CHAPTER XIX

POLYNOMIALS SYMMETRIC IN PAIRS OF VARIABLES

87. Fundamental Conceptions. Σ and S Functions. The variables $(x_1, \cdots x_n)$ which we used in the last chapter may be regarded, if we wish, not as the coördinates of a point in space of n dimensions, but rather as the coördinates of n points on a line. In fact this is the interpretation which is naturally suggested to us by the ordinary applications of the theory of symmetric functions (cf. § 86). Looked at from this point of view, it is natural to generalize the conception of symmetric functions by considering n points in a plane,

$$(1) \qquad (x_1, y_1), (x_2, y_2), \cdots (x_n, y_n).$$

DEFINITION. *A polynomial,*

$$F(x_1, y_1; x_2, y_2; \cdots x_n, y_n)$$

in the coördinates of the points (1) *is said to be a symmetric polynomial in these pairs of variables if it is unchanged by every interchange of these pairs of variables.*

As in the case of points on a line, we see that it is not necessary to consider all the possible permutations of the subscripts in order to show that a polynomial F is symmetric. It is sufficient to show that F is unchanged by the interchange of every pair of the points (1).

We will introduce the Σ notation here precisely as in the case of single variables. Thus, for example,

$$\Sigma x_1^{\alpha_1}y_1^{\beta_1} \equiv x_1^{\alpha_1}y_1^{\beta_1} + x_2^{\alpha_1}y_2^{\beta_1} + \cdots + x_n^{\alpha_1}y_n^{\beta_1},$$

$$\Sigma x_1^{\alpha_1}y_1^{\beta_1}x_2^{\alpha_2}y_2^{\beta_2} \equiv x_1^{\alpha_1}y_1^{\beta_1}x_2^{\alpha_2}y_2^{\beta_2} + x_1^{\alpha_1}y_1^{\beta_1}x_3^{\alpha_2}y_3^{\beta_2} + \cdots,$$

and so on.

As in the case of single variables, it is clear that the order in which the pairs of exponents α_1, β_1; α_2, β_2; ... are written is immaterial; and also that *every symmetric polynomial in the pairs of variables* (1) *is a linear combination of a certain number of* Σ's.

We introduce the notation

$$S_{kl} \equiv \Sigma x_1^k y_1^l \equiv x_1^k y_1^l + x_2^k y_2^l + \cdots + x_n^k y_n^l \qquad \begin{pmatrix} k = 0, 1, \cdots \\ l = 0, 1, \cdots \end{pmatrix}.$$

THEOREM. *Any symmetric polynomial* $F(x_1, y_1; \cdots x_n, y_n)$ *may be expressed as a polynomial in these* S's.

The proof of this theorem is exactly like that of Theorem 3, § 83, and is left to the reader.

88. Elementary Symmetric Functions of Pairs of Variables. Every Σ function of n pairs of variables may, by giving to the α's and β's suitable values, be written in the form

$$(1) \qquad \Sigma x_1^{\alpha_1}y_1^{\beta_1}x_2^{\alpha_2}y_2^{\beta_2} \cdots x_n^{\alpha_n}y_n^{\beta_n}.$$

DEFINITION. *The function* (1) *is said to be an elementary symmetric function of the pairs of variables* $(x_1, y_1), \cdots (x_n, y_n)$ *when, and only when,*

$$\alpha_i + \beta_i = 0 \text{ or } 1 \qquad (i = 1, 2, \cdots n),$$

but not all the α's *and* β's *are zero.*

We shall adopt the following notation for these elementary symmetric functions:

$$p_{10} \equiv \Sigma x_1, \qquad p_{01} \equiv \Sigma y_1,$$

$$p_{20} \equiv \Sigma x_1x_2, \qquad p_{11} \equiv \Sigma x_1y_2, \qquad p_{02} \equiv \Sigma y_1y_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$p_{n0} \equiv x_1x_2 \cdots x_n, \ \cdots \ p_{i,n-i} \equiv \Sigma x_1 \cdots x_i y_{i+1} \cdots y_n, \ \cdots \ p_{0n} \equiv y_1y_2 \cdots y_n.$$

It is clear that there are a finite number, $\frac{1}{2}n(n+3)$, of p_{ij}'s, but an infinite number of S_{ij}'s.

We will attach to each p a *weight with regard to the* x's equal to its first subscript and a *weight with regard to the* y's equal to its second subscript. When we speak simply of the weight of p_{ij} we will mean its *total weight*, that is, the sum of its subscripts.

THEOREM. *Any symmetric polynomial* $F(x_1, y_1; \cdots x_n, y_n)$ *may be expressed as a polynomial in the* p_{ij}'s.

Since, by the theorem in § 87, any such polynomial may be expressed as a polynomial in the S_{ij}'s, it is sufficient to show that the S_{ij}'s may be expressed as polynomials in the p_{ij}'s.

Let $\quad \xi_1 \equiv \lambda x_1 + \mu y_1, \ \xi_2 \equiv \lambda x_2 + \mu y_2, \cdots \xi_n \equiv \lambda x_n + \mu y_n,$

and form the elementary symmetric functions of these ξ's:

$$\pi_1 \equiv \Sigma \xi_1 \equiv \lambda \Sigma x_1 + \mu \Sigma y_1 \equiv \lambda p_{10} + \mu p_{01},$$
$$\pi_2 \equiv \Sigma \xi_1 \xi_2 \equiv \Sigma(\lambda x_1 + \mu y_1)(\lambda x_2 + \mu y_2)$$
$$\equiv \lambda^2 \Sigma x_1 x_2 + \lambda\mu \Sigma x_1 y_2 + \mu^2 \Sigma y_1 y_2$$
$$\equiv \lambda^2 p_{20} + \lambda\mu p_{11} + \mu^2 p_{02},$$
$$\pi_3 \equiv \Sigma \xi_1 \xi_2 \xi_3 \equiv \lambda^3 p_{30} + \lambda^2\mu p_{21} + \lambda\mu^2 p_{12} + \mu^3 p_{03},$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$\pi_n \equiv \xi_1 \xi_2 \cdots \xi_n \equiv \lambda^n p_{n0} + \lambda^{n-1}\mu p_{n-1,1} + \lambda^{n-2}\mu^2 p_{n-2,2} + \cdots + \mu^n p_{0n}.$$

Also let $\qquad \sigma_k \equiv \Sigma \xi_1^k \qquad (k = 1, 2, \cdots)$

Let α and β be positive integers, or zero, but not both zero.

$$\text{Then } \sigma_{\alpha+\beta} \equiv \Sigma \xi_1^{\alpha+\beta} \equiv \lambda^{\alpha+\beta} \Sigma x_1^{\alpha+\beta} + \lambda^{\alpha+\beta-1}\mu \Sigma x_1^{\alpha+\beta-1} y_1 + \cdots$$
$$\equiv \lambda^{\alpha+\beta} S_{\alpha+\beta,0} + \lambda^{\alpha+\beta-1}\mu S_{\alpha+\beta-1,1} + \cdots.$$

But by Theorem 1, § 84, we may write

$$\sigma_{\alpha+\beta} \equiv F(\pi_1, \pi_2, \cdots \pi_n),$$

where F is a polynomial. Hence

$$\lambda^{\alpha+\beta} S_{\alpha+\beta,0} + \lambda^{\alpha+\beta-1}\mu S_{\alpha+\beta-1,1} + \cdots \equiv \Psi(p_{10}, \cdots p_{0n}, \lambda, \mu),$$

where Ψ is a polynomial. Regarding this as an identity in (λ, μ) and equating the coefficients of the terms containing $\lambda^\alpha \mu^\beta$, we get an identity in the x's and y's,

$$S_{\alpha\beta} \equiv \Phi(p_{10}, \cdots p_{0n}),$$

where Φ is a polynomial in the p's. Thus our theorem is proved.

Theorem 3, § 84, does not hold in the case of pairs of variables, as relations between the $\frac{1}{2}n(n+3)$ p_{ij}'s do exist; for example, if $n = 2$, the polynomial

$$4p_{20}p_{02} - p_{20}p_{01}^2 - p_{10}^2 p_{02} + p_{10}p_{11}p_{01} - p_{11}^2$$

vanishes identically when the p's are replaced by their values in terms of the x's. It does not vanish identically when $n = 3$.

In view of the remark just made, it is clear that the representations of polynomials in pairs of variables in terms of the p_{ij}'s will not be unique.

For further information concerning the subjects treated in this section, the reader may consult Netto's *Algebra*, Vol. 2, p. 63.

EXERCISES

1. Prove that a polynomial symmetric in the pairs of variables (x_i, y_i) and which is homogeneous in the x's alone of degree n and in the y's alone of degree m can be expressed as a polynomial in the p_{ij}'s isobaric of weight n with regard to the x's, and m with regard to the y's.

2. Express the symmetric polynomial

$$\Sigma x_1^2 y_2 y_3$$

in terms of the p_{ij}'s by the method of undetermined coefficients, making use of Exercise 1.

3. A polynomial in $(x_1, y_1, z_1;\ x_2, y_2, z_2;\ \cdots x_n, y_n, z_n)$ which is unchanged by every interchange of the subscripts is called a symmetric polynomial in the n points (x_i, y_i, z_i).

Extend the results of this section and the last to polynomials of this sort.

89. Binary Symmetric Functions. The pairs of variables (x_1, y_1), $\cdots (x_n, y_n)$ may be regarded as the homogeneous coördinates of n points on a line as well as the non-homogeneous coördinates of n points in a plane. It will then be natural to consider only symmetric polynomials which are homogeneous in each pair of variables alone. Such polynomials we will call *binary symmetric functions*. Most of the p_{ij}'s of the last section are thus excluded. The last $n+1$ of them $(p_{n0}, p_{n-1,1}, \cdots p_{0n})$, however, are homogeneous of the first degree in each pair of variables alone. We will call them the *elementary binary symmetric functions*.

THEOREM 1. *Any binary symmetric function in $(x_1, y_1;\ \cdots x_n, y_n)$ can be expressed as a polynomial in $(p_{n0}, p_{n-1,1}, \cdots p_{0n})$.*

If we break up our binary symmetric function into Σ's, it is clear that each of these Σ's will itself be a binary symmetric function, or, as we will say for brevity, a binary Σ. It is therefore sufficient to prove that our theorem is true for every binary Σ. The general binary Σ may be written

$$\Sigma x_1^{\alpha_1} y_1^{\beta_1} x_2^{\alpha_2} y_2^{\beta_2} \cdots x_n^{\alpha_n} y_n^{\beta_n} \qquad (\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n),$$

where, if we denote by m the degree of this Σ in any one of the pairs of variables,

$$m = \alpha_1 + \beta_1 = \alpha_2 + \beta_2 = \cdots = \alpha_n + \beta_n.$$

Let us assume for the moment that none of the y's are zero, and let

$$X_1 = \frac{x_1}{y_1},\ X_2 = \frac{x_2}{y_2}, \cdots X_n = \frac{x_n}{y_n}.$$

Now consider the elementary symmetric functions of these X's:

$$P_1 = \Sigma X_1 \qquad = \frac{p_{1,n-1}}{p_{0n}},$$

$$P_2 = \Sigma X_1 X_2 \qquad = \frac{p_{2,n-2}}{p_{0n}},$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

$$P_n = X_1 X_2 \cdots X_n = \frac{p_{n0}}{p_{0n}}.$$

We may write

$$(1) \qquad \frac{\Sigma x_1^{\alpha_1} y_1^{\beta_1} x_2^{\alpha_2} y_2^{\beta_2} \cdots x_n^{\alpha_n} y_n^{\beta_n}}{p_{0n}^m} = \Sigma X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} = \Phi(P_1, \cdots P_n),$$

where, since we have assumed $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n$, Φ is a polynomial of degree α_1 in the P's (Theorem 2, § 85). Hence we may write

$$(2) \qquad \Phi(P_1, \cdots P_n) = \frac{\phi(p_{0n}, p_{1,n-1}, \cdots p_{n0})}{p_{0n}^{\alpha_1}},$$

where ϕ is a homogeneous polynomial of degree α_1.

We thus get from (1) and (2)

$$(3) \qquad \Sigma x_1^{\alpha_1} y_1^{\beta_1} \cdots x_n^{\alpha_n} y_n^{\beta_n} = p_{0n}^{\beta_1} \phi(p_{0n}, p_{1,n-1}, \cdots p_{n0}),$$

an equation which holds except when one of the y's is zero. Since each side of (3) can be regarded as a polynomial in the x's and y's, we infer, by Theorem 5, § 2, that this is an identity, and our theorem is proved.

By Theorem 1, § 85, Φ is isobaric of weight $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ in the P's. Hence $\Sigma x_1^{\alpha_1} y_1^{\beta_1} \cdots x_n^{\alpha_n} y_n^{\beta_n}$, when expressed in terms of these $(n+1)\, p_{ij}$'s, is isobaric of weight $\alpha_1 + \alpha_2 + \cdots + \alpha_n$, provided we count the weight of the p_{ij}'s with regard to the x's. Passing back now to an aggregate of a number of such Σ's, we get

THEOREM 2. *If a binary symmetric function is homogeneous in the n x's (or y's) of degree k, it will, when expressed in terms of* $p_{n0}, p_{n-1,1}, \cdots p_{0n}$, *be isobaric of weight k with regard to the x's (or y's).*

We have seen in the proof of Theorem 1 that the polynomial ϕ in (3) is a homogeneous polynomial of degree α_1 in the p's; so that $\Sigma x_1^{\alpha_1} y_1^{\beta_1} \cdots x_n^{\alpha_n} y_n^{\beta_n}$ is a homogeneous polynomial of degree $\alpha_1 + \beta_1 = m$ in the p's. Hence

THEOREM 3. *Any binary symmetric function of degree m in each pair of variables will, when written in terms of* $p_{n0}, p_{n-1,1}, \cdots p_{0n}$ *be a homogeneous polynomial of degree m in these p's.*

EXERCISES

1. Prove that no rational relation exists between $p_{n0}, \cdots p_{0n}$, and hence that a binary symmetric function can be expressed as a polynomial in them in only one way.

2. By a ternary symmetric function is meant a symmetric polynomial in n points (x_i, y_i, z_i) which is homogeneous in the coördinates of each point.

Extend the results of this section to ternary symmetric functions. Cf. Exercise 3, § 88.

90. Resultants and Discriminants of Binary Forms. It is the object of the present section to show how the subject of the resultants and discriminants of binary forms may be approached from the point of view of symmetric functions.

Let

$$f(x_1, x_2) \equiv a_0 x_1^n + a_1 x_1^{n-1} x_2 + \cdots + a_n x_2^n$$
$$\equiv (\alpha_1'' x_1 - \alpha_1' x_2)(\alpha_2'' x_1 - \alpha_2' x_2) \cdots (\alpha_n'' x_1 - \alpha_n' x_2),$$
$$\phi(x_1, x_2) \equiv b_0 x_1^m + b_1 x_1^{m-1} x_2 + \cdots + b_m x_2^m$$
$$\equiv (\beta_1'' x_1 - \beta_1' x_2)(\beta_2'' x_1 - \beta_2' x_2) \cdots (\beta_m'' x_1 - \beta_m' x_2),$$

be two binary forms. Each of these polynomials has here been written first in the unfactored and secondly in the factored form. By a comparison of these two forms we see at once that the elementary binary symmetric fractions of the n points

$$(\alpha_1', \alpha_1''), (\alpha_2', \alpha_2''), \cdots (\alpha_n', \alpha_n'')$$

are $\qquad a_0, -a_1, a_2, \cdots (-1)^n a_n;$

and of the m points $\quad (\beta_1', \beta_1''), (\beta_2', \beta_2''), \cdots (\beta_m', \beta_m'')$

are $\qquad b_0, -b_1, b_2, \cdots (-1)^m b_m.$

Let us now consider the two linear factors

$$\alpha_i'' x_1 - \alpha_i' x_2, \quad \beta_j'' x_1 - \beta_j' x_2.$$

A necessary and sufficient condition for these factors to be proportional is that the determinant

$$\alpha_i'' \beta_j' - \alpha_i' \beta_j''$$

vanish. Let us form the product of all such determinants:

$$P \equiv \left\{ \begin{matrix} (\alpha_1''\beta_1' - \alpha_1'\beta_1'')(\alpha_2''\beta_1' - \alpha_2'\beta_1'') \cdots (\alpha_n''\beta_1' - \alpha_n'\beta_1'') \\ (\alpha_1''\beta_2' - \alpha_1'\beta_2'')(\alpha_2''\beta_2' - \alpha_2'\beta_2'') \cdots (\alpha_n''\beta_2' - \alpha_n'\beta_2'') \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ (\alpha_1''\beta_m' - \alpha_1'\beta_m'')(\alpha_2''\beta_m' - \alpha_2'\beta_m'') \cdots (\alpha_n''\beta_m' - \alpha_n'\beta_m'') \end{matrix} \right\}.$$

The vanishing of this product is a necessary and sufficient condition that at least one of the linear factors of f be proportional to one of the linear factors of ϕ, that is, that f and ϕ have a common factor which is not a constant.

We may obviously reduce P to the simple form

$$P \equiv f(\beta_1', \beta_1'')f(\beta_2', \beta_2'') \cdots f(\beta_m', \beta_m'').$$

In this form it appears as a homogeneous polynomial of the mth degree in the a's, and as a symmetric polynomial in the m points (β_i', β_i''). Moreover, it is obviously a binary symmetric function which is of the nth degree in the coördinates of each point. Consequently, by Theorem 3, § 89, it can be expressed as a homogeneous polynomial of the nth degree in the elementary binary symmetric functions of the points (β_i', β_i''), that is, in the b's. Thus we have shown that *the product P can be expressed as a polynomial in the a's and b's which is homogeneous in the a's of degree m and in the b's of degree n.*

In § 72 we found another polynomial in the a's and b's, whose vanishing also gives a necessary and sufficient condition for f and ϕ to have a common factor, namely, the resultant R. We will now identify these polynomials by means of the following theorem:

THEOREM 1. *The product P differs from the resultant R of f and ϕ only by a constant factor, and the resultant is an irreducible polynomial in the a's and b's.*

We may show, in exactly the same way as in the proof of Theorem 1, § 86, that P, when expressed as a polynomial in the a's and b's, is irreducible. Since $P = 0$ and $R = 0$ each give a necessary and sufficient condition for f and ϕ to have a common factor, any set of values of the a's and b's which make $P = 0$ will also make $R = 0$. Thus by Theorem 7, § 76, P is a factor of R. We have seen that P is of degree m in the a's and n in the b's. The same is also true of R, as may easily be seen by inspection of the determinant of § 68. Hence, P being a factor of R, and of the same degree, can differ from it only by a constant factor. Thus our theorem is proved.

Let us now inquire under what conditions the binary form $f(x_1, x_2)$ has a multiple linear factor. Using the same notation as above, we see that the vanishing of the product

$$\left.\begin{matrix}(\alpha_1''\alpha_2' - \alpha_1'\alpha_2'')(\alpha_1''\alpha_3' - \alpha_1'\alpha_3'') \cdots (\alpha_1''\alpha_n' - \alpha_1'\alpha_n'') \\ (\alpha_2''\alpha_3' - \alpha_2'\alpha_3'') \cdots (\alpha_2''\alpha_n' - \alpha_2'\alpha_n'') \\ \cdots\cdots\cdots \\ (\alpha_{n-1}''\alpha_n' - \alpha_{n-1}'\alpha_n'')\end{matrix}\right\} \equiv P_1(\alpha_1', \alpha_1''; \cdots \alpha_n', \alpha_n'')$$

is a necessary and sufficient condition for this. P_1 is not symmetric in the pairs of α's, since an interchange of two subscripts changes P_1 into $-P_1$. If, however, we consider P_1^2 instead of P_1, we have a binary symmetric function which can be expressed as a polynomial in the a's

$$[P_1(\alpha_1', \alpha_1''; \cdots \alpha_n', \alpha_n'')]^2 \equiv F(a_0, \cdots a_n).$$

Moreover, F vanishes when, and only when, P_1 does. Accordingly *$F = 0$ is a necessary and sufficient condition for $f(x_1, x_2)$ to have a multiple linear factor.*

But the vanishing of the discriminant Δ (cf. § 82) of $f(x_1, x_2)$ is also a necessary and sufficient condition for this.

THEOREM 2. *F and Δ differ only by a constant factor, and are irreducible.*

The proof of this theorem, which is practically the same as that of Theorem 1, is left to the reader.

If we subject the two binary forms f and ϕ, which we may suppose written in the factored form, to the linear transformation

$$(1) \qquad \begin{cases} x_1 = c_{11}x_1' + c_{12}x_2', \\ x_2 = c_{21}x_1' + c_{22}x_2', \end{cases}$$

we get two new binary forms

$$(A_1''x_1' - A_1'x_2')(A_2''x_1' - A_2'x_2') \cdots (A_n''x_1' - A_n'x_2'),$$
$$(B_1''x_1' - B_1'x_2')(B_2''x_1' - B_2'x_2') \cdots (B_m''x_1' - B_m'x_2'),$$

where

$$A_i'' = \alpha_i''c_{11} - \alpha_i'c_{21}, \qquad B_j'' = \beta_j''c_{11} - \beta_j'c_{21},$$
$$A_i' = -\alpha_i''c_{12} + \alpha_i'c_{22}, \qquad B_j' = -\beta_j''c_{12} + \beta_j'c_{22},$$

so that

$$A_i''B_j' - A_i'B_j'' \equiv c(\alpha_i''\beta_j' - \alpha_i'\beta_j''),$$

where c is the determinant of the transformation (1).

Since the linear transformation (1) may be regarded as carrying over the α's and β's into the A's and B's, the last written identity shows us that $\alpha_i''\beta_j' - \alpha_i'\beta_j''$ is, in a certain sense, an invariant of weight 1. It can, however, not be expressed rationally in terms of the a's and b's. Such an expression is called an *irrational invariant.*

Since the resultant of f and ϕ is the product of mn such irrational invariants of weight 1, it is evident that the resultant itself is an invariant of weight mn. Thus we get a new proof of this fact, independent of the proof given in § 82.

A similar proof can be used in the case of the discriminant of a binary form.

EXERCISES

Develop the theory of the invariants of the binary biquadratic

$$f(x_1, x_2) \equiv a_0x_1^4 + 4\,a_1x_1^3x_2 + 6\,a_2x_1^2x_2^2 + 4\,a_3x_1x_2^3 + a_4x_2^4$$
$$\equiv (\alpha_1''x_1 - \alpha_1'x_2)(\alpha_2''x_1 - \alpha_2'x_2)(\alpha_3''x_1 - \alpha_3'x_2)(\alpha_4''x_1 - \alpha_4'x_2)$$

along the following lines:

1. Start from the irrational invariants of weight 2,

$$A = (\alpha_1''\alpha_2' - \alpha_1'\alpha_2'')(\alpha_3''\alpha_4' - \alpha_3'\alpha_4''),$$
$$B = (\alpha_1''\alpha_3' - \alpha_1'\alpha_3'')(\alpha_4''\alpha_2' - \alpha_4'\alpha_2''),$$
$$C = (\alpha_1''\alpha_4' - \alpha_1'\alpha_4'')(\alpha_2''\alpha_3' - \alpha_2'\alpha_3''),$$

whose sum is zero, and the negatives of whose ratios are the cross-ratios of the four points (α_1', α_1''), (α_2', α_2''), (α_3', α_3''), (α_4', α_4'').

2. Form the further irrational invariants of weight 2

$$E_1 \equiv B - C, \qquad E_2 \equiv C - A, \qquad E_3 \equiv A - B;$$

and prove that every homogeneous symmetric polynomial in E_1, E_2, E_3 is a binary symmetric function of the four points (α_i', α_i''), and therefore an integral rational invariant of f.

3. In particular

$$G_2 \equiv E_1E_2 + E_2E_3 + E_3E_1, \qquad G_3 \equiv E_1E_2E_3$$

are homogeneous integral rational invariants of weights 4 and 6, and of degrees 2 and 3 respectively. Prove that

$$G_2 \equiv -36g_2, \qquad G_3 \equiv 432g_3,$$

where

$$g_2 \equiv a_0a_4 - 4\,a_1a_3 + 3\,a_2^2,$$
$$g_3 \equiv a_0a_2a_4 + 2\,a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3.$$

These expressions g_2 and g_3 are the simplest invariants of f.*

4. Prove that the discriminant Δ of f is given by the formula

$$\Delta \equiv g_2^3 - 27g_3^2.$$

5. If $\Delta \neq 0$, prove that $g_3 = 0$ is a necessary and sufficient condition that the four points $f = 0$ form a harmonic range; and that $g_2 = 0$ is a necessary and sufficient condition that they form an equianharmonic range. (Cf. Exercise 3, § 33.)

6. Prove that $g_2 = g_3 = 0$ is a necessary and sufficient condition that f have at least a threefold linear factor.†

* They are among the oldest examples of invariants, having been found by Cayley and Boole in 1845.

† Notice that we here have a projective property of the locus $f = 0$ expressed by the vanishing of two integral rational invariants; cf. the closing paragraph of § 81.

7. If λ is the absolute irrational invariant

$$\lambda = -\frac{A}{B},$$

i.e. one of the cross-ratios of the points $f = 0$, prove that the absolute rational invariant

$$I = \frac{g_2^3}{\Delta}$$

can be expressed in the form

$$I = \frac{4}{27}\frac{(\lambda^2 - \lambda + 1)^3}{(\lambda - 1)^2\lambda^2}.$$

8. Prove that a necessary and sufficient condition for the equivalence of two biquadratic binary forms neither of whose discriminants is zero is that the invariant I have the same value for the two forms.

9. Prove that a necessary and sufficient condition for the equivalence with regard to linear transformations with determinant $+1$ of two biquadratic binary forms for which g_2 and g_3 are both different from zero is that the values of g_2 and g_3 be the same for one form as for the other.

10. Prove that if the discriminant of a biquadratic binary form is not zero, the form can be reduced by means of a linear transformation of determinant $+1$ to the normal form

$$4\,x_1^3x_2 - g_2x_1x_2^3 - g_3x_2^4.$$

11. Prove that every integral rational invariant of a biquadratic binary form is a polynomial in g_2 and g_3.

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12. Develop the theory of the invariants of a pair of binary quadratic forms along the same lines as those just sketched for a single biquadratic form.

13. Prove that every integral rational invariant of a pair of quadratic forms in n variables is an integral rational function of the invariants $\Theta_0, \cdots \Theta_n$ of § 57.

[SUGGESTION. Show first that, provided a certain integral rational function of the coefficients of the quadratic form does not vanish, there exists a linear transformation of determinant $+1$ which reduces the pair of forms to

$$\alpha_1x_1^2 + \alpha_2x_2^2 + \cdots + \alpha_nx_n^2,$$
$$\beta_1x_1^2 + \beta_2x_2^2 + \cdots + \beta_nx_n^2.$$

Then show that every integral rational invariant of the pair of quadratic forms can be expressed as a binary symmetric function of (α_1, β_1), (α_2, β_2), $\cdots$ (α_n, β_n), and that the Θ's are precisely the elementary binary symmetric functions.]