Finally, let us note the geometric meaning to be associated with the invariants and covariants which have been mentioned in this section. We confine our attention to the case of four variables. The vanishing of the resultant of four linear forms gives a necessary and sufficient condition that the four planes determined by setting the forms equal to zero meet in a point. The vanishing of the covariant of Theorem 3 is a necessary and sufficient condition that the four points lie in a plane. The vanishing of the covariant of Theorem 4 is a necessary and sufficient condition that the point (y_1, y_2, y_3, y_4) lie on the surface $f=0$. It will be seen that in all cases we are thus led to a projective property; cf. §§80, 81.

32. Some Theorems Concerning Linear Forms.

THEOREM 1. *Two systems of n linear forms in n variables are equivalent with regard to non-singular linear transformations if neither resultant is zero.*

Let

$$(1)\ \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1}x_1 + \cdots + a_{nn}x_n \end{cases} \qquad (2)\ \begin{cases} b_{11}x_1 + \cdots + b_{1n}x_n \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ b_{n1}x_1 + \cdots + b_{nn}x_n \end{cases}$$

be the two systems, whose resultants,

$$a = \begin{vmatrix} a_{11}\cdots a_{1n} \\ \cdot\quad \cdot \\ \cdot\quad \cdot \\ a_{n1}\cdots a_{nn} \end{vmatrix}, \qquad b = \begin{vmatrix} b_{11}\cdots b_{1n} \\ \cdot\quad \cdot \\ \cdot\quad \cdot \\ b_{n1}\cdots b_{nn} \end{vmatrix},$$

are, by hypothesis, not zero. Applying the transformations

$$\mathbf{a}\begin{cases} x_1' = a_{11}x_1 + \cdots + a_{1n}x_n, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_n' = a_{n1}x_1 + \cdots + a_{nn}x_n, \end{cases} \qquad \mathbf{b}\begin{cases} x_1' = b_{11}x_1 + \cdots + b_{1n}x_n, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_n' = b_{n1}x_1 + \cdots + b_{nn}x_n, \end{cases}$$

to (1) and (2) respectively, they are both reduced to the normal form

$$(3)\quad \begin{cases} x_1', \\ \quad x_2', \\ \cdot\ \cdot\ \cdot\ \cdot \\ \qquad\quad x_n'. \end{cases}$$

Now, since neither a nor b is zero, the transformations $\mathbf{a}$ and $\mathbf{b}$ have inverses, which when applied to (3) carry it back into (1) and (2) respectively. Hence the transformation $\mathbf{b}^{-1}\mathbf{a}$ carries (1) into (2).

THEOREM 2. *A system of n linear forms in n variables has no integral rational invariants* other than constant multiples of powers of the resultant.*

Let (1) be the given system and a its resultant, and let c be the determinant of a non-singular linear transformation which carries (1) over into

$$(4)\quad \begin{cases} a_{11}'x_1' + \cdots + a_{1n}'x_n', \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1}'x_1' + \cdots + a_{nn}'x_n'. \end{cases}$$

If we call the resultant of (4) a', we have

$$a' \equiv ac.$$

Let $I(a_{11}, \cdots a_{nn})$ be any integral rational invariant of the system (1) of weight μ, and write

$$I' = I(a_{11}', \cdots a_{nn}').$$

Then

$$I' \equiv c^{\mu} I.$$

Now let us assume for a moment that $a \neq 0$, and consider the special transformation which carries over (1) into the normal form (3). In this special case we have $a' = 1$; hence, as may also be seen directly, $ac = 1$. Calling the constant value which I' has in this particular case k, we have

$$k = c^{\mu} I = a^{-\mu} I,$$

or

$$(5)\qquad I = ka^{\mu}.$$

This equality, in which k is independent of the coefficients a_{ij}, has been established so far merely for values of the a_{ij}'s for which $a \neq 0$. Since μ is not negative (cf. Theorem 5, § 31), we can now infer that (5) is an identity, by making use of Theorem 5, § 2. Thus we see that I is merely a constant multiple of a power of the resultant, as was to be proved.

COROLLARY. *A system of m linear forms in n variables has no integral rational invariants (other than constants) when $m < n$.*

For such an invariant would also be an integral rational invariant of the system of n linear forms obtained by adding $n - m$ new forms to the given system; and hence it would be a constant multiple of a

* It has the arithmetical invariant mentioned in Theorem 2, § 30.

power of the resultant of this system. This power must be zero, and hence the invariant must be a mere constant, as otherwise it would involve the coefficients of the added forms, and hence would not be an invariant of the system of original forms.

EXERCISES

1. Prove that if we have two systems of $n+1$ linear forms in n variables whose matrices are both of rank n, a necessary and sufficient condition that these two systems be equivalent with regard to non-singular linear transformation is that the resultants of the forms of one set taken n at a time be proportional to the resultants of the corresponding forms of the other set.

2. Generalize the preceding theorem.

3. Prove that every integral rational invariant of a system of m linear forms in n variables $(m > n)$ is a homogeneous polynomial in the resultants of these forms taken n at a time.

4. State and prove the theorems analogous to the theorems of the present section, including the three preceding exercises, when the system of linear forms is replaced by a system of points.

33. Cross-ratio and Harmonic Division. Let us consider any four distinct points on a line

(1) $$(x_1, t_1),\ (x_2, t_2),\ (x_3, t_3),\ (x_4, t_4).$$

We have seen, in § 31, Theorem 3, that each of the six determinants

(2) $$\begin{matrix} x_1t_2 - x_2t_1, & x_1t_3 - x_3t_1, & x_1t_4 - x_4t_1, \\ x_3t_4 - x_4t_3, & x_4t_2 - x_2t_4, & x_2t_3 - x_3t_2, \end{matrix}$$

is a covariant of weight -1. The ratio of two of these determinants is therefore an absolute covariant, and we might be tempted, by analogy with the examples of absolute covariants in Exercise 1, § 28, to expect that it might have a geometric meaning. It will be readily seen, however, that this is not the case, for the value of the ratio of two of the determinants (2) will be changed if the two coördinates of one of the four points are multiplied by the same constant, and this does not affect the position of the points.

It is easy, however, to avoid this state of affairs by forming such an expression as the following:*

(3) $$(1, 2, 3, 4) = \frac{(x_1t_2 - x_2t_1)(x_3t_4 - x_4t_3)}{(x_2t_3 - x_3t_2)(x_4t_1 - x_1t_4)},$$

* The reversal of sign of the second factor in the denominator is not essential, but is customary for a reason which will presently be evident.

which is also an absolute covariant of the four points (1), and is called their *cross-ratio* or *anharmonic ratio*. More accurately it is called the cross-ratio of these four points when taken in the order written in (1).*

In order to determine the geometric meaning of the cross-ratio of four points, let us first suppose the four points to be finite so that $t_1t_2t_3t_4 \neq 0$. Dividing numerator and denominator of (1, 2, 3, 4) by this product, we find the following expression for the cross-ratio in terms of the non-homogeneous coördinates X_i of the points,

(4) $$(1, 2, 3, 4) = \frac{(X_1 - X_2)(X_3 - X_4)}{(X_2 - X_3)(X_4 - X_1)}.$$

Finally, denoting the points by P_1, P_2, P_3, P_4, we may write

(5) $$(1, 2, 3, 4) = \frac{P_1P_2}{P_3P_2}\Big/\frac{P_1P_4}{P_3P_4} = \frac{P_2P_1}{P_4P_1}\Big/\frac{P_2P_3}{P_4P_3}.$$

In words, this formula tells us that the cross-ratio of four finite points is the ratio of the ratio in which the second divides the first and third and the ratio in which the fourth divides the first and third; and that it is also the ratio of the ratios in which the first and third divide the second and fourth.

In this statement, it must be remembered that we have taken the ratio in which C divides the points A, B as AC/BC, so that the ratio is negative if C divides AB internally, positive if it divides it externally.

If we agree that the point at infinity on a line shall be said to divide any two finite points A, B of this line in the ratio $+1$ (and this is a natural convention since the more distant a point the more nearly does it divide AB in the ratio $+1$) it is readily seen, by going back to formula (3), that the first statement following (5) still holds if the second or fourth point is at infinity, while the second statement holds if the first or third is at infinity. Thus we have in all cases a simple geometric interpretation of the cross-ratio of four distinct points.

The special case in which four points P_1, P_2, P_3, P_4 are so situated that $(1, 2, 3, 4) = -1$ is of peculiar importance. In this case we have

$$\begin{aligned}(1, 2, 3, 4) &= (1, 4, 3, 2) = (3, 2, 1, 4) = (3, 4, 1, 2) = (2, 1, 4, 3) \\ &= (2, 3, 4, 1) = (4, 1, 2, 3) = (4, 3, 2, 1) = -1.\end{aligned}$$

* If these four points are taken in other orders, we get different cross-ratios: (1, 2, 4, 3), (1, 4, 3, 2), etc. Cf. Exercise 1 at the end of this section.

The relation is therefore merely a relation between the two pairs of points P_1, P_3 and P_2, P_4 taken indifferently in either order, and we say that these two pairs of points divide each other harmonically. From the geometric meaning of cross-ratio, we see that, if all four points are finite, the pairs P_1, P_3 and P_2, P_4 divide each other harmonically when, and only when, P_2 and P_4 divide P_1, P_3 internally and externally in the same ratio; and also when, and only when, P_1 and P_3 divide P_2, P_4 internally and externally in the same ratio. If P_2 or P_4 lies at infinity, the first of these statements alone has a meaning, while if P_1 or P_3 lies at infinity, it is the second statement to which we must confine ourselves.

It is easily seen that the case in which three of the four points, say P_1, P_2, P_3, coincide, while P_4 is any point on the line, may be regarded as a limiting form of two pairs of points which separate one another harmonically. It is convenient to include this case under the term *harmonic division*, and we will therefore lay down the definition:

DEFINITION. *Two pairs of points P_1, P_3 and P_2, P_4 on a line are said to divide one another harmonically if they are distinct and their cross-ratio taken in the order P_1, P_2, P_3, P_4 is -1, and also if at least three of them coincide.*

It will be seen that the property of two pairs of points dividing each other harmonically is a projective property in space of one dimension.

The most important applications of cross-ratio come in geometry of two, three, or more dimensions where the points are not determined as above by two coördinates (or one non-homogeneous coördinate), but by more. Suppose, for instance, we have four distinct finite points on a line in space of three dimensions. Let the points be P_1, P_2, Q_1, Q_2, and suppose the coördinates of P_1, P_2 are (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) respectively. Then the coördinates of Q_1, Q_2 may be written

$$(x_1+\lambda x_2, y_1+\lambda y_2, z_1+\lambda z_2, t_1+\lambda t_2), (x_1+\mu x_2, y_1+\mu y_2, z_1+\mu z_2, t_1+\mu t_2).$$

Now, let

(6) $$Ax+By+Cz+Dt=0$$

be any plane through Q_1 but not through P_2, and we have

$$(Ax_1+By_1+Cz_1+Dt_1)+\lambda(Ax_2+By_2+Cz_2+Dt_2)=0,$$

or, since P_2 does not lie on (6),

$$\frac{Ax_1+By_1+Cz_1+Dt_1}{Ax_2+By_2+Cz_2+Dt_2}=-\lambda.$$

Changing to non-homogeneous coördinates, we have

$$\frac{AX_1+BY_1+CZ_1+D}{AX_2+BY_2+CZ_2+D}=-\lambda\frac{t_2}{t_1}.$$

If P_1M_1 and P_2M_2 are the perpendiculars from P_1 and P_2 on the plane (6), we have

$$\frac{P_1Q_1}{P_2Q_1}=\frac{P_1M_1}{P_2M_2}=\frac{AX_1+BY_1+CZ_1+D}{AX_2+BY_2+CZ_2+D}=-\lambda\frac{t_2}{t_1}.$$

In exactly the same way we get

$$\frac{P_1Q_2}{P_2Q_2}=-\mu\frac{t_2}{t_1}.$$

Consequently $$\frac{P_1Q_1}{P_2Q_1}\Big/\frac{P_1Q_2}{P_2Q_2}=\frac{\lambda}{\mu}.$$

This is the cross-ratio of the four points taken in the order P_1, Q_1, P_2, Q_2.

It is readily seen that if one of the two points Q_1 or Q_2 lies at infinity, all that is essential in the above reasoning remains valid, and the cross-ratio is still λ/μ.

The case in which one of the two points P_1 or P_2 is at infinity may be reduced to the case just considered by writing for the coördinates of Q_1 and Q_2, $(\xi_1, \eta_1, \zeta_1, \tau_1)$ and $(\xi_2, \eta_2, \zeta_2, \tau_2)$. The coördinates of P_1 and P_2 are then

$$\left(\xi_1-\frac{\lambda}{\mu}\xi_2, \eta_1-\frac{\lambda}{\mu}\eta_2, \zeta_1-\frac{\lambda}{\mu}\zeta_2, \tau_1-\frac{\lambda}{\mu}\tau_2\right),$$

$$(\xi_1-\xi_2, \eta_1-\eta_2, \zeta_1-\zeta_2, \tau_1-\tau_2).$$

Accordingly, from what has just been proved, we see that the cross-ratio of the four points taken in the order Q_1, P_1, Q_2, P_2 is λ/μ. But this change of order does not change the cross-ratio. Hence in all cases we have the result:

THEOREM 1. *The cross-ratio of the four distinct points*

$$P_1 \quad (x_1, y_1, z_1, t_1),$$
$$P_2 \quad (x_2, y_2, z_2, t_2),$$
$$Q_1 \quad (x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2, t_1 + \lambda t_2),$$
$$Q_2 \quad (x_1 + \mu x_2, y_1 + \mu y_2, z_1 + \mu z_2, t_1 + \mu t_2),$$

taken in the order P_1, Q_1, P_2, Q_2, *is* λ/μ.

From this theorem we easily deduce the further result:

THEOREM 2. *The cross-ratio of four points on a line is invariant with regard to non-singular collineations of space.**

For the four points P_1, P_2, Q_1, Q_2 of Theorem 1 are carried over by a non-singular collineation into the four points

$$P_1' \quad (x_1', y_1', z_1', t_1'),$$
$$P_2' \quad (x_2', y_2', z_2', t_2'),$$
$$Q_1' \quad (x_1' + \lambda x_2', y_1' + \lambda y_2', z_1' + \lambda z_2', t_1' + \lambda t_2'),$$
$$Q_2' \quad (x_1' + \mu x_2', y_1' + \mu y_2', z_1' + \mu z_2', t_1' + \mu t_2'),$$

whose cross-ratio, when taken in the order P_1', Q_1', P_2', Q_2', is also λ/μ.

Theorems similar to Theorems 1 and 2 hold in space of two, and in general in space of n, dimensions and may be proved in the same way.

EXERCISES

1. Denote the six determinants (2) by

$$(1, 2), \quad (1, 3), \quad (1, 4), \quad (3, 4), \quad (4, 2), \quad (2, 3),$$

and write

$$A = (1, 2)(3, 4), \quad B = (1, 3)(4, 2), \quad C = (1, 4)(2, 3).$$

Prove that six, and only six, cross-ratios can be formed from four points by taking them in different orders, namely the negatives of the six ratios which can be formed from A, B, C taken two and two.

2. Prove that $A + B + C = 0$, and hence show that if λ is one of the cross ratios of four points, the other five will be

$$\frac{1}{\lambda}, \; 1 - \lambda, \; \frac{1}{1-\lambda}, \; \frac{\lambda - 1}{\lambda}, \; \frac{\lambda}{\lambda - 1}.$$

* This also follows from Exercise 5, § 24

3. Prove that the six cross-ratios of four distinct points are all different from each other except in the following two cases:

(α) The case of four harmonic points, where the values of the cross-ratios are $-1, 2, \frac{1}{2}$.

(β) The case known as four equianharmonic points, in which the values of the cross-ratios are $-\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$.

4. Prove Theorem 2, § 24, by making use of the fact that the cross-ratio of four points on a line is unchanged by non-singular projective transformations of the line.

5. By the cross-ratio of four planes which meet in a line is understood the cross-ratio of the four points in which these planes are met by any line which does not meet their line of intersection.

Justify this definition by proving that if the equations of the four planes are

$$p_1 = 0, \; p_1 + \lambda p_2 = 0, \; p_2 = 0, \; p_1 + \mu p_2 = 0$$

(p_1 and p_2 homogeneous linear polynomials in x, y, z, t), the cross-ratio of the four points in which *any* line which does not meet the line of intersection of the planes is met by the planes is λ/μ.

6. Prove that the cross-ratio of four planes which meet in a line is invariant with regard to non-singular collineations.

34. Plane-Coördinates and Contragredient Variables. If u_1, u_2, u_3, u_4 are constants, and x_1, x_2, x_3, x_4 are the homogeneous coördinates of a point in space, the equation

$$(1) \qquad u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0$$

represents a plane. Since the values of the u's determine the position of this plane, the u's may be regarded as coördinates of the plane. We will speak of them as *plane-coördinates*, just as the x's (each set of which determines a point) are called *point-coördinates*. And just as we speak of the point (x_1, x_2, x_3, x_4) so we will speak of the plane (u_1, u_2, u_3, u_4). The u's are evidently analogous to homogeneous coördinates in that if they be all multiplied by the same constant, the plane which they determine is not changed.

Suppose now that we consider the x's as constants and allow the u's to vary, taking on all possible sets of values which, with the fixed set of values of the x's, satisfy (1). This equation will now represent a family of planes, infinite in number, each one of which is determined by a particular set of values of the u's and all of which pass through the fixed point (x_1, x_2, x_3, x_4). The equation (1) may therefore be regarded as the *equation of a point in plane-coördinates*, since it is satisfied by the coördinates of a moving plane which envelops this point, just as when the x's vary and the u's are constant, it is the *equation of a plane in point-coördinates*, since it is satisfied by the coördinates of a moving point whose locus is the plane.*

In the same way, a homogeneous equation of degree higher than the first in the u's will be satisfied by the coördinates of a moving plane which will, in general, envelop a surface. The equation will then be called the equation of this surface in plane-coördinates.†

Let us now subject space to the collineation

$$\text{c} \qquad x'_i = c_{i1}x_1 + c_{i2}x_2 + c_{i3}x_3 + c_{i4}x_4 \qquad (i = 1, 2, 3, 4).$$

We will assume that the determinant c of this transformation is not zero; and we will denote the cofactors in this determinant by C_{ij}. Then the inverse of the transformation c may be written

$$\text{c}^{-1} \qquad x_i = \frac{C_{1i}}{c}x'_1 + \frac{C_{2i}}{c}x'_2 + \frac{C_{3i}}{c}x'_3 + \frac{C_{4i}}{c}x'_4 \qquad (i = 1, 2, 3, 4).$$

Substituting these expressions, we see that the plane (1) goes over into

$$(2) \qquad u'_1x'_1 + u'_2x'_2 + u'_3x'_3 + u'_4x'_4 = 0,$$

where

$$\text{d} \qquad u'_i = \frac{C_{i1}}{c}u_1 + \frac{C_{i2}}{c}u_2 + \frac{C_{i3}}{c}u_3 + \frac{C_{i4}}{c}u_4 \qquad (i = 1, 2, 3, 4).$$

We thus see that the u's have also suffered a linear transformation, though a different one from the x's, namely, the transformation whose matrix is the conjugate (cf. § 7, Definition 2) of c^{-1}. This transformation d of the plane-coördinates is merely another way of expressing the collineation which we have commonly expressed by the transformation c of the point-coördinates. The two sets of variables x and u are called *contragredient variables* according to the following

Definition 1. *Two sets of n variables each are called contragredient if, whenever one is subjected to a non-singular linear transformation, the other is subjected to the transformation which has as its matrix the conjugate of the inverse of the matrix of the first.*

* Similarly, in two dimensions, the equation

$$u_1x_1 + u_2x_2 + u_3x_3 = 0$$

represents a *line* in the point-coördinate (x_1, x_2, x_3) if u_1, u_2, u_3 are constants, or a *point* in the line-coördinates (u_1, u_2, u_3) if x_1, x_2, x_3 are constants.

† An example of this will be found in § 53.

Precisely the reasoning used above in the case of four variables establishes here also the theorem:

Theorem.* *If the two sets of contragredient variables $x_1, \cdots x_n$ and $u_1, \cdots u_n$ are carried over by a linear transformation into $x'_1, \cdots x'_n$ and $u'_1, \cdots u'_n$, then*

$$u_1x_1 + u_2x_2 + \cdots + u_nx_n$$

will go over into

$$u'_1x'_1 + u'_2x'_2 + \cdots + u'_nx'_n.$$

In connection with this subject of contragredient variables it is customary to introduce the conception of *contravariants*, just as the conception of covariants was introduced in connection with the subject of cogredient variables. For this purpose we lay down the

Definition 2. *If we have a system of forms in $(x_1, \cdots x_n)$ and also a number of sets of variables, $(u'_1, \cdots u'_n)$, $(u''_1, \cdots u''_n)$, $\cdots$, contragredient to the x's, any rational function of the u's and the coefficients of the forms which is unchanged by a non-singular linear transformation of the x's except for being multiplied by the μth power (μ an integer) of the determinant of this transformation is called a contravariant of weight μ.*

Thus the theorem that the resultant of n linear forms in n variables is an invariant of weight 1 may, if we prefer, be stated in the form: If we have n sets of n variables each, $(u'_1, \cdots u'_n), \cdots (u^{[n]}_1, \cdots u^{[n]}_n)$, each of which is contragredient to the variables $(x_1, \cdots x_n)$, the determinant of the u's is a contravariant of weight 1.†

It will be seen that the conception of contravariant, though sometimes convenient, is unnecessary, since the contragredient variables may always be regarded as the coefficients of linear forms, and, when so regarded, the contravariant is merely an invariant.

Similarly, the still more general conception of *mixed concomitants*, in which, besides the coefficients of forms and the contragredient variables, certain sets of cogredient variables are involved,‡ reduces to the familiar conception of covariants if we regard the contragredient variables as coefficients of linear forms.

* This is really a special theorem in the theory of bilinear forms. Cf. the next chapter.

† For other examples of contravariants in which coefficients also occur, see Chap. XII.

‡ An example is $u_1x_1 + u_2x_2 + \cdots + u_nx_n$, the theorem above stating that this is an *absolute* mixed concomitant.

35. Line-Coördinates in Space. A line is determined by two points (y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4) which lie on it. It is clear that these eight coördinates are not all necessary to determine the line, and it will be seen presently that the following six quantities determine the line completely, and may be used as *line-coördinates*,

$$p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23},$$

where

(1) $$p_{ij} = \begin{vmatrix} y_i & y_j \\ z_i & z_j \end{vmatrix}.$$

In other words, the p's are the two-rowed determinants of the matrix

$$\begin{Vmatrix} y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{Vmatrix},$$

except that the sign of the determinant obtained by striking out the first and third column has been changed. These six p's are not all zero if, as we assume, the two points y and z are distinct.

These six p's are connected by the relation

(2) $$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0,*$$

as may be seen either directly or by expanding the vanishing determinant

$$\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

by Laplace's method in terms of the minors of the first two rows.

That the p's may really be used as line-coördinates is shown by the following two theorems:

THEOREM 1. *When a line is given, its line-coördinates p_{ij} are completely determined except for an arbitrary factor different from zero by which they may all be multiplied.*

The definition (1) of the p's shows that they may all be multiplied by an arbitrary factor different from zero without affecting the position of the line, since the y's (and also the z's) may be multiplied by such a factor without affecting the position of the point.

* Cf. Exercise 2, § 33.

In order to prove our theorem it is sufficient to show that, if instead of the two points used above for determining the p's we use two other points of the line,

$$(Y_1, Y_2, Y_3, Y_4), \quad (Z_1, Z_2, Z_3, Z_4),$$

the line-coördinates

$$P_{ij} = \begin{vmatrix} Y_i & Y_j \\ Z_i & Z_j \end{vmatrix}$$

thus determined will be proportional to the p_{ij}'s. Since the points Y and Z are collinear with the distinct points y, z, they are linearly dependent upon them and we may write

$$Y_i = c_1 y_i + c_2 z_i, \qquad Z_i = k_1 y_i + k_2 z_i \qquad (i = 1, 2, 3, 4).$$

Accordingly

$$P_{ij} = \begin{vmatrix} c_1 & c_2 \\ k_1 & k_2 \end{vmatrix} \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} = Kp_{ij},$$

where $K \neq 0$, as Y and Z are distinct points.

THEOREM 2. *Any six constants p_{ij} satisfying the relation* (2) *and not all zero are the line-coördinates of one, and only one, line.*

That they cannot be the coördinates of more than one line may be seen as follows: Suppose the p_{ij}'s to be the coördinates of a line, and take two distinct points y and z on the line. The coördinates of these points may then be so determined that relations (1) hold. Let us suppose, for definiteness, that $p_{12} \neq 0$.* Now, consider the point whose coördinates are $c_1 y_i + c_2 z_i$. By assigning to c_1 and c_2 first the values $-z_1$ and y_1, then the values $-z_2$ and y_2, we get the two points

(3) $$(0, p_{12}, p_{13}, p_{14}), \qquad (p_{21}, 0, p_{23}, p_{24}),$$

where, by definition, $p_{ij} = -p_{ji}$.

These two points are distinct, since for the first of them the first coördinate is zero and the second is not, while for the second the second coördinate is zero and the first is not. These points accordingly determine the line, and since they, in turn, are determined by the p's, we see that the line is uniquely determined by the p's.

* By a slight modification of the formulæ this proof will apply to the case in which any one of the p's is assumed different from zero.

It remains, then, merely to show that any set of p_{ij}'s, not all zero, which satisfy (2) really determine a line. For this purpose we again assume $p_{12} \neq 0$ * and consider the two points (3) which, as above, are distinct. The line determined by them has as its coördinates

$$p_{12}^2,\ p_{12}p_{13},\ p_{12}p_{14},\ -p_{13}p_{42}-p_{14}p_{23},\ p_{12}p_{42},\ p_{12}p_{23}.$$

Using the relation (2), the fourth of these quantities reduces to $p_{12}p_{34}$, so that, remembering that the coördinates of a line may be multiplied by any constant different from zero, we see that we really have a line whose coördinates are p_{ij}.

In a systematic study of three-dimensional geometry these line-coördinates play as important a part as the point- or plane-coördinates; and in the allied algebraic theories we shall have to consider expressions having the invariant property, into which these line-coördinates enter just as point-coördinates occur in covariants and plane-coördinates in contravariants. We may, if we please, regard these expressions as ordinary covariants, since the line-coördinates are merely functions of the coördinates of two points, but the covariants we get in this way are covariants of a special sort, since the coördinates of the two points occur only in the combinations (1).

As an example, let us consider four points

$$(x_i,\ y_i,\ z_i,\ t_i) \qquad (i=1,\ 2,\ 3,\ 4).$$

The determinant of these sixteen coördinates is, by Theorem 3, § 31, a covariant of weight -1. Let us denote by p'_{ij} and p''_{ij} the coördinates of the lines determined by the first two and the last two points respectively. Expanding the four-rowed determinant just referred to by Laplace's method according to the two-rowed determinants of the first two rows, we get

$$(4) \qquad p'_{12}p''_{34}+p''_{12}p'_{34}+p'_{13}p''_{42}+p''_{13}p'_{42}+p'_{14}p''_{23}+p''_{14}p'_{23}.$$

This, then, is an expression having the invariant property and involving only line-coördinates.

Since the vanishing of the four-rowed determinant from which we started gave the condition that the four points lie in a plane, it follows that the vanishing of (4) gives a necessary and sufficient condition that the two lines p' and p'' lie in a plane, or, what amounts to the same thing, that they meet in a point.

* By a slight modification of the formulæ, this proof will apply to the case in which any one of the p's is assumed different from zero.

EXERCISES

1. Prove that, if the point-coördinates are subjected to the linear transformation

$$x'_i = c_{i1}x_1 + c_{i2}x_2 + c_{i3}x_3 + c_{i4}x_4 \qquad (i=1,2,3,4),$$

the line-coördinates will be subjected to the linear transformation

$$p'_{ij} = (c_{i1}c_{j2}-c_{i2}c_{j1})p_{12} + (c_{i1}c_{j3}-c_{i3}c_{j1})p_{13} + (c_{i1}c_{j4}-c_{i4}c_{j1})p_{14} + (c_{i3}c_{j4}-c_{i4}c_{j3})p_{34} + (c_{i4}c_{j2}-c_{i2}c_{j4})p_{42} + (c_{i2}c_{j3}-c_{i3}c_{j2})p_{23}.$$

2. A plane is determined by three points

$$(y_1,\ y_2,\ y_3,\ y_4), \qquad (z_1,\ z_2,\ z_3,\ z_4), \qquad (w_1,\ w_2,\ w_3,\ w_4).$$

Prove that the three-rowed determinants of the matrix of these three points may be used as coördinates of this plane, and that these coördinates are not distinct from the plane-coördinates defined in § 34.

3. A line determined by two of its points may be called a *ray*, and the line-coördinates of the present section may therefore be called ray-coördinates. A line determined as the intersection of two planes may be called an *axis*. If $(u_1,\ u_2,\ u_3,\ u_4)$ and $(v_1,\ v_2,\ v_3,\ v_4)$ are two planes given by their plane-coördinates, discuss the *axis-coördinates* of their intersection,

$$q_{12},\ q_{13},\ q_{14},\ q_{34},\ q_{42},\ q_{23},$$

where

$$q_{ij} = u_iv_j - u_jv_i.$$

4. Prove that ray-coördinates and axis-coördinates are not essentially different by showing that, for any line, the q's, taken in the order written in Exercise 3, are proportional to the p's taken in the order

$$p_{34},\ p_{42},\ p_{23},\ p_{12},\ p_{13},\ p_{14}.$$

5. A point is determined as the intersection of three planes

$$(u_1,\ u_2,\ u_3,\ u_4), \qquad (v_1,\ v_2,\ v_3,\ v_4), \qquad (w_1,\ w_2,\ w_3,\ w_4).$$

Prove that the three-rowed determinants of the matrix of these planes may be used as coördinates of this point, and that they do not differ from the ordinary point-coördinates.

Hence, show that all covariants may be regarded as invariants.