For suppose the form (4) were definite and non-singular; and that $a_{ii}=0$. Then the form would vanish at the point

$$x_1=\cdots=x_{i-1}=x_{i+1}=\cdots=x_n=0,\ x_i=1;$$

and this is impossible, since this is not the point $(0,0,\cdots 0)$.

EXERCISES

1. DEFINITION. *By an orthogonal transformation* is understood a linear transformation which carries over the variables* $(x_1,\cdots x_n)$ *into the variables* $(x_1',\cdots x_n')$ *in such a way that*

$$x_1^2+x_2^2+\cdots+x_n^2\equiv x_1'^2+x_2'^2+\cdots+x_n'^2.$$

Prove that every orthogonal transformation is non-singular, and, in particular, that its determinant must have the value $+1$ or -1.

2. Prove that all orthogonal transformations in n variables form a group; and that the same is true of all orthogonal transformations in n variables of determinant $+1$.

3. Prove that a necessary and sufficient condition that a linear transformation be orthogonal is that it leave the "distance"

$$\sqrt{(y_1-z_1)^2+(y_2-z_2)^2+\cdots+(y_n-z_n)^2}$$

between every pair of points $(y_1,\cdots y_n)$, $(z_1,\cdots z_n)$ invariant.

4. Prove that if $n=3$, and if x_1, x_2, x_3 be interpreted as non-homogeneous rectanglar coördinates in space, an orthogonal transformation represents either a rigid displacement which leaves the origin fixed, or such a displacement combined with reflection in a plane through the origin.

Show that the first of these cases will occur when the determinant of the transformation is $+1$, the second when this determinant is -1.

5. If the coefficients of a linear transformation are denoted in the usual way by c_{ij}, prove that a necessary and sufficient condition that the transformation be orthogonal is that

$$c_{1i}^2+c_{2i}^2+\cdots+c_{ni}^2=1 \qquad (i=1,2,\cdots n),$$

$$c_{1i}c_{1j}+c_{2i}c_{2j}+\cdots+c_{ni}c_{nj}=0 \qquad \begin{cases} i=1,2,\cdots n \\ j=1,2,\cdots n \end{cases} i\neq j.$$

Show that these will still be necessary and sufficient conditions for an orthogonal transformation if the two subscripts of every c be interchanged.†

* The matrix of such a transformation is called an orthogonal matrix, and its determinant an orthogonal determinant.

† We have here $\frac{1}{2}n(n+1)$ relations between the n^2 coefficients of the transformation. This suggests that it should be possible to express all the coefficients in terms of

$$n^2-\frac{n(n+1)}{2}=\frac{n(n-1)}{2}$$

of them, or if we prefer in terms of $\frac{1}{2}n(n-1)$ other parameters. For Cayley's discussion of this question cf. Pascal's book, *Die Determinanten*, § 47. Cayley's formulæ, however, do not include all orthogonal transformations except as limiting cases.

CHAPTER XII

THE SYSTEM OF A QUADRATIC FORM AND ONE OR MORE LINEAR FORMS

53. Relations of Planes and Lines to a Quadric Surface. If the plane

$$u_1x_1+u_2x_2+u_3x_3+u_4x_4=0 \tag{1}$$

is a true tangent plane to the quadric surface

$$\sum_1^4 a_{ij}x_ix_j=0, \tag{2}$$

there will be a point (y_1, y_2, y_3, y_4) (namely the point of contact) lying in (1) and such that its polar plane

$$\sum_1^4 a_{ij}x_iy_j=0 \tag{3}$$

coincides with (1). From elementary analytic geometry we know that a necessary and sufficient condition that two equations of the first degree represent the same plane is that their coefficients be proportional. Accordingly, from the coincidence of (1) and (3), we deduce the equations

$$\begin{cases} a_{11}y_1+a_{12}y_2+a_{13}y_3+a_{14}y_4-\rho u_1=0, \\ a_{21}y_1+a_{22}y_2+a_{23}y_3+a_{24}y_4-\rho u_2=0, \\ a_{31}y_1+a_{32}y_2+a_{33}y_3+a_{34}y_4-\rho u_3=0, \\ a_{41}y_1+a_{42}y_2+a_{43}y_3+a_{44}y_4-\rho u_4=0. \end{cases} \tag{4}$$

From the fact that the point y lies on (1), we infer the further relation

$$u_1y_1+u_2y_2+u_3y_3+u_4y_4=0. \tag{5}$$

These equations (4) and (5) have been deduced on the supposition that (1) is a true tangent plane to (2). They still hold if it is a pseudo-tangent plane; for then the quadric must be a cone, and a vertex of this cone must lie on (1). Taking the point y as this vertex, equation (5) is fulfilled. Moreover, since now the first member of (3) is identically zero, equations (4) will also be fulfilled if we let $\rho = 0$. Thus we have shown in all cases, that if (1) is a tangent plane to (2), there exist five constants, y_1, y_2, y_3, y_4, ρ, of which the first four are not all zero, and which satisfy equations (4) and (5). Hence

$$(6)\qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & u_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & u_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & u_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & u_4 \\ u_1 & u_2 & u_3 & u_4 & 0 \end{vmatrix} = 0.$$

Conversely, if this last equation is fulfilled, there exist five constants, y_1, y_2, y_3, y_4, ρ, not all zero, and which satisfy equations (4) and (5). We can go a step farther and say that y_1, y_2, y_3, y_4 cannot all be zero, as otherwise, from equations (4) and the fact that the u's are not all zero, ρ would also be zero. Thus we see that if equation (6) is fulfilled, there exists a point (y_1, y_2, y_3, y_4) in the plane (1) whose coördinates, together with a certain constant ρ, satisfy (4). If $\rho = 0$, this shows that the quadric is a cone with y as a vertex, and hence that (1) is at least a pseudo-tangent plane. If $\rho \neq 0$, equations (4) show us that the polar plane (3) of y coincides with the plane (1). Moreover we see, either geometrically, or by multiplying equations (4) by y_1, y_2, y_3, y_4 respectively and adding, that the point y lies on the quadric; so that, in this case, (1) is a true tangent plane.

We have thus established the theorem:

THEOREM 1. *Equation* (6) *is a necessary and sufficient condition that the plane* (1) *be tangent to the quadric* (2).

It will be seen that this theorem gives us no means of distinguishing between true and pseudo-tangent planes of quadric cones. In the case of non-singular quadrics, pseudo-tangent planes are impossible, and therefore equation (6) may, in this case, be regarded as the equation of the quadric in plane-coördinates.

In the case of a quadric surface of rank 3, that is, of a cone with a single vertex, the coördinates (u_1, u_2, u_3, u_4) of every plane through this vertex satisfy equation (6), so that in this case this equation represents a single point, and not the quadric cone.*

* In fact a cone cannot be represented by a single equation in plane-coördinates.

If the rank of (2) is less than 3, the coördinates of every plane in space should satisfy (6), since every such plane passes through a vertex and is therefore a tangent plane. This fact may be verified by noticing that equation (6) may also be written

$$\sum_1^4 A_{ij}u_iu_j = 0,$$

where the A's are the cofactors in the discriminant of (2) according to our usual notation.

We pass now to the condition that a straight line touch the quadric (2). This line we will determine as the intersection of the two planes (1) and

$$(7)\qquad v_1x_1 + v_2x_2 + v_3x_3 + v_4x_4 = 0.$$

If the line of intersection of these planes is a true tangent to (2), there will be a point (y_1, y_2, y_3, y_4), namely the point of contact, lying upon it, and such that its polar plane (3) contains the line. It must therefore be possible to write the equation of this polar plane in the form

$$(8)\qquad \sum_1^4 (\mu u_i + \nu v_i)x_i = 0;$$

and, in fact, by properly choosing the constants μ and ν, the coefficients of (8) may be made not merely proportional, but equal to the coefficients of (3):

$$(9)\qquad \begin{cases} a_{11}y_1 + a_{12}y_2 + a_{13}y_3 + a_{14}y_4 - \mu u_1 - \nu v_1 = 0, \\ a_{21}y_1 + a_{22}y_2 + a_{23}y_3 + a_{24}y_4 - \mu u_2 - \nu v_2 = 0, \\ a_{31}y_1 + a_{32}y_2 + a_{33}y_3 + a_{34}y_4 - \mu u_3 - \nu v_3 = 0, \\ a_{41}y_1 + a_{42}y_2 + a_{43}y_3 + a_{44}y_4 - \mu u_4 - \nu v_4 = 0. \end{cases}$$

Since the point y lies on the line of intersection of the planes (1) and (7), we also have the relations

$$(10)\qquad \begin{cases} u_1y_1 + u_2y_2 + u_3y_3 + u_4y_4 = 0, \\ v_1y_1 + v_2y_2 + v_3y_3 + v_4y_4 = 0. \end{cases}$$

Since the six equations (9) and (10) are satisfied by six constants $y_1, y_2, y_3, y_4, \mu, \nu$ not all zero, we infer finally the relation

$$(11)\qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & u_1 & v_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & u_2 & v_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & u_3 & v_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & u_4 & v_4 \\ u_1 & u_2 & u_3 & u_4 & 0 & 0 \\ v_1 & v_2 & v_3 & v_4 & 0 & 0 \end{vmatrix} = 0.$$

We have deduced this equation on the supposition that the line of intersection of (1) and (7) is a true tangent to (2). We leave it to the reader to show that (11) holds if this line is a pseudo-tangent, and also if it is a ruling of (2).

We also leave it for him to show that if (11) holds, the line of intersection of (1) and (7) will be either a true tangent, a pseudo-tangent, or a ruling, and thus to establish the theorem:

THEOREM 2. *A necessary and sufficient condition that the line of intersection of the planes* (1) *and* (7) *be either a tangent or a ruling of* (2) *is that equation* (11) *be fulfilled.*

On expanding the determinant in (11), it will be seen that it is a quadratic form in the six line-coördinates q_{ij} (cf. Exercise 3, § 35). Equation (11) may therefore be regarded as the equation of the quadric surface in line-coördinates if the surface is not a cone, or is a cone with a single vertex. If the rank of (2) is 2, so that the quadric consists of two planes, (11) is the equation of the line of intersection of these planes. While if the rank is 1 or 0, (11) is identically fulfilled.

EXERCISES

1. Two planes are said to be conjugate with regard to a non-singular quadric surface if each passes through the pole of the other.

Prove that if (2) is a non-singular quadric, a necessary and sufficient condition that the planes (1) and (7) be conjugate with regard to it is the vanishing of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & u_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & u_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & u_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & u_4 \\ v_1 & v_2 & v_3 & v_4 & 0 \end{vmatrix} = -\sum_1^4 A_{ij}u_iv_j.$$

How must this definition of conjugate planes be extended in order that this theorem be true for singular quadrics also?

2. Prove that if (2) is a non-singular quadric, a necessary and sufficient condition that the point of intersection of three planes lie on (2) is the vanishing of the seven-rowed determinant formed by bordering the discriminant of (2) with the coefficients of the three planes.

3. Admitting it to be obvious geometrically that a necessary and sufficient condition that a line touch a non-singular quadric is that the two tangent planes which can be passed through this line should coincide, prove that, if (2) is non-singular, a necessary and sufficient condition that the line of intersection of (1) and (7) touch (2) is

$$\Big(\sum_1^4 A_{ij}u_iu_j\Big)\Big(\sum_1^4 A_{ij}v_iv_j\Big) - \Big(\sum_1^4 A_{ij}u_iv_j\Big)^2 = 0.$$

4. Show algebraically that the condition of Exercise 3 is equivalent to (11).

54. The Adjoint Quadratic Form and Other Invariants. Passing now to the case of n variables, we begin by considering the system consisting of a quadratic form and a single linear form

$$\sum_1^n a_{ij}x_ix_j, \tag{1}$$

$$\sum_1^n u_ix_i. \tag{2}$$

The geometrical considerations of the last section suggest that we form the expression

$$\sum_1^n A_{ij}u_iu_j \equiv -\begin{vmatrix} a_{11} & \cdots & a_{1n} & u_1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} & u_n \\ u_1 & \cdots & u_n & 0 \end{vmatrix}. \tag{3}$$

This, it will be seen, is a quadratic form in the variables $(u_1, \cdots u_n)$ whose matrix is the adjoint of the matrix of (1). We will speak of (3) as the *adjoint* of (1).

The invariance of (3) is at once suggested by the fact that in the case $n = 4$ the vanishing of (3) gave a necessary and sufficient condition for a projective relation. In fact we will prove the theorem:

THEOREM 1. *The adjoint form* (3) *is an invariant of weight two of the pair of forms* (1), (2).

Inasmuch as the u's are, as we saw in § 34, contragredient to the x's, we may also call (3) a contravariant (cf. Definition 2, § 34).

In order to prove this theorem we must subject the x's to a linear transformation,

$$\begin{cases} x_1 = c_{11}x_1' + \cdots + c_{1n}x_n' \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_n = c_{n1}x_1' + \cdots + c_{nn}x_n', \end{cases} \tag{4}$$

whose determinant we will call c. Let us denote by a_{ij}' and u_i' respectively the coefficients of the quadratic and linear form into which this transformation carries (1) and (2).

Let us now introduce an auxiliary variable t, and consider the quadratic form in $x_1, \cdots x_n, t$,

$$\sum_1^n a_{ij}x_ix_j + 2t(u_1x_1 + \cdots + u_nx_n). \tag{5}$$

The discriminant of this form is precisely the determinant in (3), that is, the negative of the adjoint of (1).

Let us now perform on the variables $x_1, \cdots x_n, t$ the linear transformation given by formulæ (4) and the additional formula

$$t = t'. \tag{6}$$

The determinant of this transformation is c, and it carries over the form (5) into
$$\sum_1^n a'_{ij}x'_ix'_j + 2\,t'(u'_1x'_1 + \cdots + u'_nx'_n).$$

From the fact that the discriminant of (5) is an invariant of weight 2, we infer the relation we wished to obtain:

$$\begin{vmatrix} a'_{11} & \cdots & a'_{1n} & u'_1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a'_{n1} & \cdots & a'_{nn} & u'_n \\ u'_1 & \cdots & u'_n & 0 \end{vmatrix} \equiv c^2 \begin{vmatrix} a_{11} & \cdots & a_{1n} & u_1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} & u_n \\ u_1 & \cdots & u_n & 0 \end{vmatrix}.$$

The method just used admits of immediate extension to the proof of the following more general theorem:

THEOREM 2. *If we have a system consisting of a quadratic form in* n *variables and* p *linear forms, the* $(n+p)$*-rowed determinant formed by bordering the discriminant of the quadratic form by* p *rows and* p *columns each of which consists of the coefficients of one of the linear forms is an invariant of weight* 2.

We leave the details of the proof of this theorem to the reader.

If the discriminant a of the quadratic form (1) is not zero, we may form a new quadratic form whose matrix is the inverse of the matrix of (1). This quadratic form, which is known as the *inverse* or *reciprocal* of (1), is simply the adjoint of (1) divided by the discriminant a. We will prove the following theorem concerning it:

THEOREM 3. *If the quadratic form* (1) *is non-singular, it will be carried over into its inverse by the non-singular transformation*

$$x'_i = a_{i1}x_1 + \cdots + a_{in}x_n \qquad (i = 1, 2, \cdots n). \tag{7}$$

For we have
$$\sum_1^n a_{ij}x_ix_j \equiv \sum_1^n x_ix'_i.$$

But from (7) we have
$$x_i = \frac{A_{i1}}{a}x'_1 + \cdots + \frac{A_{in}}{a}x'_n$$
and therefore
$$\sum_1^n a_{ij}x_ix_j \equiv \sum_1^n \frac{A_{ij}}{a}x'_ix'_j,$$
as was to be proved.

It will be noticed that if (1) is a real quadratic form, the transformation (7) is real; and from this follows

THEOREM 4. *A real non-singular quadratic form and its inverse have the same signature.*

EXERCISES

1. Given a quadratic form $\Sigma a_{ij}x_ix_j$ and two linear forms Σu_ix_i, Σv_ix_i. Prove that
$$\sum_1^n A_{ij}u_iv_j \equiv - \begin{vmatrix} a_{11} & \cdots & a_{1n} & u_1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} & u_n \\ v_1 & \cdots & v_n & 0 \end{vmatrix}$$
is an invariant of the system of weight 2.

2. Generalize the theorem of Exercise 1 to the case in which we have more than two linear forms.

3. Prove that if a first quadratic form is transformed into a second by the linear transformation of matrix c, then the adjoint of the first will be transformed into the adjoint of the second by the linear transformation whose matrix is the conjugate of the adjoint of c.

4. Prove a similar theorem for bilinear forms.

5. State and prove a theorem for bilinear forms analogous to Theorem 3.

55. The Rank of the Adjoint Form. Suppose the discriminant a of the quadratic form $\sum_1^n a_{ij}x_ix_j$ is of rank r, and that the discriminant A of its adjoint $\sum_1^n A_{ij}u_iu_j$ is of rank R. Then, if $r < n-1$, all the $(n-1)$-rowed determinants of a are zero; but these are the elements of A, hence $R = 0$. If $r = n-1$, at least one of the elements of A is not zero, and all two-rowed determinants of A are zero (since by § 11 each of them contains a as a factor), hence $R = 1$. If $r = n$, $R = n$; for if R were less than n we should have $A = 0$, and therefore $a = 0$ (since $A = a^{n-1}$). But this is impossible, since by hypothesis $r = n$. We have then:

THEOREM 1. *If the rank of a quadratic form in* n *variables and of its adjoint are* r *and* R *respectively, then*

$$\begin{aligned} &\textit{if } r = n, && R = n, \\ &\textit{if } r = n-1, && R = 1, \\ &\textit{if } r < n-1, && R = 0. \end{aligned}$$

Let us consider further the case $r=n-1$. Here we have seen that $R=1$, that is, that the adjoint is the square of a linear form,

$$\sum_1^n A_{ij}u_iu_j \equiv (\sum_1^n c_iu_i)^2 \equiv \sum_1^n c_ic_ju_iu_j.$$

Comparing coefficients, we see that

$$A_{ij}=c_ic_j.$$

All the c's cannot be zero, as otherwise we should have $R=0$. Let $c_\lambda \neq 0$. Then since

$$A_{\lambda\lambda}=c_\lambda^2 \neq 0$$

we see that not all the quantities $(A_{\lambda 1}, \cdots A_{\lambda n})$ are zero. Accordingly (cf. § 44) the point $(A_{\lambda 1}, A_{\lambda 2}, \cdots A_{\lambda n})$, and therefore also the point $(c_1, \cdots c_n)$, is a vertex of the original quadratic form. Thus we have the theorem :

THEOREM 2. *If the rank of a quadratic form in n variables is $n-1$, its adjoint is the square of a linear form, and the coefficients of this linear form are the coördinates of a vertex of the original form.*

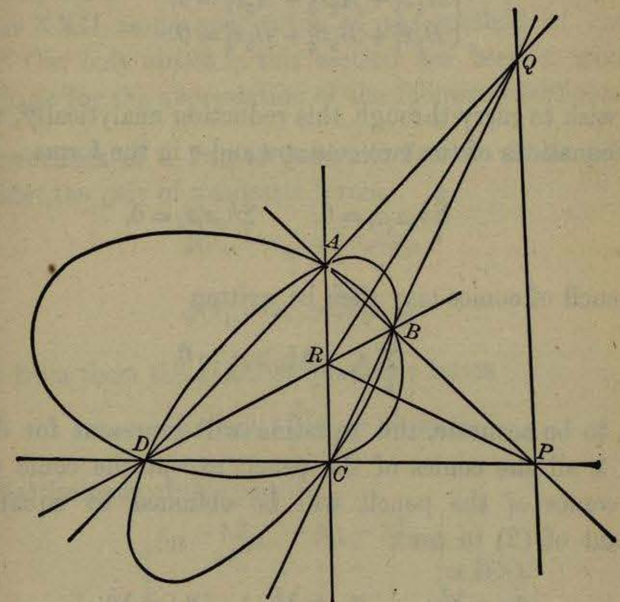
Since, in the case we are considering, all the vertices of the quadratic form are linearly dependent on any one, this theorem completely determines the linear form in question except for a constant factor.

CHAPTER XIII

PAIRS OF QUADRATIC FORMS

56. Pairs of Conics. We will give in this section a short geometrical introduction to the study of pairs of quadratic forms, confining ourselves, for the sake of brevity, to two dimensions.

Let u and v be two conics which we will assume to be so situated that they intersect in four, and only four, distinct points, A, B, C, D. Consider all conics through these four points. These conics, we will



say, form a *pencil*. It is obvious that there are three and only three singular conics (*i.e.* conics which consist of pairs of lines) in this pencil, namely, the three pairs of lines AB, CD; BC, DA; AC, BD. Let us call the "vertices" of these conics P, Q, and R respectively.

From the harmonic properties of the complete quadrilateral* we see that the secants PAB and PCD are divided harmonically by the

* Cf. any book on modern geometry.

line QR. Accordingly QR is the polar of P with regard to every conic of the pencil. In a similar manner PR is the polar of Q, and PQ the polar of R with regard to every conic of the pencil. Thus, we see that the triangle PQR is a self-conjugate triangle (see § 41) with regard to every conic of the pencil. Accordingly, if we perform a collineation which carries over P, Q, R into the origin and the points at infinity on the axes of x and y, the equation of every conic of the pencil will be reduced to a form in which only the square terms enter. We are thus led to the result:

THEOREM. *If two conics intersect in four and only four distinct points, there exists a non-singular collineation which reduces their equations to the normal form*

$$\begin{cases} A_1x_1^2 + A_2x_2^2 + A_3x_3^2 = 0, \\ B_1x_1^2 + B_2x_2^2 + B_3x_3^2 = 0. \end{cases}$$

If we wish to carry through this reduction analytically, we shall write the equations of the two conics u and v in the forms

$$(1) \qquad \sum_1^3 a_{ij}x_ix_j = 0, \qquad \sum_1^3 b_{ij}x_ix_j = 0.$$

The pencil of conics may then be written

$$(2) \qquad \sum_1^3 (a_{ij} - \lambda b_{ij})x_ix_j = 0,$$

or rather, to be accurate, this equation will represent for different values of λ all the conics of the pencil except the conic v. The singular conics of the pencil will be obtained by equating the discriminant of (2) to zero,

$$(3) \qquad \begin{vmatrix} a_{11} - \lambda b_{11} & a_{12} - \lambda b_{12} & a_{13} - \lambda b_{13} \\ a_{21} - \lambda b_{21} & a_{22} - \lambda b_{22} & a_{23} - \lambda b_{23} \\ a_{31} - \lambda b_{31} & a_{32} - \lambda b_{32} & a_{33} - \lambda b_{33} \end{vmatrix} = 0.$$

This equation we will call the λ-equation of the two conics. When expanded, it takes the form

$$(4) \qquad -\Delta'\lambda^3 + \Theta'\lambda^2 - \Theta\lambda + \Delta = 0,$$

where Δ, Δ' are the discriminants of u and v respectively, and

$$\Theta = \begin{vmatrix} a_{11} & a_{12} & b_{13} \\ a_{21} & a_{22} & b_{23} \\ a_{31} & a_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & b_{12} & a_{13} \\ a_{21} & b_{22} & a_{23} \\ a_{31} & b_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{62} & a_{33} \end{vmatrix},$$

while Θ' can be obtained from Θ by an interchange of the letters a and b. It can readily be proved (cf. the next section) that the coefficients Θ and Θ' as well as Δ and Δ' are invariants of weight two.

Except when the discriminant Δ' of v is zero, the equation (4) is of the third degree, and its three roots, which in the case we have considered must evidently be distinct, give, when substituted in (2), the three singular conics of the pencil.

We will not stop here to show how the theory of any two conics, where no restriction as to the number of points of intersection is made, can be deduced from equation (3).* This will follow in Chapter XXII as an application of the method of elementary divisors. Our only object in this section has been to give a geometrical basis for the appreciation of the following sections.

57. Invariants of a Pair of Quadratic Forms. Their λ-Equation. We consider the pair of quadratic forms

$$\phi(x_1, \cdots x_n) \equiv \sum_1^n a_{ij}x_ix_j,$$

$$\psi(x_1, \cdots x_n) \equiv \sum_1^n b_{ij}x_ix_j,$$

and form from them the *pencil* of quadratic forms

$$\phi - \lambda\psi \equiv \sum_1^n (a_{ij} - \lambda b_{ij})x_ix_j.$$

The discriminant of this pencil,

$$\begin{vmatrix} a_{11} - \lambda b_{11} & \cdots & a_{1n} - \lambda b_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{n1} - \lambda b_{n1} & \cdots & a_{nn} - \lambda b_{nn} \end{vmatrix} \equiv F(\lambda),$$

is a polynomial in λ which is in general of degree n, and which may be written
$$F(\lambda) \equiv \Theta_0 - \Theta_1\lambda + \cdots + (-1)^n\Theta_n\lambda^n.$$

* An elementary discussion of the λ-equation of two conics (*l'équation en* λ) is regularly given in French text-books on analytic geometry. See, for instance, Briot et Bouquet, *Leçons de Géométrie analytique*, 14th ed., p. 349, or Niewenglowski, *Cours de Géométrie analytique*, Vol. I, p. 459.

The coefficients of this polynomial are themselves polynomials in the a_{ij}'s and b_{ij}'s, Θ_0 and Θ_n being merely the discriminants of ϕ and ψ respectively, while Θ_k is the sum of all the different determinants which can be formed by replacing k columns of the discriminant of ϕ by the corresponding columns of the discriminant of ψ.

THEOREM 1. *The coefficients $\Theta_0, \cdots \Theta_n$ of $F(\lambda)$ are integral rational invariants of weight two of the pair of quadratic forms ϕ, ψ.**

In order to prove this, let us consider a linear transformation of determinant c which carries over ϕ and ψ into ϕ' and ψ' respectively, where

$$\phi' \equiv \sum_1^n a'_{ij} x'_i x'_j,$$

$$\psi' \equiv \sum_1^n b'_{ij} x'_i x'_j.$$

Let us denote by Θ'_i the polynomial in the a'_{ij}'s and b'_{ij}'s obtained by putting accents to the a's and b's in Θ_i. Our theorem will then be proved if we can establish the identities

$$\Theta'_i \equiv c^2 \Theta_i \qquad (i = 0, 1, \cdots n)$$

This follows at once from the fact that $F(\lambda)$, being the discriminant of $\phi - \lambda\psi$, is an invariant of weight two, so that if we denote by $F'(\lambda)$ the discriminant of $\phi' - \lambda\psi'$, we have

$$F'(\lambda) \equiv c^2 F(\lambda).$$

This being an identity in λ as well as in the a's and b's, we can equate the coefficients of like powers of λ on the two sides, and this gives precisely the identities we wished to establish.†

The equation

$$F(\lambda) = 0$$

we will call the λ-equation of the pair of forms ϕ, ψ. Since, as we have seen, F is merely multiplied by a constant different from zero when ϕ and ψ are subjected to a non-singular linear transformation, the roots of the λ-equation will not be changed by such a transformation. These roots, however, are irrational functions of the Θ's and hence of the a's and b's. We may therefore state the result:

* Cf. Exercise 13, § 90.

† The method by which we have here arrived at invariants of the system of two quadratic forms will be seen to be of very general application. If we have an integral rational invariant I of weight μ of a single form of the kth degree in n variables, we can find a large number of invariants of the system $\phi_1, \phi_2, \cdots \phi_p$ of p forms of the kth degree in n variables by forming the invariant I for the form $\lambda_1\phi_1 + \cdots + \lambda_p\phi_p$. This will be a polynomial in the λ's, each of whose coefficients is seen, precisely as above, to be an integral rational invariant of the systems of ϕ's of weight μ.

THEOREM 2. *The roots of the λ-equation of a pair of quadratic forms are absolute irrational invariants of this pair of forms with regard to non-singular linear transformations.*

It is clear that the multiplicity of any root of the λ-equation will not be changed by a non-singular linear transformation. Hence

THEOREM 3. *The multiplicities of the roots of the λ-equation are arithmetical invariants of the pair of quadratic forms with regard to non-singular linear transformations.*

If

$$\phi \equiv a_1 x_1^2 + \cdots + a_n x_n^2,$$

$$\psi \equiv \quad x_1^2 + \cdots + x_n^2,$$

the roots of the λ-equation are $a_1, \cdots a_n$. This example shows that the absolute invariants of Theorem 2 may have any values, and also that the arithmetical invariants of Theorem 3 are subject to no other restriction than the obvious one of being positive integers whose sum is n.

58. Reduction to Normal Form when the λ-Equation has no Multiple Roots. Although our main concern in this section is with the case in which the λ-equation has no multiple roots, we begin by establishing a theorem which applies to a much more general case.

THEOREM 1. *If λ_1 is a simple root of the λ-equation of the pair ϕ, ψ of quadratic forms in n variables, then, by a non-singular linear transformation, ϕ and ψ can be reduced respectively to the forms*

$$(1) \qquad \begin{cases} \lambda_1 c_1 z_1^2 + \phi_1(z_2, \cdots z_n) \\ c_1 z_1^2 + \psi_1(z_2, \cdots z_n) \end{cases}$$

where c_1 is a constant not zero and ϕ_1, ψ_1 are quadratic forms in the $n-1$ variables $z_2, \cdots z_n$.

To prove this, we will consider the pencil of forms

$$\phi - \lambda\psi \equiv \phi - \lambda_1\psi + (\lambda_1 - \lambda)\psi.$$

Since λ_1 is a root of the λ-equation of the pair of forms ϕ, ψ, the form $\phi - \lambda_1\psi$ is singular, and can therefore, by a suitable non-singular linear transformation, be written in a form in which one of the variables, say x_1', does not enter, $\phi - \lambda_1\psi \equiv \phi'(x_2', \cdots x_n')$.

If this transformation reduces ψ to ψ', we have

$$(3) \qquad \phi - \lambda\psi \equiv \phi'(x_2', \cdots x_n') + (\lambda_1 - \lambda)\psi'(x_1', \cdots x_n').$$

The discriminant of the quadratic form which stands here on the right cannot contain $\lambda_1 - \lambda$ as a factor more than once, since λ_1 is, by hypothesis, not a multiple root of the λ-equation of ϕ and ψ. It follows from this that the coefficient of $x_1'^2$ in the quadratic form ψ' cannot be zero, for otherwise the discriminant of the right-hand side of (3) would have a zero in the upper left-hand corner, and $\lambda_1 - \lambda$ would be a factor of all the elements of its first row and also of its first column; so that it would contain the factor $(\lambda_1 - \lambda)^2$.

Since the coefficient of $x_1'^2$ in ψ' is not zero, we can by Lagrange's reduction (Formulæ (2), (3), §45) obtain a non-singular linear transformation of the form

$$\begin{cases} z_1 = \gamma_1 x_1' + \gamma_2 x_2' + \cdots + \gamma_n x_n' \\ z_2 = \qquad x_2' \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ z_n = \qquad\qquad x_n' \end{cases}$$

which reduces ψ' to the form

$$c_1 z_1^2 + \psi_1(z_2, \cdots z_n) \qquad (c_1 \neq 0).$$

This transformation carries over the second member of (3) into

$$\phi'(z_2, \cdots z_n) + (\lambda_1 - \lambda)\psi_1(z_2, \cdots z_n) + (\lambda_1 - \lambda)c_1 z_1^2.$$

Combining these two linear transformations and writing

$$\phi'(z_2, \cdots z_n) + \lambda_1\psi_1(z_2, \cdots z_n) \equiv \phi_1(z_2, \cdots z_n),$$

we have thus obtained a non-singular linear transformation which effects the reduction,

$$\phi(x_1, \cdots x_n) - \lambda\psi(x_1, \cdots x_n) \equiv \phi_1(z_2, \cdots z_n) - \lambda\psi_1(z_2, \cdots z_n) + (\lambda_1 - \lambda)c_1 z_1^2.$$

If, here, we equate the coefficients of λ on both sides, and the terms independent of λ, we see that we have precisely the reduction of the forms ϕ, ψ to the forms (1); and the theorem is proved.

Let us now assume that the form ψ is non-singular, thus insuring that the λ-equation be of degree n. We will further assume that the roots $\lambda_1, \lambda_2, \cdots \lambda_n$ of this equation are all distinct. We can then, by the theorem just proved, reduce the forms ϕ, ψ to the forms (1) by a non-singular linear transformation. The λ-equation of these two forms is seen to differ from the λ-equation of the pair of forms in $(n-1)$ variables ϕ_1, ψ_1 only by the presence of the extra factor $\lambda_1 - \lambda$. Accordingly the λ-equation of the pair of forms ϕ_1, ψ_1 has as its roots $\lambda_2, \cdots \lambda_n$ and these are all simple roots. We may therefore apply the reduction of Theorem 1 to the two forms ϕ_1, ψ_1 and thus by a non-singular linear transformation of $z_2, \cdots z_n$ reduce them to the forms

$$\lambda_2 c_2 z_2'^2 + \phi_2(z_3', \cdots z_n'),$$
$$c_2 z_2'^2 + \psi_2(z_3', \cdots z_n').$$

This linear transformation may, by means of the additional formula

$$z_1' = z_1,$$

be regarded as a non-singular linear transformation of $z_1, \cdots z_n$ which carries over ϕ, ψ into the forms

$$\lambda_1 c_1 z_1'^2 + \lambda_2 c_2 z_2'^2 + \phi_2(z_3', \cdots z_n'),$$
$$c_1 z_1'^2 + \quad c_2 z_2'^2 + \psi_2(z_3', \cdots z_n').$$

Proceeding in this way, we establish the theorem:

THEOREM 2. *If ϕ, ψ are quadratic forms in $(x_1, \cdots x_n)$ of which the second is non-singular, and if the roots $\lambda_1, \cdots \lambda_n$ of their λ-equation are all distinct, there exists a non-singular linear transformation which carries over ϕ and ψ into*

$$\lambda_1 c_1 x_1'^2 + \lambda_2 c_2 x_2'^2 + \cdots + \lambda_n c_n x_n'^2,$$
$$c_1 x_1'^2 + \quad c_2 x_2'^2 + \cdots + \quad c_n x_n'^2$$

respectively, where $c_1, \cdots c_n$ are constants all different from zero.

Since none of the c's are zero, the linear transformation

$$x_i'' = \sqrt{c_i}\, x_i' \qquad (i = 1, 2, \cdots n)$$

is non-singular. Performing this transformation, we get the further result:

THEOREM 3. *Under the same conditions as in Theorem 2, ϕ and ψ may be reduced by means of a non-singular linear transformation to the normal forms*

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2,$$
$$x_1^2 + \quad x_2^2 + \cdots + \quad x_n^2.$$

From this we infer at once

THEOREM 4. *If in the two pairs of quadratic forms ϕ, ψ and ϕ', ψ' the forms ψ and ψ' are both non-singular, and if the λ-equations of these two pairs of forms have no multiple roots, a necessary and sufficient condition for the equivalence of the two pairs of forms is that these two λ-equations have the same roots; or, what amounts to the same thing, that the invariants $\Theta_0, \Theta_1, \cdots \Theta_n$ of the first pair of forms be proportional to the invariants $\Theta_0', \Theta_1', \cdots \Theta_n'$ of the second.*

EXERCISE

Prove that, under the conditions of Theorem 3, the reduction to the normal form can be performed in essentially only one way, the only possible variation consisting in a change of sign of some of the x's in the normal form.

59. Reduction to Normal Form when ψ is Definite and Non-singular. We now consider the case of two real quadratic forms ϕ, ψ of which ψ is definite and non-singular. Our main problem is to reduce this pair of forms to a normal form by means of a real linear transformation. For this purpose we begin by proving

THEOREM 1. *The λ-equation of a pair of real quadratic forms ϕ, ψ can have no imaginary root if the form ψ is definite and non-singular.*

For, if possible, let $\alpha + \beta i$ (α and β real) be an imaginary root of this λ-equation, so that $\beta \neq 0$. Then $\phi - \alpha\psi - i\beta\psi$ will be a singular quadratic form, and can therefore be reduced by a non-singular linear transformation

$$\begin{cases} x_1' = (p_{11} + iq_{11})x_1 + \cdots + (p_{1n} + iq_{1n})x_n, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_n' = (p_{n1} + iq_{n1})x_1 + \cdots + (p_{nn} + iq_{nn})x_n \end{cases}$$

to the sum of k squares, where $k < n$,

$$(1) \qquad \phi - \alpha\psi - i\beta\psi \equiv x_1'^2 + x_2'^2 + \cdots + x_k'^2.$$

Let

$$(2) \qquad y_l = p_{l1}x_1 + \cdots + p_{ln}x_n,$$

$$(3) \qquad z_l = q_{l1}x_1 + \cdots + q_{ln}x_n,$$

so that $\qquad x_l' = y_l + iz_l.$

By equating the coefficients of i on the two sides of (1) we thus get

$$(4) \qquad -\beta\psi \equiv 2y_1z_1 + 2y_2z_2 + \cdots + 2y_kz_k.$$

Let us now determine $x_1, \cdots x_n$ so as to make the right-hand side of (4) vanish, for instance by means of the equations

$$y_1 = y_2 = \cdots = y_k = 0.$$

A reference to (2) shows that we have here a system of k real homogeneous linear equations in n unknowns, so that real values of $x_1, \cdots x_n$ not all zero can be found satisfying these equations. For these values of the variables, we see from (4) that ψ vanishes; but this is impossible (cf. the Corollary of Theorem 3, § 52), since ψ is by hypothesis non-singular and definite.

THEOREM 2. *If ψ is a non-singular definite quadratic form and ϕ is any real quadratic form, the pair of forms ϕ, ψ can be reduced by a real non-singular linear transformation to the normal form*

$$(5) \qquad \begin{cases} \phi \equiv \pm(\lambda_1 x_1'^2 + \cdots + \lambda_n x_n'^2), \\ \psi \equiv \pm(\ \ x_1'^2 + \cdots + \ \ x_n'^2), \end{cases}$$

where $\lambda_1, \cdots \lambda_n$ are the roots of the λ-equation, and the upper or lower sign is to be used in both cases according as ψ is a positive or a negative form.

The proof of this theorem is very similar to the proof of Theorem 2, § 58. We must first prove, as in Theorem 1, § 58, that ϕ, ψ can be reduced by a real non-singular linear transformation to the forms

$$(6) \qquad \begin{cases} \lambda_1 c_1 z_1^2 + \phi_1(z_2, \cdots z_n) \\ c_1 z_1^2 + \psi_1(z_2, \cdots z_n). \end{cases} \qquad (c_1 \neq 0)$$

To prove this, we consider the pencil of forms

$$\phi - \lambda\psi \equiv \phi - \lambda_1\psi + (\lambda_1 - \lambda)\psi.$$

Since λ_1 is real by Theorem 1, $\phi - \lambda_1\psi$ is a real singular quadratic form, and can therefore by a real non-singular linear transformation be reduced to a form in which one of the variables does not enter,

$$\phi - \lambda_1\psi \equiv \phi'(x_2', \cdots x_n').$$

If this transformation reduces ψ to ψ', we have

$$\phi - \lambda\psi \equiv \phi'(x_2', \cdots x_n') + (\lambda_1 - \lambda)\psi'(x_1', \cdots x_n'). \tag{7}$$

At this point comes the essential difference between the case we are now considering and the case considered in § 58, as λ_1 may now be a multiple root of the discriminant of the right-hand side of (7). We need, then, a different method for showing that the coefficient of $x_1'^2$ in ψ' is not zero. For this purpose it is sufficient to notice that ψ, and therefore also ψ', is a non-singular definite form, and that accordingly, by Theorem 4, § 52, the coefficient of none of the square terms in ψ' can be zero.

Having thus shown that the coefficient of $x_1'^2$ in ψ' is not zero, we can apply Lagrange's reduction to ψ', and thus complete the reduction of the forms ϕ, ψ to the forms (6) precisely as in the proof of Theorem 1, § 58, noticing that the transformation we have to deal with is real.

In (6), ϕ_1, ψ_1 are real quadratic forms in the $n-1$ variables $z_2, \cdots z_n$. Moreover, since

$$\psi(x_1, \cdots x_n) \equiv c_1 z_1^2 + \psi_1(z_2, \cdots z_n)$$

is non-singular and definite, it follows that the same is true of ψ_1. For, if ψ_1 were either singular or indefinite, we could find values of $z_2, \cdots z_n$ not all zero and such that $\psi_1 = 0$; and these values together with the value $z_1 = 0$ would make $\psi = 0$. This, however, is impossible by the Corollary of Theorem 3, § 52.

The λ-equation of the two forms ϕ_1, ψ_1 evidently differs from the λ-equation of ϕ, ψ only by the absence of the factor $\lambda - \lambda_1$. The roots of the λ-equation of ϕ_1, ψ_1 are therefore $\lambda_2, \cdots \lambda_n$, so that if we reduce ϕ_1 and ψ_1 by the method already used for ϕ, ψ (we have just seen that ϕ_1, ψ_1 satisfy all the conditions imposed on ϕ, ψ), we get

$$\phi_1(z_2, \cdots z_n) \equiv \lambda_2 c_2 z_2'^2 + \phi_2(z_3', \cdots z_n'),$$
$$\psi_1(z_2, \cdots z_n) \equiv \quad c_2 z_2'^2 + \psi_2(z_3', \cdots z_n').$$

Proceeding in this way, we finally reduce ϕ, ψ by a real non-singular linear transformation to the forms

$$\begin{cases} \phi \equiv \lambda_1 c_1 y_1^2 + \cdots + \lambda_n c_n y_n^2, \\ \psi \equiv \quad c_1 y_1^2 + \cdots + c_n y_n^2. \end{cases} \tag{8}$$

Since ψ is definite, the constants $c_1, \cdots c_n$ are all positive or all negative according as ψ is a positive or a negative form. By means of the further non-singular real linear transformation

$$x_i' = \sqrt{|c_i|}\, y_i \qquad (i = 1, 2, \cdots n).$$

the forms (8) may be reduced to the forms (5), and our theorem is proved.

EXERCISES

1. If ϕ is a real quadratic form in n variables of rank r, prove that it can be reduced by a real orthogonal transformation in n variables to the form

$$c_1 x_1^2 + c_2 x_2^2 + \cdots + c_r x_r^2.$$

Cf. Exercises, § 52.

2. Show that the determinant of the orthogonal transformation of Exercise 1 may be taken at pleasure as $+1$ or -1.

3. Discuss the metrical classification of real quadric surfaces along the following lines:

Assume the equation in non-homogeneous rectangular coördinates, and show that by a transformation to another system of rectangular coördinates having the same origin the equation can be reduced to a form where the terms of the second degree have one or the other of the five forms (the A's being positive constants)

$$A_1 x_1^2 + A_2 x_2^2 + A_3 x_3^2,$$
$$A_1 x_1^2 + A_2 x_2^2 - A_3 x_3^2,$$
$$A_1 x_1^2 + A_2 x_2^2,$$
$$A_1 x_1^2 - A_2 x_2^2,$$
$$A_1 x_1^2.$$

Then simplify each of the non-homogeneous equations thus obtained by further transformations of coördinates; thus getting finally the standard forms of the equations of ellipsoids, hyperboloids, paraboloids, cones, cylinders, and planes which are discussed in all elementary text-books of solid analytic geometry.