CHAPTER XI

REAL QUADRATIC FORMS

50. The Law of Inertia. We come now to the study of real quadratic forms and the effect produced on them by real linear transformations.

We notice, here, to begin with, that the only operations involved in the last chapter are rational operations (*i.e.* addition, subtraction, multiplication, and division) with the single exception of the radicals which come into formula (2), § 46. In particular the reduction of § 45 (or the alternative reduction of § 49) involves only rational operations. Consequently, since rational operations performed on real quantities give real results, we have

THEOREM 1. *A real quadratic form of rank r can be reduced by means of a real non-singular linear transformation to the form*

$$c_1x_1'^2 + c_2x_2'^2 + \cdots + c_rx_r'^2 \tag{1}$$

where $c_1, \cdots c_r$ *are real constants none of which are zero.*

As we saw in the last chapter, this reduction can be performed in a variety of ways, and the values of the coefficients $c_1, \cdots c_r$ in the reduced form will be different for the different reductions. The signs of these coefficients, apart from the order in which they occur, will not depend on the particular reduction used, as is stated in the following important theorem discovered independently by Jacobi and Sylvester and called by the latter the Law of Inertia of Quadratic Forms:

THEOREM 2. *If a real quadratic form of rank r is reduced by two real non-singular linear transformations to the forms* (1) *and*

$$k_1x_1''^2 + k_2x_2''^2 + \cdots + k_rx_r''^2, \tag{2}$$

respectively, then the number of positive c's in (1) *is equal to the number of positive k's in* (2).

In order to prove this, let us suppose that the x''s and x'''s have been so numbered that the first μ of the c's and the first ν of the k's are positive while all the remaining c's and k's are negative. Our

theorem will be established if we can show that $\mu = \nu$. If this is not the case, one of the two integers μ and ν must be the greater, and it is merely a matter of notation to assume that $\mu > \nu$. We will prove that this assumption leads to a contradiction.

If we regard the x''s and x'''s simply as abbreviations for certain linear forms in the x's, (1) and (2) are both of them identically equal to the original quadratic form, and hence to each other. This identity may be written

$$\begin{aligned} c_1x_1'^2 + \cdots + c_\mu x_\mu'^2 - |c_{\mu+1}|x_{\mu+1}'^2 - \cdots - |c_r|x_r'^2 \\ \equiv k_1x_1''^2 + \cdots + k_\nu x_\nu''^2 - |k_{\nu+1}|x_{\nu+1}''^2 - \cdots - |k_r|x_r''^2. \end{aligned} \tag{3}$$

Let us now consider the system of homogeneous linear equations in $(x_1, \cdots x_n)$,

$$x_1'' = 0, \cdots x_\nu'' = 0, \; x_{\mu+1}' = 0, \cdots x_n' = 0. \tag{4}$$

We have here $\nu + n - \mu < n$ equations. Hence, by Theorem 3, Corollary 1, § 17, we can find a solution of these equations in which all the unknowns are not zero. Let $(y_1, \cdots y_n)$ be such a solution and denote by y_i', y_i'' the values of x_i', x_i'' when the constants $y_1, \cdots y_n$ are substituted in them for the variables $x_1, \cdots x_n$. Substituting the y's for the x's in (3) gives

$$c_1y_1'^2 + \cdots + c_\mu y_\mu'^2 = -|k_{\nu+1}|y_{\nu+1}''^2 - \cdots - |k_r|y_r''^2.$$

The expression on the left cannot be negative, and that on the right cannot be positive, hence they must both be zero; and this is possible only if

$$y_1' = \cdots = y_\mu' = 0.$$

But by (4) we also have $\quad y_{\mu+1}' = \cdots = y_n' = 0.$

That is, $(y_1, \cdots y_n)$ is a solution, not composed exclusively of zeros, of the system of n homogeneous linear equations in n unknowns,

$$x_1' = 0, \; x_2' = 0, \cdots x_n' = 0.$$

The determinant of these equations must therefore be zero, that is, the linear transformation which carries over the x's into the x''s must be a singular transformation. We are here led to a contradiction and our theorem is proved.

We can thus associate with every real quadratic form two integers P and N, namely, the number of positive and negative coefficients respectively which we get when we reduce the form by any real non-singular linear transformation to the form (1). These two numbers are evidently arithmetical invariants of the quadratic form with regard to real non-singular linear transformations, since two real quadratic forms which can be transformed into one another by means of such a transformation can obviously be reduced to the *same* expression of form (1).*

The two arithmetical invariants P and N which we have thus arrived at, and the arithmetical invariant r which we had before, are not independent since we have the relation

(5) $$P+N=r.$$

One of the invariants P and N is therefore superfluous and either might be dispensed with. It is found more convenient, however, to use neither P nor N, but their difference,

(6) $$s=P-N,$$

which is called the *signature* of the quadratic form.

DEFINITION. *By the signature of a real quadratic form is understood the difference between the number of positive and the number of negative coefficients which we obtain when we reduce the form by any real non-singular linear transformation to the form* (1).

Since the integers P and N used above were arithmetical invariants, their difference s will also be an arithmetical invariant. It should be noticed, however, that s is not necessarily a *positive* integer. We have thus proved

THEOREM 3. *The signature of a quadratic form is an arithmetical invariant with regard to real non-singular linear transformations.*

EXERCISES

1. Prove that the rank r and the signature s of a quadratic form are either both even or both odd; and that
$$-r \leq s \leq r.$$

2. Prove that any two integers r and s (r positive or zero) satisfying the conditions of Exercise 1 may be the rank and signature respectively of a quadratic form.

* P is sometimes called the *index of inertia* of the quadratic form.

3. Prove that a necessary and sufficient condition that a real quadratic form of rank r and signature s be factorable into two real linear factors is that

either $\quad r<2;$

or $\quad r=2, s=0.$

4. A quadratic form of rank r shall be said to be *regularly arranged* (cf. § 20, Theorem 4) if the x's are so numbered that no two consecutive A's are zero in the set

$$A_0=1, A_1=a_{11}, A_2=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix}, \dots A_r=\begin{vmatrix} a_{11} & \cdots & a_{1r} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{r1} & \cdots & a_{rr}\end{vmatrix},$$

and that $A_r \neq 0$. Prove that if the form is real and any one of these A's is zero, the two adjacent A's have opposite signs.

[SUGGESTION. In this exercise and the following ones, the work of § 49 should be consulted.]

5. Prove that the signature of a regularly arranged real quadratic form is equal to the number of permanences minus the number of variations of sign in the sequence of the A's, if the A's which are zero are counted as positive or as negative at pleasure.

6. Defining the expression sgn x (read *signum* x) by the equations

$$\begin{aligned} \operatorname{sgn} x &= +1 \qquad x>0, \\ \operatorname{sgn} x &= \ \ 0 \qquad x=0, \\ \operatorname{sgn} x &= -1 \qquad x<0, \end{aligned}$$

show that the signature of a regularly arranged real quadratic form of rank r is

$$\operatorname{sgn}(A_0A_1)+\operatorname{sgn}(A_1A_2)+\cdots+\operatorname{sgn}(A_{r-1}A_r).$$

51. Classification of Real Quadratic Forms. We saw in the last section that a real quadratic form has two invariants with regard to real non-singular linear transformations, — its rank and its signature. The main result to be established in the present section (Theorem 2) is that these two invariants form a complete system.

If in § 46 the c's and k's are real, the transformation (2) will be real when, but only when, each c has the same sign as the corresponding k. All that we can infer from the reasoning of that section now is, therefore, that if a real quadratic form of rank r can be reduced by a real non-singular linear transformation to the form

$$c_1x_1^2+\cdots+c_rx_r^2,$$

it can also be reduced by a real non-singular linear transformation to the form

$$k_1x_1^2+\cdots+k_rx_r^2,$$

where the k's are arbitrarily given real constants, not zero, subject to the condition that each k has the same sign as the corresponding c. Using the letters P and N for the number of positive and negative c's respectively, the transformation can be so arranged that the first P c's are positive, the last N negative. Accordingly the first P k's can be taken as $+1$, the last N as -1. From equations (5) and (6) of § 50, we see that P and N may be expressed in terms of the rank and signature of the form by the formulæ

(1) $$P=\frac{r+s}{2}, \qquad N=\frac{r-s}{2}.$$

Thus we have the theorem:

THEOREM 1. *A real quadratic form of rank r and signature s can be reduced by a real non-singular linear transformation to the normal form*

(2) $$x_1^2+\cdots+x_P^2-x_{P+1}^2-\cdots-x_r^2$$

where P is given by (1).

We are now able to prove the fundamental theorem:

THEOREM 2. *A necessary and sufficient condition that two real quadratic forms be equivalent with regard to real non-singular linear transformations is that they have the same rank and the same signature.*

That this is a necessary condition is evident from the invariance of rank and signature. That it is sufficient follows from the fact that if the two forms have the same rank and signature, they can both be reduced to the same normal form (2).

DEFINITION. *All real quadratic forms, equivalent with regard to real non-singular linear transformations to a given form, and therefore to each other, are said to form a class.**

Thus, for instance, since every real non-singular quadratic form in four variables can be reduced to one or the other of the five normal forms,

(3) $$\begin{cases} x_1^2+x_2^2+x_3^2+x_4^2, \\ x_1^2+x_2^2+x_3^2-x_4^2, \\ x_1^2+x_2^2-x_3^2-x_4^2, \\ x_1^2-x_2^2-x_3^2-x_4^2, \\ -x_1^2-x_2^2-x_3^2-x_4^2, \end{cases}$$

* This term may be used in a similar manner whenever the conception of equivalence is involved.

we see that all such forms belong to one or the other of five classes characterized by the values

$$s=4,\ 2,\ 0,\ -2,\ -4, \qquad r=4.$$

If, however, as is the case in many problems in geometry, we are concerned not with quadratic forms, but with the equations obtained by equating these forms to zero, the number of classes to be distinguished will be reduced by about one half, since two equations are the same if their first members differ merely in sign.

Thus there are only three classes of non-singular quadric surfaces with real equations, whose normal forms are obtained by equating the first three of the forms (3) to zero. These equations written in non-homogeneous coördinates are

$$X^2+Y^2+Z^2=-1,$$
$$X^2+Y^2+Z^2=1,$$
$$X^2+Y^2-Z^2=1.$$

The first of these represents an imaginary sphere, the second a real sphere, and the third an unparted hyperboloid generated by the revolution of a rectangular hyperbola about its conjugate axis. It may readily be proved that this last surface may also be generated by the revolution of either of the lines $$Y=1,\ X=\pm Z$$

about the axis of Z. We may therefore say:

THEOREM 3. *There are three, and only three, classes of non-singular quadric surfaces with real equations. In the first the surfaces are imaginary; in the second real, but their rulings are imaginary; in the third they are real, and the rulings through their real points are real.**

This classification is complete from the point of view we have adopted of regarding quadric surfaces as equivalent if one can be transformed into the other by a real non-singular collineation. The more familiar classification does not adopt this projective view, but distinguishes in our second class between ellipsoids, biparted hyperboloids, and elliptic paraboloids; and in the third class between unparted hyperboloids and hyperbolic paraboloids.

* If, as here, we consider not real quadratic forms, but real homogeneous quadratic equations we must use, not s, but $|s|$ as an invariant. In place of $|s|$ we may use what is known as the *characteristic* of the quadratic form, that is the smaller of the two integers P N This characteristic is simply $\frac{1}{2}(r-|s|)$.

EXERCISES

1. Prove that there are $\frac{1}{2}(n+1)(n+2)$ classes of real quadratic forms in n variables.

2. Give a complete classification of singular quadric surfaces with real equations from the point of view of the present section.

52. Definite and Indefinite Forms.

DEFINITION. *By an indefinite quadratic form is understood a real quadratic form such that, when it is reduced to the normal form* (2), § 51, *by a real non-singular linear transformation, both positive and negative signs occur. All other real quadratic forms are called definite;* and we distinguish between positive and negative definite forms according as the terms in the normal form are all positive or all negative.*

In other words, a real quadratic form of rank r and signature s is definite if $s = \pm r$, otherwise it is indefinite.†

The names *definite* and *indefinite* have been given on account of the following fundamental property:

THEOREM 1. *An indefinite quadratic form is positive for some real values of the variables, negative for others. A positive definite form is positive or zero for all real values of the variables; a negative definite form, negative or zero.*

The part of this theorem which relates to definite forms follows directly from the definition. To prove the part concerning indefinite forms, suppose the form reduced by a real non-singular linear transformation to the normal form

$$x_1'^2 + \cdots + x_P'^2 - x_{P+1}'^2 - \cdots - x_r'^2 \tag{1}$$

Regarding the x''s as abbreviations for certain real linear forms in the x's, let us consider the system of $n-P$ homogeneous linear equations

$$x'_{P+1} = 0,\ x'_{P+2} = 0, \cdots x'_n = 0. \tag{2}$$

Since these equations are real, and their number is less than the number of unknowns, they have a real solution not consisting

* Some writers reserve the name *definite* for non-singular forms, and call the singular definite forms *semidefinite*.

† Otherwise stated, the condition for a definite form is that the *characteristic* be zero. Cf. the footnote to Theorem 3, § 51.

exclusively of zeros. Let $(y_1, \cdots y_n)$ be such a solution. This solution cannot satisfy all the equations

$$x_1' = 0, \cdots x_P' = 0, \tag{3}$$

for equations (2) and (3) together form a system of n homogeneous linear equations in n unknowns whose determinant is not zero, since it is the determinant of the linear transformation which reduces the given quadratic form to the normal form (1). Accordingly, if we substitute $(y_1, \cdots y_n)$ for the variables $(x_1, \cdots x_n)$ in the given quadratic form, this form will have a positive value, as we see from the reduced form (1).

Similarly, by choosing for the x's a real solution of the equations

$$x_1' = 0, \cdots x_P' = 0, \qquad x'_{r+1} = 0, \cdots x'_n = 0,$$

which does not consist exclusively of zeros, we see that the quadratic form takes on a negative value.

We pass now to some theorems which will be better appreciated by the reader if he considers their geometrical meaning in the case $n = 4$.

THEOREM 2. *If an indefinite quadratic form is positive at the real point $(y_1, \cdots y_n)$ and negative at the real point $(z_1, \cdots z_n)$, then there are two real points linearly dependent on these two, but linearly independent of each other, at which the quadratic form is zero, and neither of which is a vertex of the form.*

The condition that the quadratic form

$$\sum_1^n a_{ij}x_ix_j \tag{4}$$

vanish at the point $(y_1 + \lambda z_1, \cdots y_n + \lambda z_n)$ is

$$\sum_1^n a_{ij}y_iy_j + 2\lambda\sum_1^n a_{ij}y_iz_j + \lambda^2\sum_1^n a_{ij}z_iz_j = 0.$$

This quadratic equation in λ has two real distinct roots, since, from our hypothesis that (4) is positive at y and negative at z, it follows that

$$\left(\sum_1^n a_{ij}y_iz_j\right)^2 - \left(\sum_1^n a_{ij}y_iy_j\right)\left(\sum_1^n a_{ij}z_iz_j\right) > 0.$$

Let us call these roots λ_1 and λ_2. Then the points

$$(y_1 + \lambda_1 z_1, \cdots y_n + \lambda_1 z_n), \qquad (y_1 + \lambda_2 z_1, \cdots y_n + \lambda_2 z_n) \tag{5}$$

are two real points linearly dependent on the points y and z at which (4) vanishes.

Next notice that

$$(6)\qquad \begin{vmatrix} y_i+\lambda_1 z_i & y_j+\lambda_1 z_j \\ y_i+\lambda_2 z_i & y_j+\lambda_2 z_j \end{vmatrix} = \begin{vmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{vmatrix} \cdot \begin{vmatrix} y_i & y_j \\ z_i & z_j \end{vmatrix}.$$

Since the points y and z are linearly independent, the integers i, j can be so chosen that the last determinant on the right of (6) is not zero. Then the determinant on the left of (6) is not zero; and, consequently, the points (5) are linearly independent.

In order, finally, to prove that neither of the points (5) is a vertex, denote them for brevity by

$$(Y_1, \cdots Y_n), \qquad (Z_1, \cdots Z_n).$$

Letting $\lambda_1 - \lambda_2 = 1/\mu$, we have

$$z_i = \mu Y_i - \mu Z_i \qquad (i = 1, 2, \cdots n).$$

Therefore

$$(7)\qquad \sum_1^n a_{ij} z_i z_j = \mu^2 \sum_1^n a_{ij} Y_i Y_j - 2\,\mu^2 \sum_1^n a_{ij} Y_i Z_j + \mu^2 \sum_1^n a_{ij} Z_i Z_j.$$

Since the points Y and Z have been so determined that (4) vanishes at them, the first and last terms on the right of (7) are zero. If either Y or Z were a vertex, the middle term would also be zero; but this is impossible since the left-hand member of (7) is, by hypothesis, negative. Thus our theorem is proved.

For the sake of completeness we add the corollary, whose truth is at once evident:

COROLLARY. *The only points linearly dependent on y and z at which the quadratic form vanishes are points linearly dependent on one or the other of the points referred to in the theorem; and none of these are vertices.*

We come now to a theorem of fundamental importance in the theory of quadratic forms.

THEOREM 3. *A necessary and sufficient condition that a real quadratic form be definite is that it vanish at no real points except its vertices and the point* $(0, 0, \cdots 0)$.

Suppose, first, that we have a real quadratic form which vanishes at no real points except its vertices and the point $(0, 0, \cdots 0)$. If it were indefinite, we could (Theorem 1) find two real points y, z, at one of which it is positive, at the other, negative. Hence (Theorem 2) we could find two real points linearly dependent on y and z, at which the quadratic form vanishes. Neither of these will be the point $(0, 0, \cdots 0)$, since, by Theorem 2, they are linearly independent. Moreover, they are neither of them vertices. Thus we see that the form must be definite, and the sufficiency of the condition is established.

It remains to be proved that a definite form can vanish only at its vertices and at the point $(0, 0, \cdots 0)$.

Suppose (4) is definite and that $(y_1, \cdots y_n)$ is any real point at which it vanishes. Then,

$$\sum_1^n a_{ij}(x_i + \lambda y_i)(x_j + \lambda y_j) \equiv \sum_1^n a_{ij} x_i x_j + 2\,\lambda \sum_1^n a_{ij} x_i y_j.$$

If y were neither a vertex nor the point $(0, 0, \cdots 0)$, $\Sigma a_{ij} x_i y_j$ would not vanish identically, and we could find a real point $(z_1, \cdots z_n)$ such that

$$k = \sum_1^n a_{ij} z_i y_j \neq 0.$$

If we let

$$c = \sum_1^n a_{ij} z_i z_j,$$

we have

$$(8)\qquad \sum_1^n a_{ij}(z_i + \lambda y_i)(z_j + \lambda y_j) = c + 2\,\lambda k.$$

For a given real value of λ, the left-hand side of this equation is simply the value of the quadratic form (4) at a certain real point. Accordingly, for different values of λ it will not change sign, while the right-hand side of (8) has opposite signs for large positive and large negative values of λ. Thus the assumption that y was neither a vertex nor the point $(0, 0, \cdots 0)$ has led to a contradiction; and our theorem is proved.

COROLLARY. *A non-singular definite quadratic form vanishes, for real values of the variables, only when its variables are all zero.*

As a simple application of the last corollary we will prove

THEOREM 4. *In a non-singular definite form, none of the coefficients of the square terms can be zero.*

For suppose the form (4) were definite and non-singular; and that $a_{ii}=0$. Then the form would vanish at the point

$$x_1=\cdots=x_{i-1}=x_{i+1}=\cdots=x_n=0,\ x_i=1;$$

and this is impossible, since this is not the point $(0, 0, \cdots 0)$.

EXERCISES

1. DEFINITION. *By an orthogonal transformation* is understood a linear transformation which carries over the variables* $(x_1, \cdots x_n)$ *into the variables* $(x_1', \cdots x_n')$ *in such a way that*

$$x_1^2+x_2^2+\cdots+x_n^2 \equiv x_1'^2+x_2'^2+\cdots+x_n'^2.$$

Prove that every orthogonal transformation is non-singular, and, in particular, that its determinant must have the value $+1$ or -1.

2. Prove that all orthogonal transformations in n variables form a group; and that the same is true of all orthogonal transformations in n variables of determinant $+1$.

3. Prove that a necessary and sufficient condition that a linear transformation be orthogonal is that it leave the "distance"

$$\sqrt{(y_1-z_1)^2+(y_2-z_2)^2+\cdots+(y_n-z_n)^2}$$

between every pair of points $(y_1, \cdots y_n)$, $(z_1, \cdots z_n)$ invariant.

4. Prove that if $n=3$, and if x_1, x_2, x_3 be interpreted as non-homogeneous rectanglar coördinates in space, an orthogonal transformation represents either a rigid displacement which leaves the origin fixed, or such a displacement combined with reflection in a plane through the origin.

Show that the first of these cases will occur when the determinant of the transformation is $+1$, the second when this determinant is -1.

5. If the coefficients of a linear transformation are denoted in the usual way by c_{ij}, prove that a necessary and sufficient condition that the transformation be orthogonal is that

$$c_{1i}^2+c_{2i}^2+\cdots+c_{ni}^2=1 \qquad (i=1, 2, \cdots n),$$

$$c_{1i}c_{1j}+c_{2i}c_{2j}+\cdots+c_{ni}c_{nj}=0 \qquad \begin{cases} i=1, 2, \cdots n \\ j=1, 2, \cdots n \end{cases} i \neq j.$$

Show that these will still be necessary and sufficient conditions for an orthogonal transformation if the two subscripts of every c be interchanged.†

* The matrix of such a transformation is called an orthogonal matrix, and its determinant an orthogonal determinant.

† We have here $\frac{1}{2}n(n+1)$ relations between the n^2 coefficients of the transformation. This suggests that it should be possible to express all the coefficients in terms of

$$n^2-\frac{n(n+1)}{2}=\frac{n(n-1)}{2}$$

of them, or if we prefer in terms of $\frac{1}{2}n(n-1)$ other parameters. For Cayley's discussion of this question cf. Pascal's book, *Die Determinanten*, § 47. Cayley's formulæ, however, do not include all orthogonal transformations except as limiting cases.

CHAPTER XII

THE SYSTEM OF A QUADRATIC FORM AND ONE OR MORE LINEAR FORMS

53. Relations of Planes and Lines to a Quadric Surface. If the plane

$$(1) \qquad u_1x_1+u_2x_2+u_3x_3+u_4x_4=0$$

is a true tangent plane to the quadric surface

$$(2) \qquad \sum_1^4 a_{ij}x_ix_j=0,$$

there will be a point (y_1, y_2, y_3, y_4) (namely the point of contact) lying in (1) and such that its polar plane

$$(3) \qquad \sum_1^4 a_{ij}x_iy_j=0$$

coincides with (1). From elementary analytic geometry we know that a necessary and sufficient condition that two equations of the first degree represent the same plane is that their coefficients be proportional. Accordingly, from the coincidence of (1) and (3), we deduce the equations

$$(4) \qquad \begin{cases} a_{11}y_1+a_{12}y_2+a_{13}y_3+a_{14}y_4-\rho u_1=0, \\ a_{21}y_1+a_{22}y_2+a_{23}y_3+a_{24}y_4-\rho u_2=0, \\ a_{31}y_1+a_{32}y_2+a_{33}y_3+a_{34}y_4-\rho u_3=0, \\ a_{41}y_1+a_{42}y_2+a_{43}y_3+a_{44}y_4-\rho u_4=0. \end{cases}$$

From the fact that the point y lies on (1), we infer the further relation

$$(5) \qquad u_1y_1+u_2y_2+u_3y_3+u_4y_4=0.$$

These equations (4) and (5) have been deduced on the supposition that (1) is a true tangent plane to (2). They still hold if it is a pseudo-tangent plane; for then the quadric must be a cone, and a vertex of this cone must lie on (1). Taking the point y as this vertex, equation (5) is fulfilled. Moreover, since now the first