses, a value of 3 per cent of the strength at 28 days, or, for 2 000 pounds concrete, 60 pounds per square inch may be allowed.*

Since tests, however, show that web reinforcement can be introduced to increase shearing resistance to a value at least three times as great as when the bars are all horizontal, for beams thoroughly reinforced for shear a limiting value based on the section of the beam of 6 per cent of the strength at 28 days, or 120 pounds for 2 000 pounds concrete, may be used.

In calculating web reinforcement, when the total shear is limited as above, the concrete may be counted upon as carrying $\frac{1}{3}$ of the shear† that is, for concrete having a crushing strength of 2 000 pounds at 28 days, 40 pounds per square inch may be allowed on the concrete and the balance of the shear taken by the reinforcement.

Following these recommendations of the Joint Committee and assuming that the distance between centers of compression and tension, jd, is approximately $\frac{7}{8}d$:

a. Stirrups are required with horizontal bars only when $\frac{V}{bd}$ is greater than $\frac{7}{8}(40) = 35$.

b. Stirrups are required in rectangular beams where a part of the horizontal reinforcement is bent up and arranged with due respect to the shearing stresses (but not computed as taking diagonal tension) when $\frac{V}{bd}$ is greater than 52.

* This is an arbitrary assumption based on observations of experiments by members of the Joint Committee.

† Although it might be expected that the concrete, since it is assumed to have no tensile value, should not be assumed to assist in carrying diagonal tension, tests by Prof. Withey at the University of Wisconsin (Bulletin, Vol. 4, No. 2) indicate that the rule given is amply safe.