We have tacitly assumed that it is possible to find points y, z, w, constructed as indicated above, and not lying on the quadric surface. We leave it for the reader to show that, if the quadric surface is not a cone, this will always be possible in an infinite number of ways. A cone, however, has no self-conjugate tetrahedron, and in this case the above reduction is impossible.

EXERCISES

1. Prove that if the discriminant of a quadric surface is zero, the equation of the surface can always be reduced, by a suitable collineation, to a form in which the coördinate x_4 does not enter.

[SUGGESTION. Show, by using the results of this chapter, that if the vertex of a quadric cone is at the origin, $a_{14} = a_{24} = a_{34} = a_{44} = 0$.]

2. Show that, provided the cone has a finite vertex, the collineation of Exercise 1 may be taken in the form

$$x_1' = x_1 + \alpha x_4,$$
$$x_2' = x_2 + \beta x_4,$$
$$x_3' = x_3 + \gamma x_4,$$
$$x_4' = x_4.$$

[SUGGESTION. Use non-homogeneous coördinates.]

CHAPTER X

QUADRATIC FORMS

42. The General Quadratic Form and its Polar. The general quadratic form in n variables is

$$(1)\qquad \sum_1^n a_{ij}x_ix_j \equiv \begin{aligned} &a_{11}x_1^2 + a_{12}x_1x_2 + \dots + a_{1n}x_1x_n \\ &+ a_{21}x_2x_1 + a_{22}x_2^2 + \dots + a_{2n}x_2x_n \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ &+ a_{n1}x_nx_1 + a_{n2}x_nx_2 + \dots + a_{nn}x_n^2, \end{aligned}$$

where $a_{ij} = a_{ji}$.* The bilinear form $\sum_1^n a_{ij}y_iz_j$ is called the *polar form* of (1). Subjecting (1) to the linear transformation

$$\text{C}\quad \begin{cases} x_1 = c_{11}x_1' + \dots + c_{1n}x_n', \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ x_n = c_{n1}x_1' + \dots + c_{nn}x_n', \end{cases}$$

we get a new quadratic form

$$(2)\qquad \sum_1^n a_{ij}'x_i'x_j'.$$

The polar form of (2) is $\Sigma a_{ij}'y_i'z_j'$. If we transform the y's and z's of the polar form of (1) by the same transformation C, we get a new bilinear form $\sum_1^n \bar{a}_{ij}y_i'z_j'$. We will now prove that $\bar{a}_{ij} = a_{ij}'$.

We have the identities

$$(3)\qquad \sum_1^n a_{ij}x_ix_j \equiv \sum_1^n a_{ij}'x_i'x_j',$$

$$(4)\qquad \sum_1^n a_{ij}y_iz_j \equiv \sum_1^n \bar{a}_{ij}y_i'z_j'.$$

* It should be clearly understood that this restriction is a matter of convenience, not of necessity. If it were not made, the quadratic form would be neither more nor less general.

Each of these we may regard as identities in the x''s, y''s, z' s, the x's, y's, z's being merely abbreviations for certain polynomials in the corresponding primed letters. The last written identity reduces, when we let $y_i' = z_i' = x_i'$ $(i = 1, 2, \dots n)$, to

$$\sum_1^n a_{ij}x_ix_j \equiv \sum_1^n \bar{a}_{ij}x_i'x_j'.$$

Combining this with (3) gives

$$\sum_1^n \bar{a}_{ij}x_i'x_j' \equiv \sum_1^n a_{ij}'x_i'x_j'.$$

Hence $$\bar{a}_{ii} = a_{ii}' \text{ and } \bar{a}_{ij} + \bar{a}_{ji} = a_{ij}' + a_{ji}'.$$

We have assumed that $a_{ij}' = a_{ji}'$, these being merely the coefficients of a certain quadratic form, and we proved, in Theorem 4, § 36, that $\bar{a}_{ij} = \bar{a}_{ji}$. Hence we infer that $\bar{a}_{ij} = a_{ij}'$.

From this fact and from (4) we get at once the further result:

$$\sum_1^n a_{ij}'y_i'z_j' \equiv \sum_1^n a_{ij}y_iz_j.$$

That is:

THEOREM. *The polar form*

$$\sum_1^n a_{ij}y_iz_j$$

is an absolute covariant of the system composed of the quadratic form

$$\sum_1^n a_{ij}x_ix_j$$

and the two points $(y_1, \cdots y_n)$, $(z_1, \cdots z_n)$.

43. The Matrix and the Discriminant of a Quadratic Form.

DEFINITION. *The matrix*

$$\mathrm{a} = \left\| \begin{matrix} a_{11} \cdots a_{1n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{n1} \cdots a_{nn} \end{matrix} \right\|$$

is called the matrix of the quadratic form

(1) $$\sum_1^n a_{ij}x_ix_j.$$

The determinant of a *is called the discriminant of* (1); *and the rank of* a, *the rank of* (1). *If the discriminant vanishes,* (1) *is called singular.*

The matrix of (1) is the matrix of its polar form. Moreover, as was shown in the last section, if the x's in (1) are subjected to a linear transformation, and the y's and z's in the polar of (1) are subjected to the same transformation, the matrix of the new quadratic form will be the same as the matrix of the new bilinear form. But we saw, in Theorem 1, § 36, how the matrix of a bilinear form is changed by linear transformations of the variables. Thus we have the theorem:

THEOREM 1. *If in the quadratic form* (1) *with matrix* a *we subject the x's to a linear transformation with matrix* c, *we obtain a new quadratic form with matrix* c′ac, *where* c′ *is the conjugate of* c.

From this there follow at once, precisely as in § 36, the further results:

THEOREM 2. *The rank of a quadratic form is not changed by non-singular linear transformation.*

THEOREM 3. *The discriminant of a quadratic form is a relative invariant of weight two.*

44. Vertices of Quadratic Forms.

DEFINITION. *By a vertex of the quadratic form*

(1) $$\sum_1^n a_{ij}x_ix_j,$$

we understand a point $(c_1, \cdots c_n)$ *where the c's are not all zero, such that*

(2) $$\sum_1^n a_{ij}x_ic_j \equiv 0.$$

A quadratic form clearly vanishes at all of its vertices.

It is merely another way of stating this definition when we say:

THEOREM 1. *A necessary and sufficient condition that* $(c_1, \cdots c_n)$ *be a vertex of* (1) *is that it be a solution, not consisting exclusively of zeros, of the system of equations*

(3) $$\begin{matrix} a_{11}c_1 + \cdots + a_{1n}c_n = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{n1}c_1 + \cdots + a_{nn}c_n = 0. \end{matrix}$$

Since the resultant of (3) is the discriminant of (1), we may add:

THEOREM 2. *A necessary and sufficient condition for a quadratic form to have a vertex is that its discriminant be zero; and if the rank of the form is r, it has n − r linearly independent vertices, and every point linearly dependent on these is a vertex*

In particular, we note that if the discriminant of a quadratic form is zero and if the cofactors of the elements of this determinant are denoted in the ordinary way by A_{ij}, then $(A_{i1}, \cdots A_{in})$ is a vertex, provided all these A's are not zero.

The following identity is of great importance (cf. formula (2), § 38),

$$(4)\qquad \sum_1^n a_{ij}(z_i+\lambda y_i)(z_j+\lambda y_j) \equiv \sum_1^n a_{ij}z_iz_j + 2\lambda\sum_1^n a_{ij}z_iy_j + \lambda^2\sum_1^n a_{ij}y_iy_j.$$

This may be regarded as an identity in all the letters involved.

If $(c_1, \cdots c_n)$ is a vertex of the quadratic form $\sum_1^n a_{ij}x_ix_j$, and these c's are substituted in (4) in place of the y's, the last two terms of the second member of this identity are zero, and we have

$$(5)\qquad \sum_1^n a_{ij}(z_i+\lambda c_i)(z_j+\lambda c_j) \equiv \sum_1^n a_{ij}z_iz_j;$$

and conversely, if (5) holds, $(c_1, \cdots c_n)$ is a vertex; for subtracting (5) from (4), after substituting the c's for the y's in (4), we have

$$2\lambda\sum_1^n a_{ij}z_ic_j + \lambda^2\sum_1^n a_{ij}c_ic_j \equiv 0,$$

and, this being an identity in λ as well as in the z's, we have

$$\sum_1^n a_{ij}z_ic_j \equiv 0.$$

Thus we have proved the following theorem:

THEOREM 3. *A necessary and sufficient condition that* $(c_1, \cdots c_n)$ *be a vertex of the quadratic form* (1) *is that* $z_1, \cdots z_n$ *and* λ *being independent variables, the identity* (5) *be fulfilled.*

EXERCISES

1. Prove that if $(c_1, \ldots c_n)$ is a vertex of (1), and $(y_1, \ldots y_n)$ is any point at which the quadratic form vanishes, then the quadratic form vanishes at every point linearly dependent on c and y.

2. State and prove a converse to 1.

45. Reduction of a Quadratic Form to a Sum of Squares. If in the quadratic form

$$(1)\qquad \phi(x_1, \cdots x_n) \equiv \sum_1^n a_{ij}x_ix_j$$

the coefficient a_{ii} is not zero, we may simplify the form by the following transformation due to Lagrange.

The difference

$$\phi(x_1, \cdots x_n) - \frac{1}{a_{ii}}(a_{i1}x_1 + \cdots + a_{in}x_n)^2$$

is evidently independent of x_i. Denoting it by ϕ_1, we have

$$\phi \equiv \frac{1}{a_{ii}}(a_{i1}x_1 + \cdots + a_{in}x_n)^2 + \phi_1.$$

If, then, we perform the non-singular linear transformation

$$(2)\qquad \begin{cases} x_1' = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \\ x_2' = \qquad x_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_i' = \quad x_1 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_n' = \qquad\qquad x_n, \end{cases}$$

the quadratic form ϕ is reduced to the form

$$(3)\qquad \frac{1}{a_{ii}}x_1'^2 + \phi_1(x_2', \cdots x_n'),$$

in which all the terms in x_1' are wanting except the term in $x_1'^2$.

It will be seen that this reduction can in general be performed in a variety of ways. It becomes impossible only when the coefficients of all the square terms in the original quadratic form are zero.

Unless, in the new quadratic form ϕ_1, the coefficients of all the square terms are zero, we can apply the same reduction to this form by subjecting the variables $x_2', \cdots x_n'$ to a suitable non-singular linear transformation. This transformation may also be regarded as a non-singular linear transformation of all the x''s: $(x_1', x_2', \cdots x_n')$ if we write $x_1'' = x_1'$. We thus reduce (3) to the form

$$(4)\qquad \frac{1}{a_{ii}}x_1''^2 + \frac{1}{a_{jj}'}x_2''^2 + \phi_2(x_3'', \cdots x_n'').$$

Applying this reduction now to ϕ_2, and proceeding as before, we see that by a number of successive non-singular transformations the form ϕ can finally be reduced to the form:

$$(5)\qquad c_1x_1^2 + c_2x_2^2 + \cdots + c_nx_n^2.$$

These successive transformations can now be combined into a single non-singular linear transformation, and we are thus led to the

THEOREM. *Every quadratic form in n variables can be reduced to the form* (5) *by a non-singular linear transformation.*

The proof of this theorem is not yet complete; for if at any stage of the reduction the quadratic form ϕ_i has the peculiarity that all its square terms are wanting, the next step in the reduction will be impossible by the method we have used. Before considering this point, we will illustrate the method of reduction by a numerical case.

Example.

$$\phi \equiv \left\{\begin{array}{l} 2x_1^2 + x_1x_2 + 8x_1x_3 \\ + x_2x_1 - 3x_2^2 + 9x_2x_3 \\ + 8x_3x_1 + 9x_3x_2 + 2x_3^2 \end{array}\right\} \equiv \tfrac{1}{2}(2x_1 + x_2 + 8x_3)^2 + \phi_1$$

where

$$\phi_1 \equiv -\tfrac{1}{2}(x_2 + 8x_3)^2 - 3x_2^2 + 18x_2x_3 + 2x_3^2 \equiv \left\{\begin{array}{l} -\tfrac{7}{2}x_2^2 + 5x_2x_3 \\ + 5x_3x_2 - 30x_3^2 \end{array}\right\}$$

$$\equiv -\tfrac{2}{7}(\tfrac{7}{2}x_2 + 5x_3)^2 - \tfrac{160}{7}x_3^2.$$

Accordingly, by means of the non-singular linear transformation

$$\begin{cases} x_1' = 2x_1 + x_2 + 8x_3, \\ x_2' = \quad -\tfrac{7}{2}x_2 + 5x_3, \\ x_3' = \quad x_3, \end{cases}$$

the form ϕ reduces to $\tfrac{1}{2}x_1'^2 - \tfrac{2}{7}x_2'^2 - \tfrac{160}{7}x_3'^2$.

We have given here merely *one* method of reduction. Three different methods were open to us at the first step and two at the second.

We proceed now to complete the proof of the general theorem. Let us suppose that the coefficients of all the square terms in ϕ are zero,* but that $a_{12} \neq 0$. Then

$$\phi(x_1, \cdots x_n) \equiv 2a_{12}x_1x_2 + 2x_1(a_{13}x_3 + \cdots + a_{1n}x_n) + 2x_2(a_{23}x_3 + \cdots + a_{2n}x_n) + \sum_3^n a_{ij}x_ix_j$$

$$\equiv \frac{2}{a_{12}}(a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n)(a_{21}x_1 + a_{23}x_3 + \cdots + a_{2n}x_n) + \phi_1,$$

where $\phi_1 \equiv -\dfrac{2}{a_{12}}(a_{13}x_3 + \cdots + a_{1n}x_n)(a_{23}x_3 + \cdots + a_{2n}x_n) + \sum_3^n a_{ij}x_ix_j.$

* This method may be used whenever $a_{11} = a_{22} = 0$ whether all the other coefficients a_{ii} are zero or not.

The non-singular linear transformation

$$\begin{cases} x_1' = \quad a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n \\ x_2' = a_{21}x_1 \quad + a_{23}x_3 + \cdots + a_{2n}x_n \\ x_3' = \quad x_3 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_n' = \quad x_n \end{cases}$$

thus reduces ϕ to the form

$$\frac{2}{a_{12}}x_1'x_2' + \phi_1(x_3', \cdots x_n').$$

The further non-singular transformation

$$\begin{cases} x_1'' = x_1' + x_2', \\ x_2'' = x_1' - x_2', \\ x_3'' = \quad x_3', \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_n'' = \quad x_n' \end{cases}$$

reduces ϕ to the form

$$\frac{1}{2a_{12}}x_1''^2 - \frac{1}{2a_{12}}x_2''^2 + \phi_1(x_3'', \cdots x_n'').$$

The above reduction was performed on the supposition that $a_{12} \neq 0$. It is clear, however, that only a slight change in notation would be necessary to carry through a similar reduction if $a_{12} = 0$ but $a_{ij} \neq 0$. The only case to which the reduction does not apply is, therefore, the one in which all the coefficients of the quadratic form are zero, a case in which no further reduction is necessary or possible.

We thus see that whenever Lagrange's reduction fails, the method last explained will apply, and thus our theorem is completely established.

EXERCISES

1. Given a quadratic form in which $n = 5$ and $a_{ij} = |i - j|$. Reduce to the form (5).

2. Reduce the quadratic form

$$9x^2 - 6y^2 - 8z^2 + 6xy - 14xz + 18xw + 8yz + 12yw - 4zw$$

to the form (5).

3. Prove that if $(y_1, \cdots y_n)$ is any point at which a given quadratic form is not zero, a linear transformation can be found (and that in an infinite number of ways) which carries this point into the point $(0, \cdots 0, 1)$ and its polar into kx_n; and show that this linear transformation eliminates from the quadratic form all terms in x_n except the term in x_n^2 which then has a coefficient not zero.

4. Prove that the transformations described in Exercise 3 are the only ones which have the effect there described.

5. Show how the two methods of reduction explained in this section come as special cases under the transformation of Exercise 3.

46. A Normal Form, and the Equivalence of Quadratic Forms. In the method of reduction explained in the last section, it may happen that, after we have taken a number of steps, and thus reduced ϕ to the form

$$c_1x_1^2 + \cdots + c_kx_k^2 + \phi_k(x_{k+1}, \cdots x_n),$$

the form ϕ_k is identically zero. In this case no further reduction would be necessary and the form (5) of the last section to which ϕ is reduced would have the peculiarity that $c_{k+1} = c_{k+2} = \cdots = c_n = 0$, while all the earlier c's are different from zero. It is easy to see just when this case will occur.

For this purpose, consider the matrix

$$\begin{Vmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & c_n \end{Vmatrix}$$

of the reduced form (5) of § 45. It is clear that the rank of this matrix is precisely equal to the number of c's different from zero; and, since the rank of this reduced form is the same as that of the original form, we have the result:

THEOREM 1. *A necessary and sufficient condition that it be possible to reduce a quadratic form by means of a non-singular linear transformation to the form*

$$c_1x_1^2 + \cdots + c_rx_r^2, \tag{1}$$

where none of the c's are zero, is that the rank of the quadratic form be r.

This form (1) involves r coefficients $c_1, \cdots c_r$. That the values of these coefficients, apart from the fact that none of them are zero, are immaterial will be seen if we consider the effect on (1) of the transformation

$$(2) \qquad \begin{cases} x_1 = \sqrt{\dfrac{k_1}{c_1}}x_1', \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_r = \sqrt{\dfrac{k_r}{c_r}}x_r', \\ x_{r+1} = x_{r+1}', \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_n = x_n', \end{cases}$$

where $k_1, \cdots k_r$ are arbitrarily given constants none of which, however, is to be zero. The transformation (2) is non-singular, and reduces (1) to the form

$$k_1x_1'^2 + \cdots + k_rx_r'^2. \tag{3}$$

Thus we have proved

THEOREM 2. *A quadratic form of rank r can be reduced by means of a non-singular linear transformation to the form* (3), *where the values of the constants $k_1, \cdots k_r$ may be assigned at pleasure provided none of them are zero.*

If, in particular, we assign to all the k's the value 1, we get

THEOREM 3. *Every quadratic form of rank r can be reduced to the normal form*

$$x_1^2 + \cdots + x_r^2 \tag{4}$$

by means of a non-singular linear transformation.

From this follows

THEOREM 4. *A necessary and sufficient condition that two quadratic forms be equivalent with regard to non-singular linear transformations is that they have the same rank.*

That this is a necessary condition is evident from the fact that the rank is an invariant. That it is a sufficient condition follows from the fact that, if the ranks are the same, both forms can be reduced to the same normal form (4).

The normal form (4) has no special advantage, except its symmetry, over any other form which could be obtained from (3) by assigning to the k's particular numerical values. Thus, for instance, a normal form which might be used in place of (4) is

$$x_1^2 + \cdots + x_{r-1}^2 - x_r^2.$$

This form would have the advantage, in geometrical work, of giving rise to a real locus.

Finally we note that the transformations used in this section are not necessarily real, even though the form we start with be real.

EXERCISE

Apply the results of this section to the study of quadric surfaces.

47. Reducibility. A quadratic form is called *reducible* when it is identically equal to the product of two linear forms, that is, when

$$(1) \qquad \sum_1^n a_{ij}x_ix_j \equiv (b_1x_1 + b_2x_2 + \cdots + b_nx_n)(c_1x_1 + c_2x_2 + \cdots + c_nx_n).$$

Let us seek a necessary and sufficient condition that this be the case. We begin by supposing the identity (1) to hold, and we consider in succession the case in which the two factors in the right-hand member of (1) are linearly independent, and that in which they are proportional. In the first case the b's are not all proportional to the corresponding c's, and by a mere change of notation we may insure b_1, b_2 not being proportional to c_1, c_2. This being done, the transformation

$$\begin{cases} x_1' = b_1x_1 + b_2x_2 + \cdots + b_nx_n \\ x_2' = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ x_3' = \qquad\qquad x_3 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_n' = \qquad\qquad\qquad x_n \end{cases}$$

is non-singular and carries our quadratic form over into the form

$$x_1'x_2'.$$

The matrix of this form is readily seen to be of rank 2, hence the original form was of rank 2.

Turning now to the case in which the two factors in (1) are proportional to each other, we see that (1) may be written

$$\sum_1^n a_{ij}x_ix_j \equiv C(b_1x_1 + \cdots + b_nx_n)^2 \qquad \text{where } C \neq 0.$$

Unless all the b's are zero (in which case the rank of the quadratic form is zero) we may without loss of generality suppose $b_1 \neq 0$, in which case the linear transformation

$$\begin{cases} x_1' = b_1x_1 + \cdots + b_nx_n \\ x_2' = \qquad x_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_n' = \qquad\qquad x_n \end{cases}$$

will be non-singular and will reduce the quadratic form to

$$Cx_1'^2,$$

which is of rank 1.

Thus we have shown that if a quadratic form is reducible, its rank is 0, 1, or 2. We wish now, conversely, to prove that every quadratic form whose rank has one of these values is reducible.

A quadratic form of rank zero is obviously reducible.

A form of rank 1 can be reduced by a non-singular linear transformation to the form $x_1'^2$, that is,

$$\sum_1^n a_{ij}x_ix_j \equiv x_1'^2.$$

If here we substitute for x_1' its value in terms of the x's, it is clear that the form is reducible.

A form of rank 2 can be reduced to the form $x_1'^2 + x_2'^2$, that is,

$$\sum_1^n a_{ij}x_ix_j \equiv x_1'^2 + x_2'^2 \equiv (x_1' + \sqrt{-1}\, x_2')(x_1' - \sqrt{-1}\, x_2').$$

Here again, replacing x_1' and x_2' by their values in terms of the x's, the reducibility of the form follows. Hence,

THEOREM. *A necessary and sufficient condition that a quadratic form be reducible is that its rank be not greater than* 2.

48. Integral Rational Invariants of a Quadratic Form. We have seen that the discriminant a of a quadratic form is an invariant of weight 2. Any integral power of a, or more generally, any constant multiple of such a power, will therefore also be an invariant. We will now prove conversely the

THEOREM. *Every integral rational invariant of a quadratic form is a constant multiple of some power of the discriminant.*

Let us begin by assuming that the quadratic form

(1) $$\sum_1^n a_{ij}x_ix_j$$

is non-singular, and let c be the determinant of a linear transformation which carries it over into the normal form

(2) $$x_1'^2 + x_2'^2 + \cdots + x_n'^2.$$

Let $I(a_{11}, \cdots a_{nn})$ be any integral rational invariant of (1) of weight μ, and denote by k the value of this invariant when formed from (2). It is clear that k is a constant, that is, independent of the coefficients a_{ij} of (1). Then
$$k = c^\mu I.$$

Moreover, the discriminant a being of weight 2, and having for (2) the value 1, we have
$$1 = c^2 a.$$

Raising the last two equations to the powers 2 and μ respectively, we get
$$k^2 = c^{2\mu} I^2, \qquad 1 = c^{2\mu} a^\mu.$$
From which follows

(3) $$I^2 = k^2 a^\mu.$$

This formula has been established so far for all values of the coefficients a_{ij} for which $a \neq 0$. That it is really an identity in the a_{ij}'s is seen at once by a reference to Theorem 5, § 2. The polynomial on the right-hand side of (3) is of degree μ in a_{11};* hence we see that μ must be an even number, since I^2 is of even degree in a_{11}. Letting $\mu = 2\nu$, we infer from (3) (cf. Exercise 1, § 2) that one or the other of the identities
$$I \equiv ka^\nu, \qquad I \equiv -ka^\nu$$
must hold, and either of these identities establishes our theorem.

A comparison of the result of this section with Theorem 4, § 46 will bring out clearly the essential difference between the two conceptions of a *complete system of invariants* mentioned in § 29. It will be seen that the rank of a quadratic form is in itself a complete system of invariants for this form in the sense of Definition 2, § 29; while the discriminant of the form is in itself a complete system in the sense of the footnote appended to this definition.

* We assume here that $k \neq 0$, as otherwise the truth of the theorem would be obvious.

49. A Second Method of Reducing a Quadratic Form to a Sum of Squares. By the side of Lagrange's method of reducing a quadratic form to a sum of squares, there are many other methods of accomplishing the same result, one of the most useful of which we proceed to explain. It depends on the following three theorems. The proof of the first of these theorems is due to Kronecker and establishes, in a remarkably simple manner, the fact that any quadratic form of rank r can be written in terms of r variables only, a fact which has already been proved by another method in Theorem 1, § 46.

THEOREM 1. *If the rank of the quadratic form*

(1) $$\phi(x_1, \cdots x_n) \equiv \sum_1^n a_{ij}x_ix_j$$

is $r > 0$, and if the variables $x_1, \cdots x_n$ are so numbered that the r-rowed determinant in the upper left-hand corner of its matrix is not zero, new variables $x_1', \cdots x_n'$ can be introduced by means of a non-singular linear transformation such that*
$$x_i' = x_i \qquad (i = r+1, \cdots n),$$
and such that (1) *reduces to the form*
$$\sum_1^r a_{ij}x_i'x_j'.$$

This, it will be noticed, is a quadratic form in r variables in which the coefficients, so far as they go, are the same as in the given form (1).

In order to prove this theorem, we begin by finding a vertex $(c_1, \cdots c_n)$ of the form (1) by means of Equations (3), § 44. Since the r-rowed determinant which stands in the upper left-hand corner of the matrix of these equations is not zero, the values of $c_{r+1}, \cdots c_n$ may be chosen at pleasure, and the other c's are then completely determined. If we let $c_{r+1} = c_{r+2} = \cdots = c_{n-1} = 0$, $c_n = 1$, we get a vertex
$$(c_1, \cdots c_r, 0, \cdots 0, 1).$$

Using this vertex in the identity (5), § 44, we have
$$\phi(x_1 + \lambda c_1, \cdots x_r + \lambda c_r, x_{r+1}, \cdots x_{n-1}, x_n + \lambda) \equiv \phi(x_1, \cdots x_n).$$

If we let $\lambda = -x_n$, this identity reduces to
$$\phi(x_1 - c_1x_n, \cdots x_r - c_rx_n, x_{r+1}, \cdots x_{n-1}, 0) \equiv \phi(x_1, \cdots x_n).$$

* That such an arrangement is possible is evident from Theorem 3, § 20.

Accordingly, if we perform the non-singular linear transformation*

$$\begin{cases} x_i' = x_i - c_i x_n & (i = 1, \cdots r), \\ x_i' = x_i & (i = r+1, \cdots n), \end{cases}$$

the quadratic form (1) reduces to

$$\phi(x_1', \cdots x_{n-1}', 0) \equiv \sum_1^{n-1} a_{ij} x_i' x_j'.$$

This, being a quadratic form in $n-1$ variables of rank r and so arranged that the r-rowed determinant which stands in the upper left-hand corner of its matrix is not zero, can be reduced, by the method just explained, to the form

$$\sum_1^{n-2} a_{ij} x_i'' x_j'',$$

where the linear transformation used is non-singular and such that

$$x_i'' = x_i' \qquad (i = r+1, \cdots n-1).$$

By adding the formula $\quad x_n'' = x_n',$

we may regard this as a non-singular linear transformation in the n variables. This transformation may then be combined with the one previously used, thus giving a non-singular transformation in which

$$x_i'' = x_i, \qquad (i = r+1, \cdots n),$$

and such that it reduces (1) to the form

$$\sum_1^{n-2} a_{ij} x_i'' x_j''.$$

Proceeding in this way step by step, our theorem is at last proved.

In the next two theorems we denote by A_{ij} in the usual way the cofactor of a_{ij} in the discriminant a of the quadratic form (1).

THEOREM 2. *If $A_{nn} \neq 0$, new variables $x_1', \cdots x_n'$ can be introduced by a non-singular transformation in such a way that*

$$x_n' = x_n$$

and that (1) *takes the form*

$$\sum_1^{n-1} a_{ij} x_i' x_j' + \frac{a}{A_{nn}} x_n'^2.$$

* This transformation should be compared with Exercise 2, § 41.

To prove this we consider the quadratic form

$$\sum_1^n a_{ij} x_i x_j - \frac{a}{A_{nn}} x_n^2.$$

Its discriminant is

$$\begin{vmatrix} a_{11} & \cdots & a_{1,n-1} & a_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-1,1} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & \cdots & a_{n,n-1} & a_{nn} - \dfrac{a}{A_{nn}} \end{vmatrix} = a - A_{nn}\frac{a}{A_{nn}} = 0.$$

Hence by means of a non-singular transformation of the kind used in the last theorem, an essential point being that $x_n' = x_n$, we get

$$\sum_1^n a_{ij} x_i x_j - \frac{a}{A_{nn}} x_n^2 \equiv \sum_1^{n-1} a_{ij} x_i' x_j',$$

or

$$\sum_1^n a_{ij} x_i x_j \equiv \sum_1^{n-1} a_{ij} x_i' x_j' + \frac{a}{A_{nn}} x_n'^2.$$

THEOREM 3. *If*

$$A_{nn} = A_{n-1,n-1} = 0, \qquad A_{n,n-1} \neq 0,$$

new variables $x_1', \cdots x_n'$ can be introduced by a non-singular transformation in such a way that $\quad x_{n-1}' = x_{n-1}, \quad x_n' = x_n,$

and that (1) *takes the form*

$$\sum_1^{n-2} a_{ij} x_i' x_j' + \frac{2a}{A_{n,n-1}} x_n' x_{n-1}'.$$

Let us denote by B the determinant obtained by striking out the last two rows and columns of a. Then (cf. Corollary 3, § 11) we have

(2) $$aB = \begin{vmatrix} A_{n-1,n-1} & A_{n-1,n} \\ A_{n,n-1} & A_{nn} \end{vmatrix} = -A_{n,n-1}^2 \neq 0.$$

Consider, now, the quadratic form

(3) $$\sum_1^n a_{ij} x_i x_j - \frac{2a}{A_{n,n-1}} x_n x_{n-1}.$$

Its discriminant is

$$
(4)\qquad \begin{vmatrix}
a_{11} & \cdots & a_{1,n-1} & a_{1n} \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
a_{n-1,1} & \cdots & a_{n-1,n-1} & a_{n-1,n} - \dfrac{a}{A_{n,n-1}} \\
a_{n1} & \cdots & a_{n,n-1} - \dfrac{a}{A_{n,n-1}} & a_{nn}
\end{vmatrix}
$$

$$
= a - A_{n,n-1}\frac{a}{A_{n,n-1}} - A_{n,n-1}\frac{a}{A_{n,n-1}} - B\left(\frac{a}{A_{n,n-1}}\right)^2,
$$

which has the value zero, as we see by making use of (2). Not only does the determinant (4) vanish, but its principal minors obtained by striking out its last row and column and its next to the last row and column are zero, being A_{nn} and $A_{n-1,n-1}$ respectively. The minor obtained by striking out the last two rows and columns from (4) is B, and, by (2), this is not zero. Thus we see (cf. Theorem 1, § 20) that the determinant (4) is of rank $n-2$. Hence, by Theorem 1, we can reduce (3) by a non-singular linear transformation in which $x'_{n-1} = x_{n-1}$, $x'_n = x_n$ to the form

$$
\sum_1^n a_{ij}x_ix_j - \frac{2a}{A_{n,n-1}}x_nx_{n-1} \equiv \sum_1^{n-2} a_{ij}x'_ix'_j.
$$

Hence

$$
\sum_1^n a_{ij}x_ix_j \equiv \sum_1^{n-2} a_{ij}x'_ix'_j + \frac{2a}{A_{n,n-1}}x'_nx'_{n-1}.
$$

COROLLARY. *Under the conditions of Theorem* 3, *the quadratic form* (1) *can be reduced to the form*

$$
\sum_1^{n-2} a_{ij}x'_ix'_j + \frac{2a}{A_{n,n-1}}\left(x'^2_{n-1} - x'^2_n\right)
$$

by a non-singular linear transformation.

To see this we have merely first to perform the reduction of Theorem 3, and then to follow this by the additional non-singular transformation

$$
\begin{cases}
x'_i = x''_i & (i = 1, 2, \cdots n-2), \\
x'_{n-1} = x''_{n-1} - x''_n, \\
x'_n = x''_{n-1} + x''_n.
\end{cases}
$$

Having thus established these three theorems, the method of reducing a quadratic form completely is obvious. If the form (1) is singular, we begin by reducing it by Theorem 1 to

$$
\sum_1^r a_{ij}x_ix_j,
$$

where r is the rank of the form. Unless all the principal $(r-1)$-rowed minors of the discriminant of this form are zero, the order of the variables $x_1, \cdots x_r$ can be so arranged that the reduction of Theorem 2 is possible, a reduction which may be regarded as a non-singular linear transformation of all n variables. If all the principal $(r-1)$-rowed minors are zero, there will be at least one of the cofactors A_{ij} which is not zero, and, by a suitable rearrangement of the order of the variables, this may be taken as $A_{r,r-1}$. The reduction of Theorem 3, Corollary, will then be possible. Proceeding in this way, we finally reach the result, precisely as in Theorem 1, § 46, that a quadratic form of rank r can always be reduced by a non-singular linear transformation to the form

$$
c_1x_1^2 + \cdots + c_rx_r^2.
$$

It may be noticed that the arrangement of the transformation of this section is in a certain sense precisely the reverse of that of § 45, inasmuch as we here leave at each step the *coefficients* of the unreduced part of the form unchanged, but change the *variables* which enter into this part; while in § 45 we change the *coefficients* of the unreduced part, but leave the *variables* in it unchanged.