CHAPTER VII

INVARIANTS. FIRST PRINCIPLES AND ILLUSTRATIONS

28. Absolute Invariants, Geometric, Algebraic, and Arithmetical. If we subject a geometric figure to a transformation, we find that, while many properties of the figure have been altered, others have not. If we consider, not a single transformation, but a set of transformations, then those properties of figures which are not changed by any of the transformations of the set are said to be invariant properties with regard to this set of transformations. Thus if our set of transformations is the group of all displacements, the property of two lines being parallel or perpendicular to each other and the property of a curve being a circle are invariant properties, since after the transformation the lines will still be parallel or perpendicular and the curve will still be a circle. If, however, we consider, not the group of displacements, but the group of all non-singular collineations, none of the properties just mentioned will be invariant properties. Properties invariant with regard to all non-singular collineations have played such an important part in the development of geometry that a special name has been given to them, and they are called *projective* or *descriptive* properties. As examples of such projective properties we mention the collinearity and complanarity of points, the complanarity and concurrence of lines, etc.; or, on the other hand, the contact of a line with a curve or a surface or the contact of two curves or of two surfaces, or of a curve and a surface.

DEFINITION 1. *If there is associated with a geometric figure a quantity which is unchanged by all the transformations of a certain set, then this quantity is called an invariant with regard to the transformations of the set.*

For instance, if our set of transformations is the group of displacements, the distance between two points and the angle between two lines would be two examples of invariants.

The geometric invariants so far considered lead up naturally to the subject of algebraic invariants. Thus let us consider the two polynomials

$$(1)\qquad \begin{cases} A_1x + B_1y + C_1, \\ A_2x + B_2y + C_2, \end{cases}$$

and subject the variables (x, y) to the transformations of the set

$$(2)\qquad \begin{cases} x' = x\cos\theta + y\sin\theta + \alpha, \\ y' = -x\sin\theta + y\cos\theta + \beta, \end{cases}$$

where α, β, θ are parameters which may have any values. The transformation (2) carries over the polynomials (1) into two new polynomials:

$$(3)\qquad \begin{cases} A_1'x' + B_1'y' + C_1', \\ A_2'x' + B_2'y' + C_2'. \end{cases}$$

The coefficients of (3) may be readily expressed in terms of the coefficients of (1) and the parameters α, β, θ. Using these expressions, we easily obtain the formulæ

$$(4)\qquad \begin{cases} A_1'B_2' - A_2'B_1' = A_1B_2 - A_2B_1, \\ A_1'A_2' + B_1'B_2' = A_1A_2 + B_1B_2. \end{cases}$$

We shall therefore speak of the two expressions

$$(5)\qquad A_1B_2 - A_2B_1, \quad A_1A_2 + B_1B_2$$

as invariants of the system of polynomials (1) with regard to the set of transformations (2) according to the following general definition:

DEFINITION 2. *If we have a system of polynomials in the variables* $(x, y, z, \ldots)$ *and a set of transformations of these variables, then any function of the coefficients of the polynomials is called an invariant* (*or more accurately an absolute invariant*) *with regard to these transformations if it is unchanged when the polynomials are subjected to all the transformations of the set.*

The relation of the example considered above to the subject of geometric invariants becomes obvious when we notice that the algebraic transformations (2) may be regarded as expressing the displacements of plane figures in their plane when (x, y) are rectangular coördinates of points in the plane. If now we consider, not the polynomials (1), but the two lines determined by setting them equal to zero, we have to deal with the displacements of these two lines The invariants (5) have themselves no geometric significance, but by equating them to zero, we get the necessary and sufficient conditions that the two lines be respectively parallel and perpendicular, and these, as we noticed above, are invariant properties with regard to displacements. Finally we may notice that the ratio of the two invariants (5) gives the tangent of the angle between the lines,—a geometric invariant.

As a second example, let us consider, not two lines, but a line and a point. Algebraically this means that we start with the system

$$(6) \qquad \begin{cases} Ax + By + C, \\ (x_1, y_1), \end{cases}$$

consisting of a polynomial and a pair of variables. We shall wish to demand here that whenever the variables (x, y) are subjected to a transformation, the variables (x_1, y_1) be subjected to the same transformation, or as we say according to Definition 3 below, that (x, y) and (x_1, y_1) be cogredient variables. If we subject the system (6) to any transformation of the set (2), we get a new system

$$(7) \qquad \begin{cases} A'x' + B'y' + C', \\ (x_1', y_1'), \end{cases}$$

and it is readily seen that

$$A'x_1' + B'y_1' + C' = Ax_1 + By_1 + C.$$

Accordingly we shall call $Ax_1 + By_1 + C$ a covariant of the system (6) according to Definition 4 below. This covariant has also no direct geometric meaning, but its vanishing gives the necessary and sufficient condition for an invariant property, namely, that the point (x_1, y_1) lie on the line $Ax + By + C = 0$.

In the light of this example we may lay down the following general definitions:

DEFINITION 3. *If we have several sets of variables*

$$(x, y, z, \cdots), \quad (x_1, y_1, z_1, \cdots), \quad (x_2, y_2, z_2, \cdots), \cdots$$

and agree that whenever one of these sets is subjected to a transformation every other set shall be subjected to the same transformation, then we say that we have sets of cogredient variables.

DEFINITION 4. *If we have a system consisting of a number of polynomials in $(x, y, z, \cdots)$ and of a number of sets of variables cogredient to $(x, y, z, \cdots)$, then any function of the coefficients of the polynomials and of the cogredient variables which is unchanged when the variables $(x, y, z, \cdots)$ are subjected to all the transformations of a certain set is called a covariant (or more accurately an absolute covariant) of this system with regard to the transformations of this set.*

It will be seen that invariants may be regarded as special cases of covariants.

Among the geometric invariants there are some which from their nature are necessarily integers, and which we will speak of as *arithmetical invariants.* An example would be the number of vertices of a polygon if our set of transformations was either the group of displacements or the group of non-singular collineations. Another example is the largest number of real points in which an algebraic curve can be cut by a line, if our set of transformations is the group of *real* non-singular collineations.

These arithmetical invariants also play, as we shall see, an important part in algebra. We mention here as an example the degree of an n-ary form, which is an invariant with regard to all non-singular linear transformations.*

EXERCISES

1. Prove that $(x_2 - x_1)^2 + (y_2 - y_1)^2$, and $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

are covariants of the system

$$(x_1, y_1), (x_2, y_2), (x_3, y_3)$$

with regard to the transformations (2).

2. Prove that $A + B$ and $B^2 - AC$

are invariants of the polynomial

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F$$

with regard to the transformations (2).

What geometric meaning can be attached to these invariants?

3. Prove that $A^2 + B^2$ is an invariant of the polynomial

$$Ax + By + C$$

with regard to the transformations (2).

Hence show that

$$\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}$$

is a covariant of the system (6). Note its geometric meaning.

* It is, in fact, an invariant with regard to all linear transformations except the one in which all the coefficients of the transformation are zero.

29. Equivalence.

Definition 1. *If A and B denote two geometric configurations or two algebraic expressions, or sets of expressions, then A and B shall be said to be equivalent with regard to a certain set of transformations when, and only when, there exists a transformation of the set which carries over A into B and also a transformation of the set which carries over B into A.*

To illustrate this definition we notice that the conception of equivalence of geometric figures with regard to displacements is identical with the Euclidean conception of the equality or congruence of figures.

Again, we see from Theorem 2, § 24, that on a straight line two sets of three points each are always equivalent with regard to non-singular projective transformations.

In both of the illustrations just mentioned the set of transformations forms a group. In such cases the condition for equivalence can be decidedly simplified, for the transformation which carries A into B has an inverse belonging to the set, and this inverse necessarily carries B into A. Thus we have the

Theorem. *A necessary and sufficient condition for the equivalence of A and B with regard to a group of transformations is that a transformation of the group carry over A into B.*

This theorem will be of great importance, as the question of equivalence will present itself to us only when the set of transformations we are considering forms a group.

Let us consider, for the sake of greater definiteness, a group of geometric transformations. If two geometric configurations are equivalent with regard to this group, every invariant of the first configuration must be equal to the corresponding invariant of the second. Thus, for instance, if two triangles are equivalent with regard to the group of displacements, all the sides and angles of the first will be equal to the corresponding sides and angles of the second. The same will be true of the altitudes, lengths of the medial lines, radius of the inscribed circle, etc., all of these being invariants. Now one of the first problems in geometry is to pick out from among these invariants of the triangle as small a number as possible whose equality for two triangles insures the equivalence of the triangles. This can be done, for instance, by taking two sides and the included angle, or two angles and the included side, or three sides. Any one of these three elements may be called *a complete system of invariants* for a triangle with regard to the group of displacements, since two triangles having these invariants in common are equivalent and therefore have all other invariants in common. The conception we have here illustrated may be defined in general terms as follows :

Definition 2. *A set of invariants of a geometric configuration or an algebraic expression are said to form a complete system of invariants if two configurations or expressions having these invariants in common are necessarily equivalent.**

It will be seen from this definition that all the invariants of a geometric configuration or of an algebraic expression are uniquely determined by any complete system of invariants.

Finally we will glance at an application to matrices of the ideas of invariants and equivalence. Let us consider matrices of the nth order,† and consider transformations of the following form which transform the matrix **A** into the matrix **B**:

$$\mathbf{aAb} = \mathbf{B}, \tag{1}$$

where **a** and **b** are any non-singular matrices of the nth order. This transformation may be denoted by the symbol $(\mathbf{a}, \mathbf{b})$, and these symbols must obviously be combined by the formula

$$(\mathbf{a}_2, \mathbf{b}_2)(\mathbf{a}_1, \mathbf{b}_1) = (\mathbf{a}_2\mathbf{a}_1, \mathbf{b}_1\mathbf{b}_2).$$

By means of this formula it may readily be shown that these transformations form a group.

According to our general definition of equivalence, two matrices **A** and **B** must therefore be said to be equivalent when, and only when, two non-singular matrices **a** and **b** exist which satisfy (1). That this definition of equivalence amounts to the same thing as our earlier definition is seen by a reference to Exercise 1, § 25.

* In the classical theory of algebraic invariants this term is used in a different and much more restricted sense. There we have to deal with integral rational relative invariants (cf. § 31). By a complete system of such invariants of a system of algebraic forms is there understood a set of such invariants in terms of which every invariant of this sort of the system of forms can be expressed integrally and rationally. Cf. for instance Clebsch, *Binäre Formen*, p. 109.

† We may, if we choose, confine our attention throughout to matrices with real elements.

30. The Rank of a System of Points or a System of Linear Forms as an Invariant. Let (x_1, y_1, z_1, t_1), (x_2, y_2, z_2, t_2), (x_3, y_3, z_3, t_3) be any three distinct collinear points, so that the rank of the matrix

$$\left\| \begin{matrix} x_1 & y_1 & z_1 & t_1 \\ x_2 & y_2 & z_2 & t_2 \\ x_3 & y_3 & z_3 & t_3 \end{matrix} \right\|$$

is two. Now subject space to a non-singular collineation and we get three new points which will also be distinct and collinear, and hence the rank of their matrix will also be two. Thus we see that in this special case the rank of the system of points is unchanged by a non-singular collineation.

Again, let

$$\begin{aligned} a_1x + b_1y + c_1z + d_1t &= 0, \\ a_2x + b_2y + c_2z + d_2t &= 0, \\ a_3x + b_3y + c_3z + d_3t &= 0, \\ a_4x + b_4y + c_4z + d_4t &= 0 \end{aligned}$$

be any four planes which have one, and only one, point in common, so that the rank of their matrix is three. After a non-singular collineation we have four new planes which will also have one, and only one, point in common, and hence the rank of the matrix of their coefficients will be three. The rank of this system of planes is therefore unchanged by such a transformation.

We proceed now to generalize these facts.

THEOREM 1. *The rank of the matrix of m points*

$$(x_1^{[i]}, x_2^{[i]}, \cdots x_n^{[i]}), \qquad (i = 1, 2, \cdots m)$$

is an invariant with regard to non-singular linear transformations.

Let

$$(1) \qquad \begin{cases} X_1 = c_{11}x_1 + \cdots + c_{1n}x_n, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ X_n = c_{n1}x_1 + \cdots + c_{nn}x_n \end{cases}$$

be a non-singular linear transformation which carries the points $(x_1^{[i]}, \cdots x_n^{[i]})$ over into the points $(X_1^{[i]}, \cdots X_n^{[i]})$. Now suppose any k of the points $(x_1^{[i]}, \cdots x_n^{[i]})$, which for convenience we will take as the first k, are linearly dependent. Then there exist k constants $(c_1, \cdots c_k)$ not all zero, such that

$$(2) \qquad c_1x_j' + c_2x_j'' + \cdots + c_kx_j^{[k]} = 0, \qquad (j = 1, 2, \cdots n).$$

By means of the transformation (1) we have

$$X_j^{[i]} = c_{j1}x_1^{[i]} + \cdots + c_{jn}x_n^{[i]},$$

hence

$$c_1X_j' + c_2X_j'' + \cdots + c_kX_j^{[k]} = c_{j1}(c_1x_1' + c_2x_1'' + \cdots + c_kx_1^{[k]}) + \cdots + c_{jn}(c_1x_n' + c_2x_n'' + \cdots + c_kx_n^{[k]}) \qquad (j = 1, 2, \cdots n).$$

Since this vanishes on account of (2), the first k of the points $(X_1^{[i]}, \cdots X_n^{[i]})$ are linearly dependent. Since (1) is a non-singular transformation, it is immaterial which set of points we consider as the initial set. Thus we have shown that if any k points of either set are linearly dependent, the corresponding k points of the other set will be, also.

Now if the rank of the matrix of the x's is r, at least one set of r of the x-points is linearly independent, but every set of $(r+1)$ of them is linearly dependent. Consequently the same is true for the X-points, and therefore their matrix must also be of rank r.

THEOREM 2. *The rank of the matrix of m linear forms*

$$f_i(x_1, \cdots x_n) \equiv a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \qquad (i = 1, 2, \cdots m)$$

is an invariant with regard to non-singular linear transformations.

The proof of this theorem, which is very similar to the proof of Theorem 1, we leave to the reader.

It will be noticed that the invariants we have considered in this section are examples of what we have called arithmetical invariants.

31. Relative Invariants and Covariants. We will begin by considering a system of n linear forms in n variables

$$(1) \qquad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n. \end{cases}$$

DEFINITION 1. *The determinant*

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

is called the resultant of the system (1).

Let us now subject the system (1) to the linear transformation

$$(2)\qquad \begin{cases} x_1 = c_{11}x_1' + \cdots + c_{1n}x_n', \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ x_n = c_{n1}x_1' + \cdots + c_{nn}x_n'. \end{cases}$$

This gives the new system of forms

$$(3)\qquad \begin{cases} a_{11}'x_1' + \cdots + a_{1n}'x_n', \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ a_{n1}'x_1' + \cdots + a_{nn}'x_n', \end{cases}$$

where
$$a_{ij}' = a_{i1}c_{1j} + a_{i2}c_{2j} + \cdots + a_{in}c_{nj}.$$

From these formulæ and the law of multiplication of matrices we infer that

$$(4)\qquad \left\| \begin{matrix} a_{11}' & \cdots & a_{1n}' \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{n1}' & \cdots & a_{nn}' \end{matrix} \right\| = \left\| \begin{matrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{matrix} \right\| \cdot \left\| \begin{matrix} c_{11} & \cdots & c_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ c_{n1} & \cdots & c_{nn} \end{matrix} \right\|.$$

This result we state as follows:

THEOREM 1. *If a system of n linear forms in n variables with matrix* a *is subjected to a linear transformation with matrix* c, *the resulting system has as its matrix* ac.

Taking the determinants of both sides of (4), we see that the resultant of (1) is not an absolute invariant. It is, however, changed in only a very simple manner by a linear transformation, namely, by being multiplied by the determinant of the transformation. This leads us to the following definition:

DEFINITION 2. *A rational function* of the coefficients of a form or system of forms which, when these forms are subjected to any non-singular linear transformation, is merely multiplied by the μth power (μ an integer †) of the determinant of the transformation is called a relative invariant of weight μ of the form or system of forms.* ‡ *The forms themselves are called the ground forms.*

* Besides these rational invariants we may also consider irrational ones (cf. § 90), in which case the exponent μ will not necessarily be an integer.

† The condition that μ be an integer need not be included as a part of our hypothesis, since it may be proved. The proof that μ cannot be a fraction is simple. The proof that μ cannot be irrational or imaginary would take us outside of the domain of algebra.

‡ From this definition it is clear that every relative invariant is an absolute invariant with regard to the group of linear transformations of determinant +1. Cf. Exercise 7, § 8

It will be seen that absolute invariants are simply relative invariants of weight zero.

The fact pointed out above concerning the resultant may now be stated in the following form:

THEOREM 2. *The resultant of a set of n linear forms in n variables is a relative invariant of weight* 1.

We pass on now to relative covariants:

DEFINITION 3. *If we have a system consisting of a number of n-ary forms and of a number of points* $(y_1, \ldots y_n)$, $(z_1, \ldots z_n)$, …, *the coördinates of each of which are cogredient with the variables* $(x_1, \ldots x_n)$ *of the forms, then any rational function of the coefficients of the forms and the coördinates of the points which is merely multiplied by the μth power (μ an integer) of the determinant of the transformation when the x's are subjected to any non-singular linear transformation is called a relative covariant of weight μ of the system of forms and points.**

We may regard an invariant as the extreme case of a covariant where the number of points is zero. The other extreme case is that in which the number of forms is zero. Here we have the theorem:

THEOREM 3. *The determinant*

$$\begin{vmatrix} x_1' & \cdots & x_n' \\ \cdot & \cdot & \cdot \\ x_1^{[n]} & \cdots & x_n^{[n]} \end{vmatrix}$$

is a relative covariant of weight − 1 *of the system of points*

$$(x_1', \cdots x_n'),\ (x_1'', \cdots x_n''),\ \cdots (x_1^{[n]}, \cdots x_n^{[n]}).$$

For applying the transformation

$$\begin{aligned} x_1 &= c_{11}X_1 + \cdots + c_{1n}X_n, \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ x_n &= c_{n1}X_1 + \cdots + c_{nn}X_n, \end{aligned}$$

* In most books where the subject of covariants is treated, the same letters $(x_1, \cdots x_n)$ are used for one of the points as for the variables of the forms. There is no objection to this, and it is sometimes convenient. We prefer to use a notation which shall make it perfectly clear that the variables of the forms have no connection with the coördinates of the points except that they are cogredient with them.

we have

$$\begin{vmatrix} x_1' & \cdots & x_n' \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ x_1^{[n]} & \cdots & x_n^{[n]} \end{vmatrix} = \begin{vmatrix} c_{11}X_1' + \cdots + c_{1n}X_n' & \cdots & c_{n1}X_1' + \cdots + c_{nn}X_n' \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ c_{11}X_1^{[n]} + \cdots + c_{1n}X_n^{[n]} & \cdots & c_{n1}X_1^{[n]} + \cdots + c_{nn}X_n^{[n]} \end{vmatrix}$$

$$= \begin{vmatrix} c_{11} & \cdots & c_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ c_{n1} & \cdots & c_{nn} \end{vmatrix} \cdot \begin{vmatrix} X_1' & \cdots & X_n' \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ X_1^{[n]} & \cdots & X_n^{[n]} \end{vmatrix}.$$

Or

$$\begin{vmatrix} X_1' & \cdots & X_n' \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ X_1^{[n]} & \cdots & X_n^{[n]} \end{vmatrix} = \begin{vmatrix} c_{11} & \cdots & c_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ c_{n1} & \cdots & c_{nn} \end{vmatrix}^{-1} \cdot \begin{vmatrix} x_1' & \cdots & x_n' \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ x_1^{[n]} & \cdots & x_n^{[n]} \end{vmatrix},$$

as was to be proved.

Another extremely simple case arises when we have a single form and a single point:

THEOREM 4. *The system consisting of the form $f(x_1, \cdots x_n)$ and the point $(y_1, \cdots y_n)$ has as an absolute covariant with regard to linear transformations*

$$f(y_1, \cdots y_n).$$

For let us denote f more explicitly as

$$f(a_1, a_2, \cdots;\ x_1, \cdots x_n),$$

where $a_1, a_2, \cdots$ are the coefficients of f. If the coefficients after the transformation are $a_1', a_2', \cdots$, we have

$$f(a_1', a_2', \cdots;\ x_1', \cdots x_n') \equiv f(a_1, a_2, \cdots;\ x_1, \cdots x_n).$$

This being true for all values of the x's, will be true if the x's are replaced by the y's. But when this is done, the x''s will be replaced by the y''s, since the x's and y's are cogredient. Accordingly

$$f(a_1', a_2', \cdots;\ y_1', \cdots y_n') \equiv f(a_1, a_2, \cdots;\ y_1, \cdots y_n),$$

as was to be proved.

The three examples of invariants and covariants which have been given in this section are all polynomials in the coefficients of the forms and the coördinates of the points. Such invariants we shall speak of as *integral rational* invariants and covariants.*

THEOREM 5. *The weight of an integral rational invariant cannot be negative.*†

Let $a_1, a_2, \cdots;\ b_1, b_2, \cdots;\ \cdots$ be the coefficients of the system of forms, and let c_{ij} be the coefficients of the transformation. It is clear that the coefficients $a_1', a_2', \cdots;\ b_1', b_2', \cdots;\ \cdots$ after the transformation are polynomials in the a's, b's, etc., and in the c_{ij}'s. Now let I be an integral rational invariant of weight μ,

$$I(a_1', a_2', \cdots;\ b_1', b_2', \cdots;\ \cdots) = c^\mu I(a_1, a_2, \cdots;\ b_1, b_2, \cdots;\ \cdots),$$

where c is the determinant of the transformation. Suppose now that μ were negative, $\mu = -\nu$. Then

$$(5) \qquad c^\nu I(a_1', \cdots;\ b_1', \cdots;\ \cdots) = I(a_1, \cdots;\ b_1, \cdots;\ \cdots).$$

This equality, like the preceding one, is known to hold for all values of the c_{ij}'s for which $c \neq 0$. Hence, since the expressions on both sides of the equality are polynomials in the a's, b's, $\cdots$ and the c_{ij}'s, we infer, by an application of Theorem 5, § 2, that we really have an identity.

Let us now assign to the a's, b's, $\cdots$ any constant values such that $I(a_1, \cdots;\ b_1, \cdots;\ \cdots) \neq 0$. Then $I(a_1', \cdots;\ b_1', \cdots;\ \cdots)$ will be a polynomial in the c_{ij}'s alone, which, from (5), cannot be identically zero. The identity (5) thus takes a form which states that the product of two polynomials in the c_{ij}'s is a constant, and since the first of these polynomials, c^ν, is of higher degree than zero, this is impossible.

We will agree in future to understand by the terms *invariant* and *covariant*, invariants or covariants (absolute, relative, or arithmetical) *with regard to all non-singular linear transformations*. If we wish to consider invariants or covariants with regard to other sets of transformations, for instance with regard to real linear transformations, this fact will be explicitly mentioned.

* All rational invariants and covariants may be formed as the quotients of such as are integral and rational; cf. Exercises 4, 5, § 78.

† It cannot be zero either; cf. Theorem 5, § 79.