CHAPTER VIII

BILINEAR FORMS

36. The Algebraic Theory. Before entering on the study of quadratic forms, which will form the subject of the next five chapters, we turn briefly to a very special type of quadratic form in $2n$ variables, known as a bilinear form, and which, as its name implies, forms a natural transition between linear and quadratic forms.

DEFINITION 1. *A polynomial in the $2n$ variables $(x_1, \cdots x_n)$, $(y_1, \cdots y_n)$ is called a bilinear form if each of its terms is of the first degree in the x's and also of the first degree in the y's.*

Thus, for $n = 3$, the most general bilinear form is

$$\begin{aligned} &a_{11}x_1y_1 + a_{12}x_1y_2 + a_{13}x_1y_3 \\ + &a_{21}x_2y_1 + a_{22}x_2y_2 + a_{23}x_2y_3 \\ + &a_{31}x_3y_1 + a_{32}x_3y_2 + a_{33}x_3y_3. \end{aligned}$$

This may be denoted, for brevity, by $\sum_1^3 a_{ij}x_iy_j$; and, in general, we may denote the bilinear form in $2n$ variables by

$$\sum_1^n a_{ij}x_iy_j. \tag{1}$$

The matrix

$$\mathbf{a} = \begin{Vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{Vmatrix}$$

is called the matrix of the form (1); its determinant, the determinant of the form; and its rank, the rank of the form.* A bilinear form is called singular when, and only when, its determinant is zero.

* It should be noticed that the bilinear form is completely determined when its matrix is given, so there will be no confusion if we speak of the bilinear form $\mathbf{a}$. If two bilinear forms have matrices $\mathbf{a}_1$ and $\mathbf{a}_2$, their sum has the matrix $\mathbf{a}_1 + \mathbf{a}_2$. The bilinear form whose matrix is $\mathbf{a}_1\mathbf{a}_2$ is *not* the product of the two forms, but is sometimes spoken of as their *symbolic product*.

Let us notice that the bilinear form (1) may be obtained by starting from the system of n linear forms in the y's of matrix $\mathbf{a}$, multiplying them respectively by $x_1, x_2, \cdots x_n$, and adding them together. It can also be obtained by starting from the system of n linear forms in the x's whose matrix is the conjugate of $\mathbf{a}$, multiplying them respectively by $y_1, y_2, \cdots y_n$, and adding them together.

Using the first of these two methods, we see (cf. Theorem 1, § 31) that if the y's are subjected to a linear transformation with matrix $\mathbf{d}$, the bilinear form is carried over into a new bilinear form whose matrix is $\mathbf{ad}$. Using the second of the above methods of building up the bilinear form from linear forms, we see that if the x's are subjected to a linear transformation with matrix $\mathbf{c}$, we get a new bilinear form the conjugate of whose matrix is $\mathbf{a'c}$, where accents are used to denote conjugate matrices. The matrix of the form itself is then (cf. Theorem 6, § 22) $\mathbf{c'a}$.*

Combining these two facts, we have

THEOREM 1. *If, in the bilinear form* (1) *with matrix $\mathbf{a}$, we subject the x's to a linear transformation with matrix $\mathbf{c}$ and the y's to a linear transformation with matrix $\mathbf{d}$, we obtain a new bilinear form with matrix $\mathbf{c'ad}$, where $\mathbf{c'}$ is the conjugate of $\mathbf{c}$.*

Considering the determinants of these matrices, we may say:

THEOREM 2. *The determinant of a bilinear form is multiplied by the product of the determinants of the transformations to which the x's and y's are subjected.*†

We also infer from Theorem 1, in combination with Theorem 7, § 25, the important result: ‡

THEOREM 3. *The rank of a bilinear form is an invariant with regard to non-singular linear transformations of the x's and y's.*

DEFINITION 2. *A bilinear form whose matrix is symmetric is called a symmetric bilinear form.*

* These results may also be readily verified without referring to any earlier theorems.

† This theorem tells us that the determinant of a bilinear form is, in a generalized sense, a relative invariant. Such invariants, where the given forms depend on several sets of variables, are known as *combinants*.

‡ This result may also be deduced from Theorem 2, § 30.

THEOREM 4. *A symmetric bilinear form remains symmetric if we subject the x's and the y's to the same linear transformation.*

For if c is the matrix of the transformation to which both the x's and the y's are subjected, the matrix of the transformed form will, by Theorem 1, be c′ac. Remembering that a, being symmetric, is its own conjugate, we see, by Theorem 6, § 22, that c′ac is its own conjugate. Hence the transformed form is symmetric.

EXERCISES

1. Prove that a necessary and sufficient condition for the equivalence of two bilinear forms with regard to non-singular linear transformations of the x's and y's is that they have the same rank.

2. Prove that a necessary and sufficient condition that it be possible to factor a bilinear form into the product of two linear forms is that its rank be zero or one.

3. Prove that every bilinear form of rank r can be reduced by non-singular linear transformations of the x's and y's to the normal form

$$x_1y_1 + x_2y_2 + \cdots + x_ry_r.$$

4. Do the statements in the preceding exercises remain correct if we confine our attention to real bilinear forms and real linear transformations?

5. Prove that a necessary and sufficient condition that the form

$$x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

should be unchanged by linear transformations of the x's and of the y's is that these be contragredient transformations.

37. A Geometric Application. Let (x_1, x_2, x_3) and (y_1, y_2, y_3) be homogeneous coördinates of points in a plane, and let us consider the bilinear equation

$$(1) \qquad \sum_1^3 a_{ij}x_iy_j = 0.$$

If (y_1, y_2, y_3) is a fixed point P, then (1), being linear in the x's, is the equation of a straight line p. The only exception is when the coefficients of (1), regarded as a linear equation in the x's, are all zero, and this cannot happen if the determinant of the form is different from zero. Thus we see that the equation (1) causes one, and only one, line p to correspond to every point P of the plane, provided the bilinear form in (1) is non-singular.

Conversely, if

$$(2) \qquad Ax_1 + Bx_2 + Cx_3 = 0$$

is a line p, there is one, and only one, point P which corresponds to it by means of (1), provided the bilinear form in (1) is non-singular. For if P is the point (y_1, y_2, y_3), the equation of the line corresponding to it is (1), and the necessary and sufficient condition that this line coincide with (2) is

$$\begin{aligned} a_{11}y_1 + a_{12}y_2 + a_{13}y_3 &= \rho A, \\ a_{21}y_1 + a_{22}y_2 + a_{23}y_3 &= \rho B, \\ a_{31}y_1 + a_{32}y_2 + a_{33}y_3 &= \rho C, \end{aligned}$$

where ρ is a constant, not zero. For a given value of ρ, this set of equations has one, and only one, solution (y_1, y_2, y_3), since the determinant a is not zero, while a change in ρ merely changes all the y's in the same ratio. Hence,

THEOREM. *If the bilinear equation* (1) *is non-singular, it establishes a one-to-one correspondence between the points and lines of the plane.*

This correspondence is called a *correlation.*

EXERCISES

1. Discuss the singular correlations of the plane, considering separately the cases in which the rank of the bilinear form is 2 and 1.

2. Examine the corresponding equation in three dimensions, that is, the equation obtained by equating to zero a bilinear form in which $n = 4$, and discuss it for all possible suppositions as to the rank of the form.

3. Show that a necessary and sufficient condition for three or more lines, which correspond to three or more given points by means of a non-singular correlation, to be concurrent is that the points be collinear.

4. Show that the cross-ratio of any four concurrent lines is the same as that of the four points to which they correspond by means of a non-singular correlation.

5. Let P be any point in a plane and p the line corresponding to it by means of a non-singular correlation. Prove that a necessary and sufficient condition for the lines corresponding to the points of p to pass through P is that the bilinear form be symmetrical.

6. State and prove the corresponding theorem for points and planes in space of three dimensions, showing that here it is necessary and sufficient that the form be *symmetrical* or *skew-symmetrical.**

* The correlation given by a symmetric bilinear equation is known as a *reciprocation.* By reference to the formulæ of the next chapter, it will be readily seen that in this case every point corresponds, in the plane, to its polar with regard to a fixed conic; in space, to its polar plane with regard to a fixed quadric surface. The skew-symmetric bilinear equation gives rise in the plane merely to a very special singular correlation. In space, however, it gives an important correlation which is in general non-singular and is known as a *null-system.* Cf. any treatment of line geometry, where, however, the subject is usually approached from another side.

CHAPTER IX

GEOMETRIC INTRODUCTION TO THE STUDY OF QUADRATIC FORMS

38. Quadric Surfaces and their Tangent Lines and Planes. If x_1, x_2, x_3 are homogeneous coördinates in a plane, we see, by reference to § 4, that the equation of any conic may be written

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 = 0.$$

Similarly, in space of three dimensions, the equation of any quadric surface may be written

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{44}x_4^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{14}x_1x_4 + 2a_{34}x_3x_4 + 2a_{42}x_4x_2 + 2a_{23}x_2x_3 = 0.$$

This form may be made still more symmetrical if, besides the coefficients $a_{12}, a_{13}, a_{14}, a_{34}, a_{42}, a_{23}$, we introduce the six other constants $a_{21}, a_{31}, a_{41}, a_{43}, a_{24}, a_{32}$, defined by the general formula

$$a_{ij} = a_{ji}.$$

The equation of the quadric surface may then be written

$$\begin{aligned}
&a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{14}x_1x_4\\
+&a_{21}x_2x_1 + a_{22}x_2^2 + a_{23}x_2x_3 + a_{24}x_2x_4\\
+&a_{31}x_3x_1 + a_{32}x_3x_2 + a_{33}x_3^2 + a_{34}x_3x_4\\
+&a_{41}x_4x_1 + a_{42}x_4x_2 + a_{43}x_4x_3 + a_{44}x_4^2 = 0,
\end{aligned}$$

or for greater brevity

(1) $$\sum_1^4 a_{ij}x_ix_j = 0.$$

Definition 1. — *The matrix of the sixteen a's taken in the order written above is called the matrix of the quadric surface* (1), *the determinant of this matrix is called the discriminant of the quadric surface, its rank is called the rank of the quadric surface, and if the discriminant vanishes, the quadric surface is said to be singular.*

A fundamental problem is the following: If (y_1, y_2, y_3, y_4) and (z_1, z_2, z_3, z_4) are two points, in what points does the line yz meet the surface (1)?

The coördinates of any point on yz, other than y, may be written

$$(z_1 + \lambda y_1,\ z_2 + \lambda y_2,\ z_3 + \lambda y_3,\ z_4 + \lambda y_4).$$

A necessary and sufficient condition for this point to lie on (1) is

$$\sum_1^4 a_{ij}(z_i + \lambda y_i)(z_j + \lambda y_j) = 0,$$

or expanded,

(2) $$\sum_1^4 a_{ij}z_iz_j + 2\lambda\sum_1^4 a_{ij}y_iz_j + \lambda^2\sum_1^4 a_{ij}y_iy_j = 0.$$

If the point y does not lie on (1), this is a quadratic equation in λ. To each root of this equation corresponds one point where the line meets the quadric. Thus we see that every line through a point y which does not lie on a quadric surface, meets this surface either in two, and only two, distinct points, or in only one point.

On the other hand, if y does lie on (1), the equation (2) reduces to an equation of the first degree, provided $\Sigma a_{ij}y_iz_j \neq 0$. In this case, also, the line meets the surface in two, and only two, distinct points, viz., the point y and the point corresponding to the root of the equation of the first degree (2).

Finally, if $\Sigma a_{ij}y_iy_j = \Sigma a_{ij}y_iz_j = 0$, the first member of equation (2) reduces to a constant, so that (2) is either satisfied by no value of λ, in which case the line meets the surface at the point y only, or by all values of λ (if $\Sigma a_{ij}z_iz_j = 0$), in which case every point on the line is also a point on the surface.

Combining the preceding results we may say:

Theorem 1. *If a quadric surface and a straight line are given, one of the following three cases must occur:*

(1) *The line meets the quadric in two, and only two, points, in which case the line is called a secant.*

(2) *The line meets the quadric in one, and only one, point, in which case it is called a tangent.**

(3) *Every point of the line is a point of the quadric. In this case the line is called a ruling of the quadric.†*

* We shall presently distinguish between true tangents and pseudo-tangents.

† Also called a *generator*, because, as will presently appear, the whole surface may be generated by the motion of such a line.

That all these three cases are possible is shown by simple examples; for instance, in the case of the surface

$$y^2 + z^2 - zt = 0,$$

the three coördinate axes illustrate the three cases.

We shall often find it convenient to say that a tangent line meets the quadric in *two coincident points.*

From the proof we have given of Theorem 1, we can also infer the further result:

THEOREM 2. *If* (y_1, y_2, y_3, y_4) *is a point on the quadric* (1), *then if*

$$(3) \qquad \sum_1^4 a_{ij}x_iy_j \equiv 0,^*$$

every line through y is either a tangent or a ruling of (1), *otherwise every line through y which lies in the plane*

$$(4) \qquad \sum_1^4 a_{ij}x_iy_j = 0$$

is a tangent or ruling of (1), *while every other line through y is a secant.*

A theorem of fundamental importance, which follows immediately from this, is:

THEOREM 3. *If there exists a point y on the quadric* (1) *such that the identity* (3) *is fulfilled, then* (1) *is a cone with y as a vertex; and, conversely, if* (1) *is a cone with y as a vertex, then the identity* (3) *is fulfilled.*

We pass now to the subject of tangent planes, which we define as follows:

DEFINITION 2. *A plane p is said to be tangent to the quadric* (1) *at one of its points P, if every line of p which passes through P is either a tangent or a ruling of* (1).

It will be seen that, according to this definition, if (1) is a cone, every plane through a vertex of (1) is tangent to (1) at this vertex. We have thus included among the tangent planes, planes which in ordinary geometric parlance would not be called tangent. The same objection applies to our definition of tangent lines. We therefore now introduce the distinction between true tangent lines or planes and pseudo-tangent lines or planes.

DEFINITION 3. *A line or plane which touches a quadric surface at a point which is not a vertex is called a true tangent; all other tangent lines and planes are called pseudo-tangents.*

* It should be noticed that, on account of the relation $a_{ij} = a_{ji}$, $\Sigma a_{ij}x_iy_j \equiv \Sigma a_{ij}y_ix_j$.

EXERCISES

1. Prove that if P is a point on a quadric surface S, which is not a vertex, and p the tangent plane at this point, one of the following three cases must occur:

(*a*) Two, and only two, lines of p are rulings of S, and these rulings intersect at P.

(*b*) One, and only one, line of p is a ruling of S, and this ruling passes through P.

(*c*) Every line of p is a ruling of S.

2. Prove that

(*a*) When case (*a*) of Exercise 1 occurs, the quadric surface is not a cone; and, conversely, if the quadric surface is not a cone, case (*a*) will always occur.

(*b*) If case (*b*) of Exercise 1 occurs, p is tangent to S at every point of the ruling which lies in p.

(*c*) If case (*b*) of Exercise 1 occurs, S is a cone with one, and only one, vertex, and this vertex is on the ruling which lies in p; and conversely, if S is a cone with one, and only one, vertex, case (*b*) will always occur.

(*d*) If case (*c*) of Exercise 1 occurs, there is a line l in p every point of which (but no other point) is a vertex of S; and S consists of two planes one of which is p, while the other intersects it in l.

39. Conjugate Points and Polar Planes. Two points are commonly said to be conjugate with regard to a quadric surface

$$(1) \qquad \sum_1^4 a_{ij}x_ix_j = 0,$$

when they are divided harmonically by the points where the line connecting them meets the surface. In order to include all limiting cases, we frame the definition as follows:

DEFINITION. *Two distinct points are said to be conjugate with regard to the surface* (1) *if*

(*a*) *The line joining them is a tangent or a secant to* (1), *and the points are divided harmonically by the points where this line meets* (1); *or*

(*b*) *The line joining them is a ruling of* (1).

Two coincident points are called conjugate if they both lie on (1).

Let the coördinates of the points be (y_1, y_2, y_3, y_4) and (z_1, z_2, z_3, z_4), and let us look first at the case in which the points are distinct and neither of them lies on (1), and in which the line connecting them is a secant of (1). The points of intersection of the line yz with (1) may therefore be written

$$(z_1 + \lambda_i y_1,\ z_2 + \lambda_i y_2,\ z_3 + \lambda_i y_3,\ z_4 + \lambda_i y_4) \qquad (i = 1, 2),$$

where λ_1 and λ_2 are the roots of Equation (2), § 38. A necessary and sufficient condition for harmonic division is that the cross-ratio λ_1/λ_2 have the value -1; that is $\lambda_1+\lambda_2=0$; or, referring back to Equation (2), § 38,

$$(2) \qquad \sum_1^4 a_{ij}y_iz_j=0.$$

We leave it for the reader to show that in all other cases in which y and z are conjugate this relation (2) is fulfilled; and that, conversely, whenever this condition is fulfilled, the points are conjugate. That is:

THEOREM 1. *A necessary and sufficient condition that the points y, z be conjugate with regard to* (1) *is that* (2) *be fulfilled.*

This theorem enables us at once to write down the equation of the locus of the point x conjugate to a fixed point y, namely,

$$(3) \qquad \sum_1^4 a_{ij}x_iy_j=0.$$

Except when the first member of this equation vanishes identically, this locus is therefore a plane called the *polar plane* of the point y. We saw in the last section that the first member of (3) vanishes identically when (1) is a cone and y is a vertex. This is the only case in which it vanishes identically; for, if y is any point, not a vertex, on a quadric surface, (3) represents the tangent plane at that point; while if y is not on (1), the first member of (3) can clearly not vanish identically, since it does not vanish when the x's are replaced by the y's. Hence the theorem:

THEOREM 2. *If* (1) *is not a cone, every point y has a definite polar plane* (3); *if* (1) *is a cone, every point except its vertices has a definite polar plane* (3), *while for the vertices the first member of* (3) *is identically zero.*

We note that the property that a plane is the polar of a given point with regard to a quadric surface is a projective property, since a collineation of space evidently carries over two conjugate points into points conjugate with regard to the transformed surface.

THEOREM 3. *If two points P_1 and P_2 are so situated that the polar plane of P_1 passes through P_2, then, conversely, the polar plane of P_2 will pass through P_1.*

For, from the hypothesis, it follows that P_1 and P_2 are conjugate points, and from this the conclusion follows.

40. Classification of Quadric Surfaces by Means of their Rank. Theorem 2 of the last section may be stated by saying that a necessary and sufficient condition that the quadric surface

$$(1) \qquad \sum_1^4 a_{ij}x_ix_j=0$$

be a cone and that (y_1, y_2, y_3, y_4) be its vertex (or one of its vertices) is that

$$(2) \qquad \sum_1^4 a_{ij}x_iy_j\equiv 0.$$

This identity (2) is equivalent to the four equations

$$(3) \qquad \begin{cases} a_{11}y_1+a_{12}y_2+a_{13}y_3+a_{14}y_4=0,\\ a_{21}y_1+a_{22}y_2+a_{23}y_3+a_{24}y_4=0,\\ a_{31}y_1+a_{32}y_2+a_{33}y_3+a_{34}y_4=0,\\ a_{41}y_1+a_{42}y_2+a_{43}y_3+a_{44}y_4=0. \end{cases}$$

A necessary and sufficient condition for this set of equations to have a common solution other than (0, 0, 0, 0) is that the determinant of their coefficients be zero. We notice that this determinant is the discriminant a of the quadric surface. Hence,

THEOREM 1. *A necessary and sufficient condition for a quadric surface to be a cone is that its discriminant vanish.*

If, then, the rank of the quadric surface is *four*, the surface is not a cone.

If the rank is *three*, the set of equations (3) has one, and, except for multiples of this, only one, solution. Hence in this case the surface is an ordinary cone with a single vertex.

If the rank is *two*, equations (3) have two linearly independent solutions (cf. § 18), on which all other solutions are linearly dependent. Hence in this case the surface is a cone with a whole line of vertices.

If the rank is *one*, equations (3) have three linearly independent solutions on which all other solutions are linearly dependent. Hence we have a cone with a whole plane of vertices.

If the rank is *zero* we have, strictly speaking, no quadric surface; but the locus of (1) may be regarded as a cone. every point in space being a vertex.

It is clear that the property of a quadric surface being a cone is a projective property; and the same is true of the property of a point being a vertex of a cone. Hence from the classification we have just given we infer

THEOREM 2. *The rank of a quadric surface is unchanged by non-singular collineations.*

EXERCISES

1. DEFINITION. *If a plane p is the polar of a point P with regard to a quadric surface, then P is called a pole of p.*

Prove that if the quadric surface is non-singular, every plane has one, and only one, pole.

2. Prove that if the quadric surface is a cone, a plane which does not pass through a vertex has no pole.

What can be said here about planes which do pass through a vertex?

41. Reduction of the Equation of a Quadric Surface to a Normal Form. Since cross-ratio is invariant under a non-singular collineation, a quadric surface S, a point P, not on S, and its polar plane with regard to S, are carried over by any non-singular collineation into a quadric surface S', a point P', and its polar plane with regard to S'. A point (y_1, y_2, y_3, y_4) not on the quadric surface $\sum_1^4 a_{ij}x_ix_j = 0$, cannot be on its own polar plane $\sum_1^4 a_{ij}x_iy_j = 0$ as we see by replacing the x's in this last equation by the y's. Now transform by a collineation so that this point becomes the origin and its polar plane the plane at infinity.* The quadric surface will now be a central quadric with center at the origin, since, if any line be drawn through the origin, the two points in which this line meets the surface are divided harmonically by the origin and the point at infinity on this line.

The equation of the polar plane of the point (y_1', y_2', y_3', y_4') with regard to the transformed quadric

$$\sum_1^4 a_{ij}'x_i'x_j' = 0$$

is

$$\sum_1^4 a_{ij}'x_i'y_j' = 0,$$

* Such a collineation can obviously be determined in an infinite number of ways by means of the theorem that there exists a collineation which carries over any five linearly independent points into any five linearly independent points; cf. Exercises 2, 3, § 24.

which reduces to the simple form

$$a_{14}'x_1' + a_{24}'x_2' + a_{34}'x_3' + a_{44}'x_4' = 0$$

when the point is the origin (0, 0, 0, 1). For this equation to represent the plane at infinity, we must have

$$a_{14}' = a_{24}' = a_{34}' = 0, \; a_{44}' \neq 0.$$

Hence the quadric surface becomes

$$\begin{aligned} a_{11}'\,x_1'^2 \quad &+ a_{12}'\,x_1'\,x_2' + a_{13}'\,x_1'\,x_3' \\ + a_{21}'\,x_2'\,x_1' &+ a_{22}'\,x_2'^2 \quad + a_{23}'\,x_2'\,x_3' \\ + a_{31}'\,x_3'\,x_1' &+ a_{32}'\,x_3'\,x_2' + a_{33}'\,x_3'^2 \\ &\qquad\qquad + a_{44}'\,x_4'^2 = 0. \end{aligned}$$

A slightly different reduction can be performed by transforming the point (y_1, y_2, y_3, y_4) to the point at infinity on the x_1-axis and its polar plane to the x_2x_3-plane. It is easy to see that we thus get rid of the terms containing x_1 except the square term.

Similarly we can get rid of the terms containing x_2 and x_3. Thus we see that *any quadric surface can be reduced by a collineation to a form where its equation contains no term in x_i except the term in x_i^2 whose coefficient then is not zero.*

According as we take for i the values 1, 2, 3, 4, we get thus four different normal forms for the equation of our quadric surface, and inasmuch as each of these forms can be obtained in a great variety of ways, the question naturally arises whether we cannot perform all four reductions simultaneously. That this can, in general, be done may be seen as follows: let y be a point not on the quadric surface, and z any point on the polar plane of y, but not on the quadric surface. Its polar plane contains y. Let w be any point on the intersection of the polar planes of y and z, but not on the quadric surface. Then its polar plane passes through y and z. These three polar planes meet in some point u, and it is readily seen that the four points y, z, w, u do not lie on a plane. The tetrahedron $yzwu$ is called a *polar* or *self-conjugate* tetrahedron of the quadric surface, since it has the property that any vertex is the pole of the opposite face.

If we transform the four points, y, z, w, u to the origin and the points at infinity on the three axes, the effect will be the same as that of the separate transformations above, that is, the equation of the quadric surface will be reduced to the form

$$a_{11}'\,x_1'^2 + a_{22}'\,x_2'^2 + a_{33}'\,x_3'^2 + a_{44}'\,x_4'^2 = 0.$$

We have tacitly assumed that it is possible to find points y, z, w, constructed as indicated above, and not lying on the quadric surface. We leave it for the reader to show that, if the quadric surface is not a cone, this will always be possible in an infinite number of ways. A cone, however, has no self-conjugate tetrahedron, and in this case the above reduction is impossible.

EXERCISES

1. Prove that if the discriminant of a quadric surface is zero, the equation of the surface can always be reduced, by a suitable collineation, to a form in which the coördinate x_4 does not enter.

[SUGGESTION. Show, by using the results of this chapter, that if the vertex of a quadric cone is at the origin, $a_{14} = a_{24} = a_{34} = a_{44} = 0$.]

2. Show that, provided the cone has a finite vertex, the collineation of Exercise 1 may be taken in the form

$$x_1' = x_1 + \alpha x_4,$$
$$x_2' = x_2 + \beta x_4,$$
$$x_3' = x_3 + \gamma x_4,$$
$$x_4' = x_4.$$

[SUGGESTION. Use non-homogeneous coördinates.]

CHAPTER X

QUADRATIC FORMS

42. The General Quadratic Form and its Polar. The general quadratic form in n variables is

$$(1) \qquad \sum_1^n a_{ij}x_ix_j \equiv \begin{array}{l} a_{11}x_1^2 \;\; + a_{12}x_1x_2 + \dots + a_{1n}x_1x_n \\ + a_{21}x_2x_1 + a_{22}x_2^2 \;\; + \dots + a_{2n}x_2x_n \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ + a_{n1}x_nx_1 + a_{n2}x_nx_2 + \dots + a_{nn}x_n^2, \end{array}$$

where $a_{ij} = a_{ji}$.* The bilinear form $\sum_1^n a_{ij}y_iz_j$ is called the *polar form* of (1). Subjecting (1) to the linear transformation

$$\text{c} \quad \begin{cases} x_1 = c_{11}x_1' + \dots + c_{1n}x_n', \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ x_n = c_{n1}x_1' + \dots + c_{nn}x_n', \end{cases}$$

we get a new quadratic form

$$(2) \qquad \sum_1^n a_{ij}'x_i'x_j'.$$

The polar form of (2) is $\Sigma a_{ij}'y_i'z_j'$. If we transform the y's and z's of the polar form of (1) by the same transformation c, we get a new bilinear form $\sum_1^n \bar{a}_{ij}y_i'z_j'$. We will now prove that $\bar{a}_{ij} = a_{ij}'$.

We have the identities

$$(3) \qquad \sum_1^n a_{ij}x_ix_j \equiv \sum_1^n a_{ij}'x_i'x_j',$$

$$(4) \qquad \sum_1^n a_{ij}y_iz_j \equiv \sum_1^n \bar{a}_{ij}y_i'z_j'.$$

* It should be clearly understood that this restriction is a matter of convenience, not of necessity. If it were not made, the quadratic form would be neither more nor less general.