For this product is a λ-matrix of the form

$$c_0\lambda^{k+l} + c_1\lambda^{k+l-1} + \cdots + c_{k+l}$$

where c_0 has the value a_0b_0 or b_0a_0 according to the order in which the two given matrices are multiplied together. By Theorem 7, § 25, neither a_0b_0 nor b_0a_0 is zero if a_0 and b_0 are not both singular.

The next theorem relates to what we may call the division of λ-matrices.

THEOREM 3. *If* a *and* b *are two λ-matrices and if* b, *when written in the form* (1), *has as the coefficient of the highest power of* λ *a non-singular matrix, then there exists one, and only one, pair of λ-matrices* q_1 *and* r_1 *for which*

$$a \equiv q_1 b + r_1$$

and such that either $r_1 \equiv 0$, *or* r_1 *is a λ-matrix of lower degree than* b; *and also one and only one pair of λ-matrices* q_2 *and* r_2 *for which*

$$a \equiv bq_2 + r_2$$

and such that either $r_2 \equiv 0$, *or* r_2 *is a λ-matrix of lower degree than* b.

The proof of this theorem is practically identical with the proof of Theorem 1, § 63.

EXERCISE

DEFINITION. *By a real matrix is understood a matrix whose elements are real; by a real λ-matrix, a matrix whose elements are real polynomials in λ; and by a real elementary transformation, an elementary transformation in which the constant in* (*b*) *and the polynomial in* (*c*), *Definition* 1, § 91, *are real.*

Show that all the results of this chapter still hold if we interpret the words *matrix*, *λ-matrix*, and *elementary transformation* to mean *real matrix*, *real λ-matrix*, and *real elementary transformation*, respectively.

CHAPTER XXI

THE EQUIVALENCE AND CLASSIFICATION OF PAIRS OF BILINEAR FORMS AND OF COLLINEATIONS

96. The Equivalence of Pairs of Matrices. The applications of the theory of elementary divisors with which we shall be concerned in this chapter and the next have reference to problems in which λ-matrices occur only indirectly. A typical problem is the theory of a pair of bilinear forms. The matrices a and b of these two forms have constant elements, and we get our λ-matrix only by considering the matrix $a - \lambda b$ of the pencil of forms determined by the two given forms. It will be noticed that this matrix is of the first degree, and in fact we shall deal, from now on, exclusively with λ-matrices of the first degree.

By the side of this simplification, a new difficulty is introduced, as will be clear from the following considerations. We shall subject the two sets of variables in the bilinear forms to two non-singular linear transformations whose coefficients we naturally assume to be constants, that is, independent of λ. These transformations have the effect of multiplying the λ-matrix, $a - \lambda b$, by certain non-singular matrices whose elements are constants (cf. § 36) and therefore, by § 94, carry it over into an equivalent λ-matrix which is evidently of the first degree. The transformations of § 94, however, were far more general than those just referred to, so that it is not at all obvious whether every λ-matrix of the first degree equivalent to the given one can be obtained by transformations of the sort just referred to or not.

These considerations show the importance of the following theorem:

THEOREM 1. *If* a_1, a_2, b_1, b_2 *are matrices with constant elements of which the last two are non-singular, and if the λ-matrices of the first degree*

$$m_1 \equiv a_1 - \lambda b_1, \qquad m_2 \equiv a_2 - \lambda b_2$$

are equivalent, then there exist two non-singular matrices, **p** *and* **q**, *whose elements are independent of* λ, *and such that*

(1) $$m_2 \equiv p m_1 q.$$

Since m_1 and m_2 are equivalent, there exist two non-singular λ-matrices, p_0 and q_0, whose determinants are constants and such that

(2) $$m_2 \equiv p_0 m_1 q_0.$$

The matrix q_0 has, therefore, an inverse, q_0^{-1}, which is also a λ-matrix.

Let us now divide p_0 by m_2 and q_0^{-1} by m_1 by means of Theorem 3, § 95, in such a way as to get matrices p_1, p, s_1, s which satisfy the relations

(3) $$p_0 \equiv m_2 p_1 + p, \qquad q_0^{-1} \equiv s_1 m_1 + s,$$

p and s being matrices whose elements are independent of λ. From (2) we get

$$p_0 m_1 \equiv m_2 q_0^{-1}.$$

Substituting here from (3), we have

$$m_2 p_1 m_1 + p m_1 \equiv m_2 s_1 m_1 + m_2 s,$$

or

(4) $$m_2(p_1 - s_1) m_1 \equiv m_2 s - p m_1.$$

From this identity we may infer that $p_1 \equiv s_1$ and therefore

(5) $$m_2 s \equiv p m_1.$$

For if $p_1 - s_1$ were not identically zero, $m_2(p_1 - s_1)$ would be a λ-matrix of at least the first degree (cf. Theorem 2, § 95), and hence the left-hand side of (4) would be a λ-matrix of at least the second degree. But this is impossible, since the right-hand side of (4) is a λ-matrix of at most the first degree.

If we knew that p and s were both non-singular, our theorem would follow at once from (5); for we could write (5) in the form

(6) $$m_2 \equiv p m_1 s^{-1}$$

and p and s^{-1} would be non-singular matrices with constant elements. Moreover, we see from (5) that p and s are either both singular or both non-singular. Our theorem will thus be proved if we can show that s is non-singular.

For this purpose let us substitute in the identity

$$I \equiv q_0 q_0^{-1}$$

for q_0^{-1} its value from (3),

(7) $$I \equiv q_0 s_1 m_1 + q_0 s.$$

Now divide q_0 by m_2 by means of Theorem 3, § 95, in such a way as to get

(8) $$q_0 \equiv q_1 m_2 + q$$

where q is a matrix with constant elements.

Substituting this value in (7), we have

$$I \equiv q_0 s_1 m_1 + q_1 m_2 s + q s.$$

Referring to (5), we see that this may be written

(9) $$I - qs \equiv (q_0 s_1 + q_1 p) m_1.$$

From this we infer that $q_0 s_1 + q_1 p$ must be identically zero, and therefore

(10) $$I = qs.$$

For if $q_0 s_1 + q_1 p$ were not identically zero, the right-hand side of (9) would be a λ-matrix of at least the first degree, while the left-hand side of (9) does not involve λ.

Equation (10) shows that s is non-singular, and thus our theorem is proved. It shows us, however, also that q is non-singular, and that $q = s^{-1}$, so that equation (6) becomes $m_2 \equiv p m_1 q$.

We may, therefore, add the following

COROLLARY. *The matrices* **p** *and* **q** *whose existence is stated in the above theorem may be obtained as the remainders in the division of* p_0 *and* q_0 *in* (2) *by* m_2 *by means of the formulæ:*

$$p_0 \equiv m_2 p_1 + p, \qquad q_0 \equiv q_1 m_2 + q.$$

From this theorem concerning λ-matrices of the first degree we can now deduce the following theorem concerning pairs of matrices with constant elements. It is this theorem which forms the main foundation for such applications of the theory of elementary divisors as we shall give.

We shall naturally speak of two pairs of matrices with constant elements a_1, b_1 and a_2, b_2 as equivalent if two non-singular matrices p and q exist for which

(11) $$a_2 = p a_1 q, \qquad b_2 = p b_1 q.$$

THEOREM 2. *If* a_1, b_1 *and* a_2, b_2 *are two pairs of matrices independent of* λ, *and if* b_1 *and* b_2 *are non-singular, a necessary and sufficient condition that these two pairs of matrices be equivalent is that the two* λ*-matrices*

$$m_1 \equiv a_1 - \lambda b_1, \qquad m_2 \equiv a_2 - \lambda b_2$$

have the same invariant factors, — or, if we prefer, the same elementary divisors.

For if the pairs of matrices are equivalent, equations (11) hold; hence, multiplying the second of these equations by λ and subtracting it from the first, we have

$$(12) \qquad m_2 \equiv p m_1 q,$$

that is the λ-matrices m_1 and m_2 are equivalent, and therefore have the same invariant factors, and the same elementary divisors. On the other hand, it follows at once from the assumption that b_1 and b_2 are non-singular, that m_1 and m_2 are non-singular, and hence have the same rank. Consequently if m_1 and m_2 have the same invariant factors, or the same elementary divisors, they are equivalent. Since they are of the first degree, there must, by Theorem 1, exist two non-singular matrices p and q, whose elements are independent of λ, which satisfy the identity (12). From this identity, the two equations (11) follow at once; and the two pairs of matrices are equivalent. Thus the proof of our theorem is complete.

A case of considerable importance is that in which the matrices b_1 and b_2 both reduce to the unit matrix I. In this case m_1 and m_2 reduce to what are known as the *characteristic matrices* of a_1 and a_2 respectively, according to the following definition:

DEFINITION. *If* a *is a matrix of the nth order with constant elements and* I *the unit matrix of the nth order, the* λ*-matrix*

$$A \equiv a - \lambda I$$

is called the characteristic matrix of a; *the determinant of* A *is called the characteristic function of* a; *and the equation of the nth degree in* λ *formed by setting this determinant equal to zero is called the characteristic equation of* a.

We can now deduce from Theorem 2 the following more special result:

THEOREM 3. *If* a_1 *and* a_2 *are two matrices independent of* λ, *a necessary and sufficient condition that a non-singular matrix* p *exist such that**

$$(13) \qquad a_2 = p a_1 p^{-1}$$

is that the characteristic matrices A_1 *and* A_2 *of* a_1 *and* a_2 *have the same invariant factors, — or, if we prefer, the same elementary divisors.*

For if A_1 and A_2 have the same invariant factors (or elementary divisors), there exist, by Theorem 2, two non-singular matrices p and q such that

$$a_2 = p a_1 q, \qquad I = p I q.$$

The second of these equations shows us that $q = p^{-1}$; and this value being substituted in the first, we see that p is the matrix whose existence our theorem asserts.

That, on the other hand, A_1 and A_2 have the same invariant factors and elementary divisors if equation (13) is fulfilled, is at once obvious.

97. The Equivalence of Pairs of Bilinear Forms. Suppose we have a pair of bilinear forms in $2n$ variables

$$\phi_1 \equiv \sum_1^n a'_{ij} x_i y_j, \qquad \psi_1 \equiv \sum_1^n b'_{ij} x_i y_j,$$

and also a second pair

$$\phi_2 \equiv \sum_1^n a''_{ij} x_i y_j, \qquad \psi_2 \equiv \sum_1^n b''_{ij} x_i y_j,$$

and let us assume that ψ_1 and ψ_2 are non-singular. We will inquire under what conditions the two pairs of forms are equivalent, that is, under what conditions a first non-singular linear transformation for the x's and a second for the y's,

$$c\begin{cases} x_1 = c_{11}x'_1 + \cdots + c_{1n}x'_n \\ \cdots\cdots\cdots \\ x_n = c_{n1}x'_1 + \cdots + c_{nn}x'_n \end{cases} \qquad d\begin{cases} y_1 = d_{11}y'_1 + \cdots + d_{1n}y'_n \\ \cdots\cdots\cdots \\ y_n = d_{n1}y'_1 + \cdots + d_{nn}y'_n \end{cases}$$

can be found which together carry over ϕ_1 into ϕ_2 and ψ_1 into ψ_2.

* Two matrices connected by a relation of the form (13) are sometimes called *similar* matrices. This conception of similarity is evidently merely a special case of the general conception of equivalence as defined in § 29, the transformations considered being of the form (13) instead of the more general form usually considered in this chapter and the last.

If we denote the conjugate of the matrix c by c' and the matrices of $\phi_1, \psi_1, \phi_2, \psi_2$ by $\mathsf{a}_1, \mathsf{b}_1, \mathsf{a}_2, \mathsf{b}_2$ respectively, we know, by Theorem 1, § 36, that the transformations c, d carry over ϕ_1 and ψ_1 into forms with matrices

$$\mathsf{c}'\mathsf{a}_1\mathsf{d}, \qquad \mathsf{c}'\mathsf{b}_1\mathsf{d}$$

respectively; so that, if these are the forms ϕ_2 and ψ_2, we have

$$(1) \qquad \mathsf{a}_2 = \mathsf{c}'\mathsf{a}_1\mathsf{d}, \qquad \mathsf{b}_2 = \mathsf{c}'\mathsf{b}_1\mathsf{d}.$$

Consequently, by Theorem 2, § 96, the two λ-matrices

$$\mathsf{a}_1 - \lambda\mathsf{b}_1, \qquad \mathsf{a}_2 - \lambda\mathsf{b}_2$$

have the same invariant factors and elementary divisors.

Conversely, by the same theorem, if these two λ-matrices have the same invariant factors (or elementary divisors), two constant matrices c' and d exist which satisfy both equations (1); and hence there exists a linear transformation of the x's and another of the y's which together carry over ϕ_1 into ϕ_2 and ψ_1 into ψ_2. Thus we have proved the

THEOREM. *If ϕ_1, ψ_1 and ϕ_2, ψ_2 are two pairs of bilinear forms in $2n$ variables of which ψ_1 and ψ_2 are non-singular, a necessary and sufficient condition that these two pairs of forms be equivalent is that the matrices of the two pencils*

$$\phi_1 - \lambda\psi_1, \qquad \phi_2 - \lambda\psi_2$$

*have the same invariant factors,—or, if we prefer, the same elementary divisors.**

EXERCISE

Prove that the theorem of this section remains true if the bilinear forms $\phi_1, \psi_1, \phi_2, \psi_2$ are real and the term *equivalent* is understood to mean *equivalent with regard to real non-singular linear transformations.*

98. The Equivalence of Collineations. A second important application of the theory of elementary divisors is to the theory of collineations. For the sake of simplicity we will consider the case of two dimensions

$$\mathsf{a} \begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ x_2' = a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ x_3' = a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \end{cases}$$

although the reasoning will be seen to be perfectly general.

* For the sake of brevity, we shall, in future, speak of these invariant factors and elementary divisors as the invariant factors and elementary divisors of the *pairs of forms* ϕ_1, ψ_1 and ϕ_2, ψ_2 respectively.

We have so far regarded a collineation merely as a means of transforming certain geometric figures. It is possible to adopt another point of view, and to study the collineation in itself with special reference to the relative position of points before and after the transformation. Thus suppose we have a figure consisting of the points $A_1, A_2, \cdots$, finite or infinite in number, and suppose these points are carried over by the collineation a into the points $A_1', A_2', \cdots$. These two sets of points together form a geometric figure. It is the properties of such figures as this that we call the properties of the collineation. Such properties may be either projective or metrical. Thus it would be a metrical property of a collineation if it carried over some particular pair of perpendicular lines into a pair of perpendicular lines; it would be a projective property of the collineation if it carried over some particular triangle into itself. We shall be concerned only with the projective properties of collineations.

As an example, let us consider the *fixed points* of the collineation, that is points whose initial and final position is the same. In order that (x_1, x_2, x_3) be a fixed point it is necessary and sufficient that

$$x_1' = \lambda x_1, \qquad x_2' = \lambda x_2, \qquad x_3' = \lambda x_3,$$

that is, substituting in a, that a constant λ exist such that,

$$(1) \qquad \begin{aligned} (a_{11} - \lambda)x_1 + \quad a_{12}x_2 + \quad a_{13}x_3 &= 0, \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \quad a_{23}x_3 &= 0, \\ a_{31}x_1 + \quad a_{32}x_2 + (a_{33} - \lambda)x_3 &= 0. \end{aligned}$$

The matrix of this system of equations is precisely what we have called the characteristic matrix of the matrix a of the linear transformation. The characteristic function is a polynomial of the third degree in λ which, when equated to zero, has one, two, or three distinct roots. Let λ_1 be one of these roots. When this is substituted in (1), these equations are satisfied by the coördinates of one or more points,—the fixed points of the collineation a. The number and distribution of these fixed points give an important example of a projective property of a collineation; and it is readily seen that collineations may have wholly different properties in this respect, one having three fixed points, another two, and still another an infinite number.

Coming back now to the two sets of points $A_1, A_2, \cdots$ and $A_1', A_2', \cdots$ which correspond to one another by means of the collineation **a** (which may be singular or non-singular), let us subject all these points to a non-singular collineation **c**, which carries over $A_1, A_2, \cdots$ into $B_1, B_2, \cdots$ and $A_1', A_2', \cdots$ into $B_1', B_2', \cdots$ respectively. The figure formed by the B's will have the same projective properties as that formed by the A's; and consequently if we can find a collineation **b** which carries over $B_1, B_2, \cdots$ into $B_1', B_2', \cdots$, this collineation will have the same projective properties as the collineation **a**. Such a collineation is clearly given by the formula

$$(2) \qquad \mathbf{b} = \mathbf{c}\mathbf{a}\mathbf{c}^{-1}$$

since $\mathbf{c}^{-1}$ carries over the points B_i into the points A_i, **a** then carries over these into A_i', and **c** carries over the points A_i' into the points B_i'.

Since two collineations **a** and **b** related by formula (2) are indistinguishable so far as their projective properties go (though they may have very different metrical properties), we will call them equivalent according to the following

DEFINITION. *Two collineations* **a** *and* **b** *shall be called equivalent if a non-singular collineation* **c** *exists such that relation* (2) *is fulfilled.*

A reference to Theorem 3, § 96, now gives us the fundamental theorem:

THEOREM. *A necessary and sufficient condition that two collineations be equivalent is that their characteristic matrices have the same invariant factors, — or, if we prefer, the same elementary divisors.*

EXERCISES

1. If $P_1, P_2, \cdots P_k$ are fixed points of a non-singular collineation in space of $n-1$ dimensions which correspond to k distinct roots of the characteristic equation, prove that these points are linearly independent.

2. Discuss the distribution of the fixed points of a collineation
   (*a*) in two dimensions,
   (*b*) in three dimensions,
for all possible cases of non-singular collineations.

3. Discuss the distribution of
   (*a*) the fixed lines of a collineation in two dimensions,
   (*b*) the fixed planes of a collineation in three dimensions,
for all possible cases of non-singular collineations; paying special attention to their relation to the fixed points.

4. Two real collineations, **a** and **b**, may be said to be equivalent if there exists a real non-singular collineation **c** such that $\mathbf{b} = \mathbf{c}\mathbf{a}\mathbf{c}^{-1}$.

With this understanding of the term *equivalence*, show that the theorem of the present section holds for real collineations.

99. Classification of Pairs of Bilinear Forms. We consider again the pair of bilinear forms

$$\phi \equiv \sum_1^n a_{ij}x_iy_j, \qquad \psi \equiv \sum_1^n b_{ij}x_iy_j,$$

of which we assume the second to be non-singular, and form the λ-matrix.

$$(1) \qquad \mathbf{a} - \lambda\mathbf{b}.$$

Using a slightly different notation from that employed in § 92, we will denote the elementary divisors of (1) by

$$(\lambda-\lambda_1)^{e_1}, \quad (\lambda-\lambda_2)^{e_2}, \quad \cdots\cdots \quad (\lambda-\lambda_k)^{e_k}, \qquad (e_1+e_2+\cdots+e_k=n),$$

so that the linear factors $\lambda - \lambda_i$ need not all be distinct from one another. The most important thing concerning these elementary divisors is, for many purposes, their *degrees*, $e_1, e_2, \cdots e_k$. When we wish to indicate these degrees without writing out the elementary divisors in full, we will use the symbol $[e_1\, e_2 \cdots e_k]$, called the *characteristic* of the λ-matrix (1), or of the pair of forms ϕ, ψ. It will be seen that this characteristic is a sort of arithmetical invariant of the pair of bilinear forms, since two pairs of bilinear forms which are equivalent necessarily have the same characteristic. The converse of this, however, is not true, since for the equivalence of two pairs of bilinear forms the identity of the elementary divisors themselves, not merely the equality of their degrees, is necessary.

All pairs of bilinear forms which have the same characteristic are said to form a *category*. Thus, for example, in the case of pairs of bilinear forms in six variables we should distinguish between three categories corresponding to the three characteristics,

$$[1\ 1\ 1], \qquad [2\ 1], \qquad [3],$$

which are obviously the only possible ones in this case. In fact, we must inquire whether these three categories really all exist. This question we answer in the affirmative by writing down the following pairs of bilinear forms in six variables which represent these three categories:

$$\text{I. } [1\,1\,1] \quad \begin{cases} \lambda_1x_1y_1 + \lambda_2x_2y_2 + \lambda_3x_3y_3, \\ \quad x_1y_1 + \quad x_2y_2 + \quad x_3y_3, \end{cases}$$

$$\left\| \begin{matrix} \lambda_1-\lambda & 0 & 0 \\ 0 & \lambda_2-\lambda & 0 \\ 0 & 0 & \lambda_3-\lambda \end{matrix} \right\|.$$

$$\text{II. } [2\ 1] \quad \begin{cases} \lambda_1 x_1 y_1 + \lambda_1 x_2 y_2 + x_1 y_2 + \lambda_2 x_3 y_3, \\ x_1 y_1 + \ \ x_2 y_2 \qquad\quad + \ \ x_3 y_3, \end{cases}$$

$$\left\| \begin{matrix} \lambda_1 - \lambda & 1 & 0 \\ 0 & \lambda_1 - \lambda & 0 \\ 0 & 0 & \lambda_2 - \lambda \end{matrix} \right\|.$$

$$\text{III. } [3] \quad \begin{cases} \lambda_1 x_1 y_1 + \lambda_1 x_2 y_2 + \lambda_1 x_3 y_3 + x_1 y_2 + x_2 y_3, \\ x_1 y_1 + \ \ x_2 y_2 + \ \ x_3 y_3, \end{cases}$$

$$\left\| \begin{matrix} \lambda_1 - \lambda & 1 & 0 \\ 0 & \lambda_1 - \lambda & 1 \\ 0 & 0 & \lambda_1 - \lambda \end{matrix} \right\|.$$

The pairs of bilinear forms we have just written down do more than merely establish the existence of our three categories. They establish the fact that not only the degrees of the elementary divisors are arbitrary (subject merely to the condition that their sum be three), but that, subject to this restriction, the elementary divisors themselves may be arbitrarily chosen. They are, moreover, *normal forms* to one or the other of which every pair of bilinear forms in six variables, of which the first is non-singular, may be reduced by non-singular linear transformations.

The general theorem here is this:

THEOREM. *If* $\lambda_1, \lambda_2, \cdots \lambda_k$ *are any constants, equal or unequal, and* $e_1, e_2, \cdots e_k$ *are any positive integers whose sum is* n, *there exist pairs of bilinear forms in* $2n$ *variables, the second form in each pair being non-singular, which have the elementary divisors*

$$(2) \qquad (\lambda - \lambda_1)^{e_1}, \quad (\lambda - \lambda_2)^{e_2}, \quad \cdots\cdots \quad (\lambda - \lambda_k)^{e_k}.$$

The proof of this theorem consists in considering the pair of bilinear forms

$$(3) \quad \begin{cases} \phi \equiv \left(\sum\limits_1^{e_1} \lambda_1 x_i y_i + \sum\limits_2^{e_1} x_{i-1} y_i \right) + \left(\sum\limits_{e_1+1}^{e_1+e_2} \lambda_2 x_i y_i + \sum\limits_{e_1+2}^{e_1+e_2} x_{i-1} y_i \right) \\ \qquad\qquad + \cdots\cdots + \left(\sum\limits_{n-e_k+1}^{n} \lambda_k x_i y_i + \sum\limits_{n-e_k+2}^{n} x_{i-1} y_i \right), \\ \psi \equiv x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \end{cases}$$

of which the second is non-singular. These forms have a **λ**-matrix which may be indicated, for brevity, as

$$(4) \qquad \left\| \begin{matrix} \mathbf{M}_1 & & & \\ & \mathbf{M}_2 & & \\ & & \ddots & \\ & & & \mathbf{M}_k \end{matrix} \right\|,$$

where the letters $\mathbf{M}_1, \cdots \mathbf{M}_k$ represent not single terms but blocks of terms; $\mathbf{M}_i$ standing for the matrix of order e_i

$$\mathbf{M}_i = \left\| \begin{matrix} \lambda_i - \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i - \lambda & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \lambda_i - \lambda \end{matrix} \right\|;$$

while all the terms of the matrix (4) are zero which do not stand in one of the blocks of terms $\mathbf{M}_i$. The elementary divisors of (4) are, as we see by a reference to § 93 (Formula (1) and Theorem 2), precisely the expressions (2). Thus our theorem is proved.

A reference to § 97 shows that formula (3) is a normal form to which every pair of bilinear forms in $2n$ variables with the elementary divisors (2) can be reduced.*

* Many other normal forms might be chosen in place of (3). Thus, for instance, we might have used in place of (3) the form

$$(3') \quad \begin{cases} \phi \equiv \left(\sum\limits_1^{e_1} \lambda_1 c_1 x_i y_{e_1 - i + 1} + \sum\limits_1^{e_1 - 1} d_1 x_i y_{e_1 - i} \right) + \left(\sum\limits_{e_1+1}^{e_1+e_2} \lambda_2 c_2 x_i y_{2e_1 + e_2 - i + 1} + \sum\limits_{e_1+1}^{e_1+e_2-1} d_2 x_i y_{2e_1 + e_2 - i} \right) \\ \qquad + \cdots + \left(\sum\limits_{n-e_k+1}^{n} \lambda_k c_k x_i y_{2n - e_k - i + 1} + \sum\limits_{n-e_k+1}^{n-1} d_k x_i y_{2n - e_k - i} \right), \\ \psi \equiv \sum\limits_1^{e_1} c_1 x_i y_{e_1 - i + 1} + \sum\limits_{e_1+1}^{e_1+e_2} c_2 x_i y_{2e_1 + e_2 - i + 1} + \sum\limits_{e_1+e_2+1}^{e_1+e_2+e_3} c_3 x_i y_{2e_1 + 2e_2 + e_3 - i + 1} \\ \qquad + \cdots + \sum\limits_{n-e_k+1}^{n} c_k x_i y_{2n - e_k - i + 1}, \end{cases}$$

where the constants $c_1, \cdots c_k, d_1, \cdots d_k$ may be chosen at pleasure provided, merely, that none of them are zero. For instance, they may all be assigned the value 1.

Let us now return to the classification of pairs of bilinear forms. For a given number, $2n$, of variables we have obviously only a finite number of categories. We may subdivide these categories into *classes* by noticing which, if any, of the elementary divisors correspond to the same linear factor. This we can indicate in the characteristic by connecting by parentheses those integers which are the degrees of elementary divisors corresponding to one and the same linear factor. Thus, in the case $n = 8$, the characteristic

$$[(2\,1)(1\,1\,1)2]$$

would indicate that the λ-matrix has just three distinct linear factors; that to one of these there correspond two elementary divisors of degrees two and one respectively, to another three elementary divisors of the first degree, and to the last a single elementary divisor of degree two.

Two pairs of bilinear forms which are equivalent belong necessarily to the same class, but two pairs of bilinear forms which belong to the same class are not necessarily equivalent.

To illustrate what has just been said, let us again consider the case $n = 3$. Here we have now, instead of three categories, six classes, which are exhibited in the following table:

	a	*b*	*c*
I.	[1 1 1]	[(1 1) 1]	[(1 1 1)]
II.	[2 1]	[(2 1)]	
III.	[3]		

The λ-matrix of this pair of forms may be written in the form (4), where, however, $\mathbf{M}_i$ now stands for the matrix of order e_i:

$$\mathbf{M}_i = \left\| \begin{matrix} 0 & \cdots & 0 & d_i & c_i(\lambda_i - \lambda) \\ 0 & \cdots & d_i & c_i(\lambda_i - \lambda) & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_i(\lambda_i - \lambda) & \cdots & 0 & 0 & 0 \end{matrix} \right\|.$$

It will be noticed that the matrices $\mathbf{M}_i$, and therefore also the bilinear forms (3′), are symmetrical, a fact which will make this normal form important when we come to the subject of quadratic forms in the next chapter.

Constants similar to the constants c_i and d_i which we have introduced in (3′) might also have been introduced in (3).

The three classes I*a*, I*b*, I*c* form together the category I, and are all represented by the normal form given for that category above, the only difference being that in class I*a* the three quantities λ_1, λ_2, λ_3 are all distinct, in class I*b* two, and only two, of them are equal, while in class I*c* they are all equal. Similarly category II is now divided into two classes, II*a* and II*b*, for both of which the normal form of category II holds good, λ_1 and λ_2 being, however, different in that normal form for class II*a* and equal for class II*b*. Finally category III consists of only a single class.

For some purposes it is desirable to carry this subdivision still farther. The second of our two bilinear forms, ψ, has been assumed throughout to be non-singular. The first, ϕ, may be singular or non-singular; and it is readily seen that a necessary and sufficient condition that ϕ be singular is that one, at least, of the constants λ, which enter into the linear factors of the λ-matrix be zero. Thus it will be seen that in a single class we shall have pairs of bilinear forms both of which are non-singular and others one of which is singular, and we may wish to separate into different sub-classes the pairs of forms which belong to one or the other of these two cases.

Let us go a step farther in this same direction, and inquire how the rank of ϕ is connected with the values of the constants λ_i. We notice that the matrix of ϕ is equal to the matrix of the pencil $\phi - \lambda\psi$ when $\lambda = 0$. Accordingly, if ϕ is of rank r, every $(r+1)$-rowed determinant of the matrix of $\phi - \lambda\psi$ will be divisible by λ, while at least one r-rowed determinant of this matrix is not divisible by λ. It is then necessary, as we see by a reference to the definition of elementary divisors (cf. the footnote to Definition 3, § 92), that just $n - r$ of the constants λ_i which enter into the elementary divisors should be zero. Since the converse of these statements is also true, we may say that *a necessary and sufficient condition that the form ϕ be of rank r is that just $n - r$ of the elementary divisors be of the form* λ^{e_i}. Let us, in the characteristic $[e_1\ e_2 \cdots e_k]$, place a small zero above each of the integers e_i which is the degree of such an elementary divisor; and regard two pairs of bilinear forms as belonging to a single class when, and only when, their characteristics coincide in the distribution of these zeros as well as in other respects. Here again two equivalent pairs of forms will always belong to the same class, but the converse will not be true.

As an illustration, let us again take the case $n=3$. We have now fourteen classes instead of six.

$$[1\,1\,1],\ [(1\,1)1],\ [(1\,1\,1)],\ [2\,1],\ [(2\,1)],\ [3], \qquad (r=3),$$
$$[1\,1\,\overset{0}{1}],\ [(1\,1)\overset{0}{1}],\ [2\,\overset{0}{1}],\ [\overset{0}{2}\,1],\ [\overset{0}{3}], \qquad (r=2),$$
$$[(\overset{0}{1}\,\overset{0}{1})\,1],\ [(\overset{0}{2}\,\overset{0}{1})], \qquad (r=1),$$
$$[(\overset{0}{1}\,\overset{0}{1}\,\overset{0}{1})], \qquad (r=0).$$

We have indicated, in each case, the rank r of the form ϕ. Thus in the first six cases ϕ is non-singular; in the next five it is of rank 2, etc.

EXERCISES

1. Prove that there exist pairs of real bilinear forms in $2n$ variables of which the second is non-singular, and which have the elementary divisors

$$(\lambda-\lambda_1)^{e_1},\quad (\lambda-\lambda_2)^{e_2},\quad \cdots\cdots\quad (\lambda-\lambda_k)^{e_k} \qquad (e_1+e_2+\cdots+e_k=n),$$

provided that such of these elementary divisors as are not real admit of arrangement in conjugate imaginary pairs. (Cf. Exercises 1, 2, § 93.)

2. Classify pairs of real bilinear forms in six variables (the second form in each pair being non-singular), distinguishing between real and imaginary elementary divisors.

100. Classification of Collineations. The classification of pairs of bilinear forms which we gave in the last section may obviously be regarded, from a more general point of view, as a classification of pairs of matrices, the second matrix of each pair being assumed to be non-singular. From this point of view it admits of application to the classification of collineations, since, as we saw in § 98, to every collineation corresponds a pair of matrices of which one is non-singular, namely the unit matrix **I** and the matrix of the linear transformation. Moreover, the normal form (3) of § 99 is precisely adapted to the treatment of the more special kind of equivalence which we have to consider here, since the matrix of the form ψ is precisely the unit matrix. We may therefore state at once the fundamental theorem:

THEOREM 1. *If $\lambda_1, \lambda_2, \cdots \lambda_k$ are any constants, equal or unequal, and $e_1, e_2, \cdots e_k$ any positive integers whose sum is n, there exists a collineation in space of $n-1$ dimensions whose characteristic matrix has the elementary divisors*

$$(\lambda-\lambda_1)^{e_1},\quad (\lambda-\lambda_2)^{e_2},\quad \cdots\cdots\ (\lambda-\lambda_k)^{e_k}.$$

To this we may add

THEOREM 2. *Every collineation of the kind mentioned in Theorem 1 is equivalent to the collineation whose matrix is*

$$\left\|\begin{array}{cccc} \mathbf{M}_1 & & & \\ & \mathbf{M}_2 & & \\ & & \ddots & \\ & & & \mathbf{M}_k \end{array}\right\|,$$

where $\mathbf{M}_i$ stands for the matrix of order e_i,

$$\mathbf{M}_i = \left\|\begin{array}{ccccc} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \lambda_i \end{array}\right\|.$$

We thus get a classification of collineations into categories and a subdivision of these categories into classes precisely as in § 99. For instance, in the case $n=3$ (collineations in the plane), we have three categories whose characteristics and representative normal forms we give:

I. [1 1 1] $\quad \begin{cases} x_1' = \lambda_1 x_1 \\ x_2' = \quad\ \lambda_2 x_2 \\ x_3' = \quad\quad\ \lambda_3 x_3, \end{cases}$

II. [2 1] $\quad \begin{cases} x_1' = \lambda_1 x_1 + x_2 \\ x_2' = \quad\ \lambda_1 x_2 \\ x_3' = \quad\quad\ \lambda_2 x_3, \end{cases}$

III. [3] $\quad \begin{cases} x_1' = \lambda_1 x_1 + x_2 \\ x_2' = \quad\ \lambda_1 x_2 + x_3 \\ x_3' = \quad\quad\ \lambda_1 x_3. \end{cases}$

These categories we should then subdivide either into six classes as on page 290 or into fourteen classes as on page 292. This latter classification is the desirable one in this case. We proceed to give a list of these fourteen classes with a characteristic property of each.

That the normal forms of the collineations have these properties will be at once evident, and from this it follows that all the collineations of the class have the property in question, since the properties mentioned are obviously all projective. That the properties mentioned are really *characteristic* properties, that is, serve to distinguish one class from another, can only be seen *a posteriori*, by noticing that no one of the properties mentioned is shared by two classes.

$[1\,1\,1]$ Three distinct non-collinear fixed points.*

$[(1\,1)\,1]$ Every point of a certain line and one point not on this line are fixed.

$[(1\,1\,1)]$ The identical collineation.

$[2\,1]$ Two distinct fixed points.

$[(2\,1)]$ Every point of a certain line is fixed.

$[3]$ One fixed point.

In all these cases the collineation is non-singular. The remaining collineations are singular. In the next three, one point P of the plane is not transformed at all, while all other points go over on to a line p which does not pass through P, and every one of whose points corresponds to an infinite number of points.

$[1\,1\,\overset{0}{1}]$ There are two fixed points on p.

$[(1\,1)\,\overset{0}{1}]$ Every point on p is fixed.

$[2\,\overset{0}{1}]$ One fixed point on p.

In the next two cases one point P is not transformed at all, while all other points go over on to a line p which passes through P, and every one of whose points corresponds to an infinite number of points.

$[\overset{0}{2}\,1]$ One fixed point.

$[\overset{0}{3}]$ No fixed point.

The remaining collineations are so simple that they are not merely characterized, but completely described, by the property we mention.

$[(\overset{0}{1}\,\overset{0}{1})\,1]$ The points on a certain line are not transformed. All other points go over into a single point which does not lie on this line.

* It should be understood here and in what follows that the fixed points which are mentioned are the only fixed points of the collineation in question.

$[(\overset{0}{2}\,\overset{0}{1})]$ The points on a certain line are not transformed. All other points go over into a single point on this line.

$[(\overset{0}{1}\,\overset{0}{1}\,\overset{0}{1})]$ No point in the plane is transformed.

This last case is of course not a transformation at all.

EXERCISES

1. Classify, in a similar manner, the projective transformations in one dimension.

2. Classify the collineations in space of three dimensions.

3. Classify the real projective transformations in space of one, two, and three dimensions. (Cf. Exercises 1, 2, § 99.)