CHAPTER XIV

SOME PROPERTIES OF POLYNOMIALS IN GENERAL

60. Factors and Reducibility. In the present section we will introduce certain conceptions of fundamental importance in our subsequent work.

DEFINITION 1. *By a factor or divisor of a polynomial f in any number of variables is understood a polynomial ϕ which satisfies an identity of the form*

$$f \equiv \phi\psi,$$

ψ being also a polynomial.

It will be noticed that every constant different from zero is a factor of every polynomial; that every polynomial is a factor of a polynomial which vanishes identically; while a polynomial which is a mere constant, different from zero, has no factors other than constants.

We note also that a polynomial in $x_1, \cdots x_n$ which is not identically zero cannot have as a factor a polynomial which actually contains any other variables.

The conception of reducibility, which we have already met in a special case (§47), we define as follows:

DEFINITION 2. *A polynomial is said to be reducible if it is identically equal to the product of two polynomials neither of which is a constant.*

In dealing with real polynomials, a narrower determination of the conception of reducibility is usually desirable. We consider, then, what we will call *reducibility in the domain of reals*, a conception which we define as follows:

DEFINITION 3. *A real polynomial is said to be reducible in the domain of reals if it is identically equal to the product of two other real polynomials neither of which is a constant.*

In many branches of algebra still another modification of the conception of reducibility plays an important part. In order to explain this, we first lay down the following definition:

DEFINITION 4. *A set of numbers is said to form a domain of rationality if, when a and b are any numbers of the set, $a+b$, $a-b$, ab, and, so far as $b \neq 0$, a/b are also numbers of the set.*

Thus all numbers, real and imaginary, form a domain of rationality, and the same is true of all real numbers. The simplest of all domains of rationality, apart from the one which contains only the single number zero, is what is known as the natural domain, that is all rational numbers, positive and negative. A more complicated domain of rationality would be the one consisting of all numbers of the form $a+b\sqrt{-1}$, where a and b are not merely real, but rational. These illustrations, which might be multiplied indefinitely, should suffice to make the scope of the above definition clear.*

DEFINITION 5. *A polynomial all of whose coefficients lie in a domain of rationality R is said to be reducible in this domain if it is identically equal to a product of two polynomials, neither of which is a constant, whose coefficients also lie in this domain.*

It will be noticed that Definitions 2 and 3 are merely the special cases of this definition in which the domain of rationality is the domain of all numbers, and the domain of all reals respectively. To illustrate these three definitions, we note that the polynomial x^2+1 is reducible according to Definition 2, since it is identically equal to $(x+\sqrt{-1})(x-\sqrt{-1})$. It is, however, not reducible in the domain of reals, nor in the natural domain. On the other hand, x^2-2 is reducible in the domain of reals, but not in the natural domain. Finally, x^2-4 is reducible in the natural domain.

Leaving these modifications of the conception of reducibility, we close this section with the following two definitions:

DEFINITION 6. *Two polynomials are said to be relatively prime if they have no common factor other than a constant.*

*By $R(a_1, a_2, \cdots a_n)$ is understood the domain of rationality consisting of all numbers which can be obtained from the given numbers $a_1, \cdots a_n$ by the rational processes (addition, subtraction, multiplication, and division). In this notation the natural domain would be most simply denoted by $R(1)$; the domain last mentioned in the text by $R(1, \sqrt{-1})$ or, even more simply, by $R(\sqrt{-1})$. This notation would not apply to all cases (*e.g.* the real domain) except by the use of an infinite number of arguments.

DEFINITION 7. *Two methods of factoring a polynomial shall be said to be not essentially different if there are the same number n of factors in each case, and these factors can be so arranged that the kth factors are proportional for all values of k, from* 1 *to n inclusive.*

EXERCISES

1. Prove that every polynomial in (x, y) is irreducible if it is of the form

$$f(x) + y,$$

where $f(x)$ is a polynomial in x alone.

Would this also be true for polynomials of the form

$$f(x) + y^2?$$

2. If f, ϕ, ψ are polynomials in any number of variables which satisfy the relation

$$f \equiv \phi\psi,$$

and if the coefficients of f and ϕ lie in a certain domain of rationality, prove that the coefficients of ψ will lie in the same domain provided $\phi \not\equiv 0$.

61. The Irreducibility of the General Determinant and of the Symmetrical Determinant.

THEOREM 1. *The determinant*

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

is an irreducible polynomial if its n^2 elements are regarded as independent variables.

For suppose it were reducible, and let

$$D \equiv f(a_{11}, \cdots a_{nn})\, \phi(a_{11}, \cdots a_{nn}),$$

where neither f nor ϕ is a constant. Expanding D according to the elements of the first row, we see that it is of the first degree in a_{11}. Hence one of the two polynomials f and ϕ must be of the first degree in a_{11}, the other of the zeroth degree. Precisely the same reasoning shows that if a_{ij} is any element of D, one of the polynomials f and ϕ will be of the first degree in a_{ij}, while the other will not involve this variable.

Let us denote by f that one of the two polynomials which involves a_{ii}, any element of the principal diagonal of D. Then ϕ does not

involve any element of the ith row or the ith column. For if it did, since f is of the first degree in a_{ii} and ϕ is of the zeroth, their product D would involve terms containing products of the form $a_{ii}a_{ij}$ or $a_{ii}a_{ji}$, which, from the definition of a determinant, is impossible. Consequently, if either one of the polynomials f and ϕ contains any element of the principal diagonal of D, it must contain all the elements standing in the same row and all those standing in the same column with this one, and none of these can occur in the other polynomial.

Now suppose f contains a_{ii} and that ϕ contains any other element of the principal diagonal, say a_{jj}. Then a_{ij} and a_{ji} can be in neither f nor ϕ, which is impossible. Hence, if f contains any one of the elements in the principal diagonal, it must contain all the others, and hence *all* the elements, and our theorem is proved.

THEOREM 2. *The symmetrical determinant*

$$D = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \qquad (a_{ij} = a_{ji})$$

is an irreducible polynomial if its $\frac{1}{2}n(n+1)$ elements be regarded as independent variables.

The proof given for the last theorem holds, almost word for word, in this case also, the only difference being that while D is of the first degree in each of the elements of its principal diagonal, it is of the *second* degree in each of the other elements. The slight changes in the proof made necessary by this difference are left to the reader.

EXERCISES

1. The general bordered determinant

$$\begin{vmatrix} a_{11} \cdots a_{1n} & u_{11} \cdots u_{1p} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ a_{n1} \cdots a_{nn} & u_{n1} \cdots u_{np} \\ v_{11} \cdots v_{1n} & 0 \cdots 0 \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ v_{p1} \cdots v_{pn} & 0 \cdots 0 \end{vmatrix}$$

is irreducible if $p < n$, the a's, u's, and v's being regarded as independent variables.

2. The symmetrical bordered determinant obtained from the determinant in Exercise 1 by letting $a_{ij} = a_{ji}$, $u_{ij} = v_{ji}$ is irreducible if $p < n$.

3. If for certain values of i and j, but not for all, $a_{ij} = a_{ji}$, but if the a's are otherwise independent, can we still say that

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

is irreducible ?

4. Prove that a skew-symmetric determinant (cf. Exercises 2, 3, § 20) is always reducible by showing that, when it is of even order, it is a perfect square.

[SUGGESTION. Use Corollary 3, § 11, and Theorem 6 and Exercise 1, § 76.]

Does this theorem require any modification if the elements are real and we consider reducibility in the domain of reals?

62. Corresponding Homogeneous and Non-Homogeneous Polynomials. It is often convenient to consider side by side two polynomials, one homogeneous and the other non-homogeneous, which bear to one another the same relation as the first members of the equations of a plane curve or of a surface in homogeneous and non-homogeneous coördinates respectively. Such polynomials we will speak of as *corresponding* to one another according to the following definition:

DEFINITION. *If we have a non-homogeneous polynomial of the kth degree in any number of variables* $(x_1, \cdots x_{n-1})$ *and form a new polynomial by multiplying each term of the old by the power of a new variable* x_n *necessary to bring up the degree of this term to* k, *the homogeneous polynomial thus formed shall be said to* **correspond** *to the given non-homogeneous polynomial.*

Thus the two polynomials

$$(1)\qquad 2x^3 + 3x^2y - 5xz^2 - yz + 2z^2 + x - 3y - 9,$$

$$(2)\qquad 2x^3 + 3x^2y - 5xz^2 - yzt + 2z^2t + xt^2 - 3yt^2 - 9t^3,$$

correspond to each other.

It may be noticed that if $\phi\,(x_1, \cdots x_{n-1})$ is the non-homogeneous polynomial of degree k, the corresponding homogeneous polynomial may be written

$$x_n^k\,\phi\left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \cdots \frac{x_{n-1}}{x_n}\right).$$

To every non-homogeneous polynomial there corresponds one, and only one, homogeneous polynomial. Conversely, however, to a homogeneous polynomial in n variables there correspond in general n different non-homogeneous polynomials which are obtained by setting one of the variables equal to 1. For instance, in the example given above, to (2) corresponds not only (1) but also

$$(3)\qquad 2 + 3y - 5z^2 - yzt + 2z^2t + t^2 - 3yt^2 - 9t^3,$$

$$(4)\qquad 2x^3 + 3x^2 - 5xz^2 - zt + 2z^2t + xt^2 - 3t^2 - 9t^3,$$

$$(5)\qquad 2x^3 + 3x^2y - 5x - yt + 2t + xt^2 - 3yt^2 - 9t^3.$$

It should be noticed, however, that if one of the variables enters into every term of a homogeneous polynomial, the result of setting this variable equal to unity is to give, not a *corresponding* non-homogeneous polynomial, but a polynomial of lower degree. In fact, in the extreme case in which every variable enters into every term of the homogeneous polynomial, there is no corresponding non-homogeneous polynomial; as, for instance, in the case of the polynomial

$$x^2yz + xy^2z + xyz^2.$$

THEOREM 1. *If one of two corresponding polynomials is reducible, then the other is, also, and the factors of each polynomial correspond to the factors of the other.*

For let $\phi(x_1, \cdots x_n)$ be a homogeneous polynomial of degree $(k+l)$, and suppose it can be factored into two factors of degrees k and l, respectively,

$$(6)\qquad \phi_{k+l}(x_1, \cdots x_n) \equiv \psi_k(x_1, \cdots x_n)\,\chi_l(x_1, \cdots x_n).$$

Now suppose the corresponding non-homogeneous polynomial in question is the one formed by setting $x_n = 1$. We have

$$(7)\qquad \phi_{k+l}(x_1, \cdots x_{n-1}, 1) \equiv \psi_k(x_1, \cdots x_{n-1}, 1)\,\chi_l(x_1, \cdots x_{n-1}, 1).$$

Since by hypothesis the degree of the polynomial on the left is unchanged by this operation, neither of the factors on the right-hand side of (6) can have had its degree reduced, hence neither of the factors on the right of (7) is a constant. Our non-homogeneous polynomial is therefore reducible; and moreover the two factors on the right of (7), being of degrees k and l respectively, are precisely the two functions corresponding to the two factors on the right of (6).

Now, let $\Phi_{k+l}(x_1, \cdots x_{n-1})$ be a non-homogeneous polynomial, and suppose

$$\Phi_{k+l}(x_1, \cdots x_{n-1}) \equiv \Psi_k(x_1, \cdots x_{n-1})\, X_l(x_1, \cdots x_{n-1}),$$

where the subscripts denote the degrees of the polynomials. Let $\phi_{k+l}, \psi_{k+l}, \chi_{k+l}$ be the homogeneous polynomials corresponding to Φ, Ψ, X. Then when $x_n \neq 0$,

$$\Phi_{k+l}\left(\frac{x_1}{x_n}, \cdots \frac{x_{n-1}}{x_n}\right) = \Psi_k\left(\frac{x_1}{x_n}, \cdots \frac{x_{n-1}}{x_n}\right) X_l\left(\frac{x_1}{x_n}, \cdots \frac{x_{n-1}}{x_n}\right).$$

Multiplying this equation by x_n^{k+l} we have

$$\phi_{k+l}(x_1, \cdots x_{n-1}, x_n) = \psi_k(x_1, \cdots x_{n-1}, x_n)\, \chi_l(x_1, \cdots x_{n-1}, x_n),$$

an equation which holds whenever $x_n \neq 0$, and, therefore, by Theorem 5, § 2, is an identity. Thus our theorem is proved.

As a simple illustration of the way in which this theorem may be applied we mention the condition for reducibility of a non-homogeneous quadratic polynomial in any number of variables. By applying the test of § 47 to the corresponding homogeneous polynomial we obtain at once a test for the reducibility of any non-homogeneous quadratic polynomial.

THEOREM 2. *If f and ϕ are non-homogeneous polynomials, and F, Φ are the corresponding homogeneous polynomials, a necessary and sufficient condition that F and Φ be relatively prime is that f and ϕ be relatively prime.*

For if f and ϕ have a common factor ψ which is not a constant, the homogeneous polynomial Ψ which corresponds to ψ is, by Theorem 1, a common factor of F and Φ, and is clearly not a constant. Conversely, if Ψ is a common factor of F and Φ which is not a constant, f and ϕ will have, by Theorem 1, a common factor which corresponds to Ψ and which therefore cannot be a mere constant.

63. Division of Polynomials. We will consider first two polynomials in one variable:

$$(1) \qquad \begin{cases} f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n, \\ \phi(x) \equiv b_0x^m + b_1x^{m-1} + \cdots + b_m. \end{cases}$$

We learn in elementary algebra how to divide f by ϕ, getting a quotient $Q(x)$ and a remainder $R(x)$. What is essential here is contained in the following theorem:

THEOREM 1. *If f and ϕ are two polynomials in x of which ϕ is not identically zero, there exists one, and only one, pair of polynomials, Q and R, which satisfy the identity*

$$(2) \qquad f(x) \equiv Q(x)\,\phi(x) + R(x),$$

and such that either $R \equiv 0$, or the degree of R is less than the degree of ϕ.*

We begin by proving that *at least* one pair of polynomials Q, R exists which satisfies the conditions of the theorem.

If f is of lower degree than ϕ (or if $f \equiv 0$), the truth of this statement is obvious, for we may then let $Q \equiv 0$, $R \equiv f$.

Suppose, then, that f is of at least as high degree as ϕ. Writing f and ϕ in the form (1), we may assume

$$a_0 \neq 0, \quad b_0 \neq 0, \quad n \geqq m.$$

LEMMA. *If ϕ is not of higher degree than f, there exist two polynomials Q_1 and R_1 which satisfy the identity*

$$f(x) \equiv Q_1(x)\,\phi(x) + R_1(x),$$

and such that either $R_1 \equiv 0$, or the degree of R_1 is less than the degree of f.

The truth of this lemma is obvious if we let

$$Q_1(x) \equiv \frac{a_0}{b_0}x^{n-m}.$$

These two polynomials Q_1 and R_1 will serve as the polynomials Q and R of our theorem if $R_1 \equiv 0$, or if the degree of R_1 is less than the degree of ϕ. If not, apply the lemma again to the two functions R_1 and ϕ, getting

$$R_1(x) \equiv Q_2(x)\,\phi(x) + R_2(x),$$

where R_2 is either identically zero or is of lower degree than R_1. We may then write

$$f(x) \equiv [\,Q_1(x) + Q_2(x)]\,\phi(x) + R_2(x).$$

If $R_2 \equiv 0$, or if the degree of R_2 is less than the degree of ϕ, we may take for the polynomials Q, R of our theorem, $Q_1 + Q_2$ and R_2. If not, we apply our lemma again to R_2 and ϕ. Proceeding in this way

* It will be remembered that, according to the definition we have adopted, a polynomial which vanishes identically has no degree.

we get a series of polynomials $R_1, R_2, \cdots$ whose degrees are constantly decreasing. We therefore, after a certain number of steps, reach a polynomial R_i which is either identically zero or of degree less than ϕ. Combining the identities obtained up to this point, we have

$$f(x) \equiv [Q_1(x) + \cdots + Q_i(x)]\,\phi(x) + R_i(x),$$

an identity which proves the part of our theorem which states that at least one pair of polynomials Q, R of the kind described exists.*

Suppose now that besides the polynomials Q, R of the theorem there existed a second pair of polynomials Q', R' satisfying the same conditions. Subtracting from (2) the similar identity involving Q', R', we have

$$(3) \qquad 0 \equiv (Q - Q')\,\phi + (R - R').$$

From this we infer, as was to be proved,

$$Q \equiv Q', \qquad R \equiv R'.$$

For if Q and Q' were not identical, the first term on the right of (3) would be of at least the mth degree, while the second involves no power of x as high as m.

Turning now to polynomials in several variables:

$$(4) \quad \begin{cases} f(x_1, \cdots x_k) \equiv a_0(x_2, \cdots x_k)x_1^n + a_1(x_2, \cdots x_k)x_1^{n-1} + \cdots + a_n(x_2, \cdots x_k), \\ \phi(x_1, \cdots x_k) \equiv b_0(x_2, \cdots x_k)x_1^m + b_1(x_2, \cdots x_k)x_1^{m-1} + \cdots + b_m(x_2, \cdots x_k), \end{cases}$$

the ordinary method of dividing f by ϕ would give us as quotient and remainder, not polynomials, but fractional rational functions. In order to avoid this, we state our theorem in the following form:

THEOREM 2. *If f and ϕ are polynomials in $(x_1, \cdots x_k)$ of which ϕ is not identically zero, there exist polynomials Q, R, P, of which the last is not identically zero and does not involve the variable x_1, which satisfy the identity,*

$$(5) \quad P(x_2, \cdots x_k)f(x_1, \cdots x_k) \equiv Q(x_1, \cdots x_k)\phi(x_1, \cdots x_k) + R(x_1, \cdots x_k),$$

and such that either $R \equiv 0$, or the degree in x_1 of R is less than the degree in x_1 of ϕ.

The proof of this theorem follows the same lines as the proof of Theorem 1.

* The reader should notice that the process just used is merely the ordinary process of long division.

If f is of lower degree in x_1 than ϕ (or if $f \equiv 0$), the truth of the theorem is obvious, for we may then let $P \equiv 1$, $Q \equiv 0$, $R \equiv f$.

Suppose, then, that f is of at least as high degree in x_1 as ϕ. Writing f and ϕ in the form (4), we may assume

$$a_0 \not\equiv 0, \quad b_0 \not\equiv 0, \quad n \geqq m.$$

LEMMA. *If ϕ is not of higher degree in x_1 than f, there exist two polynomials Q_1, R_1 which satisfy the identity,*

$$b_0(x_2, \cdots x_k)f(x_1, \cdots x_k) \equiv Q_1(x_1, \cdots x_k)\phi(x_1, \cdots x_k) + R_1(x_1, \cdots x_k),$$

and such that either $R_1 \equiv 0$, or the degree of R_1 in x_1 is less than the degree of f in x_1.

The truth of this lemma is obvious if we let

$$Q_1 \equiv a_0(x_2, \cdots x_k)\, x_1^{n-m}.$$

The polynomials Q_1, R_1, b_0 will serve as the polynomials Q, R, P of our theorem if $R_1 \equiv 0$, or if the degree of R_1 in x_1 is less than the degree of ϕ in x_1. If not, apply the lemma again to the two functions R_1 and ϕ, getting

$$b_0(x_2, \cdots x_k)R_1(x_1, \cdots x_k) \equiv Q_2(x_1, \cdots x_k)\phi(x_1, \cdots x_k) + R_2(x_1, \cdots x_k),$$

where R_2 is either identically zero or is of lower degree in x_1 than R_1. We may then write

$$b_0^2 f \equiv (b_0 Q_1 + Q_2)\phi + R_2.$$

If $R_2 \equiv 0$, or if the degree of R_2 in x_1 is less than the degree of ϕ in x_1, we may take for the polynomials Q, R, P of our theorem the functions $b_0Q_1 + Q_2$, R_2, b_0^2. If not, we apply our lemma again to R_2 and ϕ. Proceeding in this way, we get a series of polynomials $R_1, R_2, \cdots$ whose degrees in x_1 are constantly decreasing. We therefore, after a certain number of steps, reach a polynomial R_i which is either identically zero, or of degree in x_1 less than ϕ. Combining the identities obtained up to this point, we have

$$b_0^i f \equiv (b_0^{i-1}Q_1 + b_0^{i-2}Q_2 + \cdots + Q_i)\phi + R_i,$$

an identity which proves our theorem, and which also establishes the additional result:

COROLLARY. *The polynomial P whose existence is stated in our theorem may be taken as a power of b_0.*

We note that it would obviously not be correct to add to the statement of Theorem 2 the further statement that there is only one set of polynomials Q, R, P, since the identity (5) may be multiplied by any polynomial in $(x_2, \cdots x_n)$ without changing its form. Cf., however, the exercise at the end of § 73.

64. A Special Transformation of a Polynomial. Suppose that $f(x_1, x_2, x_3, x_4)$ is a homogeneous polynomial of the kth degree in the homogeneous coördinates (x_1, x_2, x_3, x_4), so that the equation $f = 0$ represents a surface of the kth degree. If, in f, the term in x_4^k has the coefficient zero, the surface passes through the origin; and if the term in x_1^k (or x_2^k, or x_3^k) has the coefficient zero, the surface passes through the point at infinity on the axis of x_1 (or x_2, or x_3). It is clear that these peculiarities of the surface can be avoided, and that, too, in an infinite variety of ways, by subjecting the surface to a non-singular collineation which carries over any four non-complanar points, no one of which lies on the surface, into the origin and the three points at infinity on the coördinate axes. It is this fact, generalized to the case of n variables, which we now proceed to prove.

LEMMA. *If $f(x_1, \cdots x_n)$ is a homogeneous polynomial of the kth degree in which the term x_m^k is wanting, there exists a non-singular linear transformation of the variables $(x_1, \cdots x_n)$ which carries f into a new form f_1, in which the term in $x_m'^k$ has a coefficient different from zero, while the coefficients of the kth powers of the other variables have not been changed.*

In proving this theorem there is obviously no real loss of generality in taking as the variable x_m the last of the variables x_n.

Let us then consider the non-singular transformation

$$\begin{aligned} x_i &= x_i' + a_i x_n' \qquad (i = 1, \cdots n-1). \\ x_n &= x_n' \end{aligned}$$

This transformation carries f into

$$f_1(x_1', \cdots x_n') \equiv f(x_1' + a_1 x_n', \cdots x_{n-1}' + a_{n-1} x_n', x_n'),$$

and evidently does not change the coefficients of the terms in $x_1^k, \cdots x_{n-1}^k$.

Now, since every term in f_1, except the term in $x_n'^k$, contains at least one of the variables $x_1', \cdots x_{n-1}'$, the coefficient of the term in $x_n'^k$ will be

$$f_1(0, \cdots 0, 1) = f(a_1, \cdots a_{n-1}, 1).$$

Our lemma will therefore be proved if we can show that the constants $a_1, \cdots a_{n-1}$ can be so chosen that this quantity is not zero.

Let us take any point $(b_1, \cdots b_n)$ for which $b_n \neq 0$; and consider a neighborhood of this point sufficiently small so that x_n does not vanish at any point in this neighborhood. Then, since f does not vanish identically, we can find a point $(c_1, \cdots c_n)$ in this neighborhood (so that $c_n \neq 0$) such that

$$f(c_1, \cdots c_n) \neq 0.$$

If now we take for $a_1, \cdots a_{n-1}$ the values $c_1/c_n, \cdots c_{n-1}/c_n$, we shall have, since f is homogeneous,

$$f(a_1, \cdots a_{n-1}, 1) \neq 0.$$

Thus our lemma is proved.

THEOREM 1. *If $f(x_1, \cdots x_n)$ is a homogeneous polynomial of the kth degree, there exists a non-singular linear transformation which carries f into a new form f_1 in which the terms in $x_1'^k, \cdots x_n'^k$ all have coefficients different from zero.*

The proof of this theorem follows at once from the preceding lemma. For we need merely to perform in succession the transformations which cause the coefficients first of x_1^k, then of x_2^k, etc., to become different from zero, and which our lemma assures us will exist and be non-singular, to obtain the transformation we want. To make sure of this it is necessary merely to notice that the coefficient of x_1^k obtained by the first transformation will not be changed by the subsequent transformations; that the same will be true of the coefficient of x_2^k obtained by the second transformation; etc.

THEOREM 2. *If $f(x_1, \cdots x_n)$ is a polynomial of the kth degree which is not necessarily homogeneous, there exists a non-singular homogeneous linear transformation of $(x_1, \cdots x_n)$ which makes this polynomial of the kth degree in each of the variables $x_1', \cdots x_n'$ taken separately.*

If f is homogeneous, this is equivalent to Theorem 1. If f is non-homogeneous, we may write it in the form

$$f(x_1, \cdots x_n) \equiv \phi_k(x_1, \cdots x_n) + \phi_{k-1}(x_1, \cdots x_n) + \cdots + \phi_1(x_1, \cdots x_n) + \phi_0,$$

where each ϕ is a homogeneous polynomial of the degree indicated by its subscript or else is identically zero. We need now merely to apply Theorem 1 to the homogeneous polynomial ϕ_k, which is, of course, not identically zero.

This theorem, and therefore also Theorem 1, which is merely a special case of it, admits the following generalization to the case of a system of functions:

THEOREM 3. *If we have a system of polynomials*

$$f_1(x_1, \cdots x_n), \quad f_2(x_1, \cdots x_n), \cdots\cdots f_m(x_1, \cdots x_n),$$

of degrees $k_1, k_2, \cdots k_m$ respectively, there exists a non-singular homogeneous linear transformation which makes these polynomials of degrees $k_1, \cdots k_m$ in each of the variables $x_1', \cdots x_n'$ taken separately.

This theorem may be proved either by the same method used in proving Theorems 1, 2; or by applying Theorem 2 to the product $f_1 f_2 \cdots f_m$.

CHAPTER XV

FACTORS AND COMMON FACTORS OF POLYNOMIALS IN ONE VARIABLE AND OF BINARY FORMS.

65. Fundamental Theorems on the Factoring of Polynomials in One Variable and of Binary Forms. Theorem 2, § 6 may be stated in the following form:

THEOREM 1. *A polynomial of the nth degree in one variable is always reducible when $n > 1$. It can be resolved into the product of n linear factors in one, and essentially in only one, way.*

By means of § 62 we can deduce from this a similar theorem in the case of the binary form

(1) $$a_0x_1^n + a_1x_1^{n-1}x_2 + \cdots + a_nx_2^n.$$

Let us first assume that $a_0 \neq 0$. Then the non-homogeneous polynomial

(2) $$a_0x_1^n + a_1x_1^{n-1} + \cdots + a_n$$

corresponds to (1), according to the definition of § 62. Factoring (2), we get

$$a_0(x_1 - \alpha_1)(x_1 - \alpha_2) \cdots (x_1 - \alpha_n),$$

or, if we take n constants $\alpha_1'', \alpha_2'', \cdots \alpha_n''$ whose product is a_0,

(3) $$(\alpha_1''x_1 - \alpha_1')(\alpha_2''x_1 - \alpha_2') \cdots (\alpha_n''x_1 - \alpha_n'),$$

where for brevity we have written

$$\alpha_i''\alpha_i = \alpha_i' \qquad (i = 1, 2, \cdots n).$$

By Theorem 1, § 62, we now infer that the binary form (1) is identically equal to

(4) $$(\alpha_1''x_1 - \alpha_1'x_2)(\alpha_2''x_1 - \alpha_2'x_2) \cdots (\alpha_n''x_1 - \alpha_n'x_2).$$