5. Prove that a projective transformation in space effects on every plane a two-dimensional, and on every line a one-dimensional, projective transformation, while at the same time the positions of the plane and line are changed.

[SUGGESTION. If p and p' are any two corresponding planes, assume in any way a pair of perpendicular axes in each of them, and denote by (x_1, y_1, t_1), and (x_1', y_1', t_1') respectively the systems of two-dimensional homogeneous coördinates based on these axes. Then show, by using the result of Exercise 4, that the transformation of one plane on the other will be expressed by writing x_1', y_1', t_1' as homogeneous linear polynomials in x_1, y_1, t_1.]

25. Further Development of the Algebra of Matrices. We proceed to establish certain further properties of matrices, leaving, however, much to the reader in the shape of exercises at the end of the section.

The theory of linear transformations suggests to us at once certain properties of matrices. The first of these is:

THEOREM 1. *The matrix*

$$\mathrm{I} = \begin{Vmatrix} 1 & 0 & \cdots\cdots & 0 \\ 0 & 1 & \cdots\cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots\cdots & 1 \end{Vmatrix},$$

has the property that if a *is any matrix whatever*

$$\mathrm{Ia} = \mathrm{aI} = \mathrm{a}.$$

For the linear transformation of which a is the matrix will evidently not be changed by being either followed or preceded by the identical transformation of which I is the matrix.

If we do not wish to use the idea of linear transformation, we may prove the theorem directly by actually forming the products Ia and aI.

This theorem tells us that I plays in the algebra of matrices the same rôle that is played by 1 in ordinary algebra. For this reason I is sometimes called the *unit matrix* or *idemfactor*.

Let us now consider any non-singular linear transformation and its inverse. These two transformations performed in succession in either order obviously lead to the identical transformation. This gives us the theorem:

THEOREM 2. *If*

$$\mathrm{a} = \begin{Vmatrix} a_{11} \cdots a_{1n} \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ a_{n1} \cdots a_{nn} \end{Vmatrix}$$

is a non-singular matrix of determinant a, and if A_{ij} denote in the ordinary way the cofactors of the elements of a, the matrix

$$\begin{Vmatrix} \dfrac{A_{11}}{a} & \cdots\cdots & \dfrac{A_{n1}}{a} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \dfrac{A_{1n}}{a} & \cdots\cdots & \dfrac{A_{nn}}{a} \end{Vmatrix},$$

called the inverse of a, *and denoted by* a^{-1}, *is a non-singular matrix which has the property that*

$$\mathrm{aa}^{-1} = \mathrm{a}^{-1}\mathrm{a} = \mathrm{I}.$$

This suggests that we define positive and negative integral powers of matrices as follows:

DEFINITION 1. *If p is any positive integer and* a *any matrix we understand by* a^p *the product* aa ⋯ a *to p factors. If* a *is a non-singular matrix, we define its negative and zero powers by the formulæ*

$$\mathrm{a}^{-p} = (\mathrm{a}^{-1})^p, \quad \mathrm{a}^0 = \mathrm{I}.$$

From this definition we infer at once

THEOREM 3. *The laws of indices*

$$\mathrm{a}^p\mathrm{a}^q = \mathrm{a}^{p+q}, \quad (\mathrm{a}^p)^q = \mathrm{a}^{pq}$$

hold for all matrices when the indices p and q are positive integers, and for all non-singular matrices when p and q are any integers.

We turn now to the question of the division of one matrix by another. We naturally define division as the inverse of multiplication, and, since multiplication is not commutative, we thus get two distinct kinds of division; a divided by b being on the one hand a matrix x such that

$$\mathrm{a} = \mathrm{bx},$$

on the other hand a matrix y such that

$$\mathrm{a} = \mathrm{yb}.$$

On account of this ambiguity, the term *division* is not ordinarily used here. We have, however, as is easily seen, the following theorem:

THEOREM 4. *If* a *is any matrix and* b *any non-singular matrix there exists one, and only one, matrix* x *which satisfies the equation*

$$a = bx,$$

and one, and only one, matrix y *which satisfies the equation*

$$a = yb,$$

and these matrices are given respectively by the formulæ

$$x = b^{-1}a, \qquad y = ab^{-1}.$$

A special class of matrices is of some importance; namely, those of the type

$$\begin{Vmatrix} k & 0 & \cdots & 0 \\ 0 & k & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & k \end{Vmatrix}.$$

Such matrices we will call *scalar matrices* for a reason which will presently appear.

If we denote by **k** the scalar matrix just written, and by **a** any matrix of the same order as **k**, we obtain readily the formula

$$(1) \qquad ka = ak = ka.$$

If now, besides the scalar matrix **k**, we have a second scalar matrix **l** in which each element in the principal diagonal is l, we have the two formulæ

$$(2) \qquad k + l = l + k = (k + l)I,$$

$$(3) \qquad kl = lk = klI.$$

Formula (1) shows that scalar matrices may be replaced by ordinary scalars when they are to be multiplied by other matrices; while formulæ (2) and (3) show that scalar matrices combined with one another not only obey all the laws of ordinary scalars, but that each scalar matrix may in such cases be replaced by the scalar which occurs in each element of its principal diagonal provided that at the end of the work the resulting scalar be replaced by the corresponding scalar matrix.

For these reasons we may, in the algebra of matrices, replace all scalar matrices by the corresponding scalars, and then consider that all scalars which enter into our work stand for the corresponding scalar matrices. If we do this, the unit matrix **I** will be represented by the symbol **1**.

DEFINITION 2. *By the adjoint* A *of a matrix* a *is understood another matrix of the same order in which the element in the ith row and jth column is the cofactor of the element in the jth row and ith column of* a.*

It will be seen that when **a** is non-singular,

$$(4) \qquad A = a\mathrm{a}^{-1},$$

but it should be noticed that while every matrix has an adjoint, only non-singular matrices have inverses.

Equation (4) may be written in the form

$$(5) \qquad Aa = aA = aI,$$

a form in which it is true not merely when **a** is non-singular, but also, as is seen by direct multiplication, when the determinant of **a** is zero.

Finally we come to a few important theorems concerning the rank of the matrix obtained by multiplying together two given matrices. In the first place, we notice that the rank of the product is not always completely determined by the ranks of the factors. This may be shown by numerous examples, for instance, in formula (5), § 22, the ranks of the factors are in general two and one, and the rank of the product is zero, while in the formula

$$\begin{Vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{Vmatrix} = \begin{Vmatrix} 0 & 0 & a_{12} \\ 0 & 0 & a_{22} \\ 0 & 0 & a_{32} \end{Vmatrix}$$

the ranks of the factors are in general the same, namely two and one, while the rank of the product is one.

But though, as this example shows, the ranks of the factors (even together with the order of the matrices) do not suffice to determine the rank of the product, there are, nevertheless, important inequalities between these ranks, one of which we now proceed to deduce.

* Notice the interchange of rows and columns here, which in the case of adjoint determinants, being immaterial and sometimes inconvenient, was not made.

For this purpose consider the two matrices

$$\mathbf{a} = \begin{Vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{Vmatrix}, \qquad \mathbf{b} = \begin{Vmatrix} b_{11} & \cdots & b_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ b_{n1} & \cdots & b_{nn} \end{Vmatrix},$$

and their product **ab**.

THEOREM 5. *Any k-rowed determinant of the matrix* **ab** *is equal to an aggregate of k-rowed determinants of* **b** *each multiplied into a polynomial in the a's, and also to an aggregate of k-rowed determinants of* **a** *each multiplied by a polynomial in the b's.*

For any k-rowed determinant of **ab** may be broken up into a sum of determinants of the kth order in such a way that each column of each determinant has one of the b's as a common factor.* After taking out these common factors from each determinant, we have left a determinant in the a's which, if it does not vanish identically, is a k-rowed determinant of **a**. Or, on the other hand, we may break up the k-rowed determinant of **ab** into a sum of determinants of the kth order in such a way that each row of each determinant has one of the a's as a common factor. After taking out these common factors from each determinant, we have left a determinant in the b's which, if it does not vanish identically, is a k-rowed determinant of **b**.

From the theorem just proved it is clear that if all the k-rowed determinants of **a** or of **b** are zero, the same will be true of all the k-rowed determinants of **ab**. Hence

THEOREM 6. *The rank of the product of two matrices cannot exceed the rank of either factor.*†

* The truth of this statement and the following will be evident if the reader actually writes out the matrix **ab**.

† Thus if r_1 and r_2 are the ranks of the two factors and R is the rank of the product, we have $R \leqq r_1$, $R \leqq r_2$. This is one half of Sylvester's "Law of Nullity," of which the other half may be stated in the form $R \geqq r_1 + r_2 - n$, where n is the order of the matrices; cf. Exercise 8 at the end of this section. Sylvester defines the nullity of a matrix as the difference between its order and its rank, so that his statement of the law of nullity is: The nullity of the product of two matrices is at least as great as the nullity of either factor, and at most as great as the sum of the nullities of the factors.

There is one important case in which this theorem enables us to determine completely the rank of the product, namely, the case in which one of the two matrices **a** or **b** is non-singular. Suppose first that **a** is non-singular, and denote the ranks of **b** and **ab** by r and R respectively. By Theorem 6, $R \leqq r$. We may, however, also regard **b** as the product of $\mathbf{a}^{-1}$ and **ab**, and hence, applying Theorem 6 again, we have $r \leqq R$. Combining these two results, we see that $r = R$.

On the other hand, if **b** is non-singular, and we denote the ranks of **a** and **ab** respectively by r and R, we get from Theorem 6, $R \leqq r$; and, applying this theorem again to the equation

$$(\mathbf{ab})\,\mathbf{b}^{-1} = \mathbf{a},$$

we have $r \leqq R$. Thus again we get $r = R$.

We have thus established the result:

THEOREM 7. *If a matrix of rank r is multiplied in either order by a non-singular matrix, the rank of the product is also r.*

EXERCISES

1. Prove that a necessary and sufficient condition that two matrices **a** and **b** of the same order be equivalent is that there exist two non-singular matrices **c** and **d** such that

$$\mathbf{dac} = \mathbf{b}.$$

Cf. § 22, Exercise 2, and § 19, Exercise 4.

2. Prove that a necessary and sufficient condition that two matrices **a** and **b** of the same order be equivalent is that there exist four matrices **c**, **d**, **e**, **f** such that

$$\mathbf{dac} = \mathbf{b}, \qquad \mathbf{a} = \mathbf{fbe}.$$

3. Prove that every matrix of rank r can be written as the sum of r matrices of rank one.*

[SUGGESTION. Notice that the special matrix mentioned in § 19, Exercise 3, can be so written.]

* A matrix of rank one has been called by Gibbs a *dyad*, since it may (cf. § 19, Ex. 5) be regarded as a product of two complex quantities $(a_1, a_2, \cdots a_n)$ and $(b_1, b_2, \cdots b_n)$. The sum of any number of dyads is called a *dyadic polynomial*, or simply a *dyadic*. Every matrix is therefore a dyadic, and *vice versa*. Gibbs's theory of dyadics, in the case $n = 3$, is explained in the *Vector Analysis* of Gibbs-Wilson, Chap. V. Geometric language is used here exclusively, the complex quantities (a_1, a_2, a_3) and (b_1, b_2, b_3) from which the dyads are built up being interpreted as vectors in space of three dimensions. This theory is equivalent to Hamilton's theory of the Linear Vector Function in Quaternions.

4. Prove that a necessary and sufficient condition that a matrix be a divisor of zero (cf. § 22, Exercise 1) is that it be singular.

[SUGGESTION. Consider equivalent matrices.]

5. Prove that the inverse of the product of any number of non-singular matrices is the product of the inverses of these matrices taken in the reverse order.

Hence deduce a similar theorem concerning the adjoint of a product of any number of matrices, whether these matrices are singular or not.

What theorem concerning determinants can be inferred?

6. Prove that the conjugate of the inverse of a non-singular matrix is the inverse of the conjugate; and that the conjugate of the adjoint of any matrix is the adjoint of the conjugate.

7. Prove that if a matrix has the property that its product with every matrix of the same order is commutative, it is necessarily a scalar matrix.

8. If r_1 and r_2 are the ranks of two matrices of order n, and R the rank of their product, prove that

$$R \geqq r_1 + r_2 - n.*$$

[SUGGESTION. Prove this theorem first on the supposition that one of the two matrices which are multiplied together is of the form mentioned in Exercise 3, § 19, using also at this point Exercise 1, § 8. Then reduce the general case to this one by means of Exercise 1 of this section.]

26. Sets, Systems, and Groups. These three words are the technical names for conceptions which are to be met with in all branches of mathematics. In fact the first two are of such generality that they may be said to form the logical foundation on which all mathematics rests.† In this section we propose, after having given a brief explanation of these three conceptions, to show how they apply to the special subjects considered in this chapter.

The objects considered in mathematics — we use the word *object* in the broadest possible sense — are of the most varied kinds. We have, on the one hand, to mention a few of the more important ones, the different kinds of *quantities* ranging all the way from the positive integers to complex quantities and matrices. Next we have in geometry not only points, lines, curves, and surfaces but also such

* Cf. the footnote to Theorem 6.

† For a popular exposition of the point of view here alluded to, see my address on *The Fundamental Conceptions and Methods of Mathematics*, St. Louis Congress of Arts and Science, 1904. Reprinted in Bull. Amer. Math. Soc., December, 1904.

things as displacements (rotations, translations, etc.), collineations, and, in fact, geometrical transformations in general. Then in various parts of mathematics we have to deal with the Theory of Substitutions, that is, with the various changes which can be made in the order of certain objects, and these substitutions themselves may be regarded as objects of mathematical study. Finally, in mechanics we have to deal with such objects as forces, couples, velocities, etc.

These objects, and all others which are capable of mathematical consideration, are constantly presenting themselves to us, not singly, but in *sets*. Such sets (or, as they are sometimes called, *classes*) of objects may consist of a finite or an infinite number of objects, or *elements*. We mention as examples:

(1) All prime numbers.

(2) All lines which meet two given lines in space.

(3) All planes of symmetry of a given cube.

(4) All substitutions which can be performed on five letters.

(5) All rotations of a plane about a given line perpendicular to it.

Having thus gained a slight idea of the generality of the conception of a set, we next notice that in many cases in which we have to deal with a set in mathematics, there are one or more rules by which pairs of elements of the set may be combined so as to give objects, either belonging to the set or not as the case may be. As examples of such rules of combination, we mention addition and multiplication both in ordinary algebra and in the algebra of matrices; the process by which two points, in geometry, determine a line; the process of combining two displacements to give another displacement, etc.

Such a set, with its associated rules of combination, we will call a *mathematical system*, or simply a system.*

We come now to a very important kind of system known as a *group*, which we define as follows:

* This definition is sufficiently general for our immediate purposes. In general, however, it is desirable to admit, not merely rules of combination, but also *relations* between the elements of a system. In fact we may have merely one or more relations and no rules of combination at all. From this point of view the positive integers with the relation of greater and less would form a system, even though we do not introduce any rule of combination such as addition or multiplication. It may be added that rules of combination may be regarded as merely relations between three objects; cf. the address referred to above.

DEFINITION. *A system consisting of a set of elements and one rule of combination, which we will denote by* $\circ$, *is called a group if the following conditions are satisfied:*

(1) *If a and b are any elements of the set, whether distinct or not,* $a \circ b$ *is also an element of the set.**

(2) *The associative law holds; that is, if a, b, c are any elements of the set,*

$$(a \circ b) \circ c = a \circ (b \circ c).$$

(3) *The set contains an element, i, called the identical element, which is such that every element is unchanged when combined with it,*

$$i \circ a = a \circ i = a.$$

(4) *If a is any element, the set also contains an element a′, called the inverse of a, such that*

$$a' \circ a = a \circ a' = i.$$

Thus, for example, the positive and negative integers with zero form a group if the rule of combination is addition. In this case zero is the identical element, and the inverse of any element is its negative. These same elements, however, do not form a group if the rule of combination is multiplication, for while conditions (1), (2), and (3) are fulfilled (the identical element being 1 in this case), condition (4) is not, since zero has no reciprocal.

Again, the set of all real numbers forms a group if the rule of combination is addition, but not if it is multiplication, since in this case zero has no inverse. If we exclude zero from the set, we have a group if the rule of combination is multiplication, but not if it is addition.

As an example of a group with a finite number of elements we mention the four numbers

$$+1, -1, +\sqrt{-1}, -\sqrt{-1}$$

with multiplication as the law of combination.

In order to get an example of a group of geometrical operations, let us consider the translations of a plane, regarded as a rigid lamina, in the directions of its own lines. Every such translation may be represented both in magnitude and in direction by the length and

* A system satisfying condition (1) is sometimes said to have "the group property." In the older works on the subject this condition was the only one to be explicitly mentioned, the others, however, being tacitly assumed.

direction of an arrow lying in the plane in question. Two such translations performed in succession are obviously equivalent to a translation of the same sort represented by the arrow obtained by combining the two given arrows according to the law of the parallelogram of forces. The set of all translations with the law of combination just explained is readily seen to form a group if we include in it the *null translation*, *i.e.* the transformation which leaves every point in the plane fixed. This null translation is then the identical element, and two translations are the inverse of each other if they are equal in magnitude and opposite in direction.

All the groups we have so far mentioned satisfy, not only the four conditions stated in the definition, but also a fifth condition, viz. that the law of combination is commutative. Such groups are called *commutative* or *Abelian groups*. In general, however, groups do not have this property. As examples of non-Abelian groups, we may mention first the group of all non-singular matrices of a given order, the rule of combination being multiplication; and secondly the group of all matrices of a given order whose determinants have the value ± 1, the rule of combination being again multiplication. This second group is called a *subgroup* of the first, since all its elements are also elements of the first group, and the rule of combination is the same in both cases. A subgroup of the group last mentioned is the group of all matrices of a given order whose determinants have the value $+1$,* the rule of combination being multiplication.

We add that non-Abelian groups may readily be built up whose elements are linear transformations, or collineations. On the other hand, Abelian groups may be formed from matrices if we take as our rule of combination addition instead of multiplication.

27. Isomorphism.

DEFINITION. *Two groups are said to be isomorphic† if it is possible to establish a one-to-one correspondence between their elements of such a*

* These are called *unimodular matrices*; or, more accurately, *properly unimodular matrices* to distinguish them from the *improperly unimodular matrices* whose determinants have the value -1. It should be noticed that these last matrices taken by themselves do not constitute a group, since they do not even have the group property.

† *Simply isomorphic* would be the more complete term. We shall, however, not be concerned with isomorphism which is not simple.

*sort that if a, b are any elements of the first group and a′, b′ the corresponding elements of the second, then a′ ∘ b′ corresponds to a ∘ b.**

We proceed to illustrate this definition by some examples, leaving to the reader the proofs of the statements we make. In each case we omit the statement of the rule of combination in the case of transformations, where no misunderstanding is possible.

First Example. (*a*) The group of the four elements

$$1, \sqrt{-1}, -1, -\sqrt{-1},$$

the rule of combination being multiplication.

(*b*) The group of four rotations about a given line through angles of 0°, 90°, 180°, 270°.

These two groups may be proved to be isomorphic by pairing the elements against one another in the order in which they have just been written.

Second Example. (*a*) The group of the four matrices

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix},$$

the rule of combination being multiplication.

(*b*) The group of the following four transformations: the identical transformation; reflection in a plane; reflection in a second plane at right angles to the first; rotation through 180° about the line of intersection of these two planes.

(*c*) The group consisting of the identical transformation and of three rotations through angles of 180° about three straight lines through a point at right angles to one another.

The two groups of Example 1 are not isomorphic with the three of Example 2 in spite of the fact that there are the same number of elements in all the groups. This follows from the presence of two elements in the groups of Example 1 whose squares are not the identical element.

* This idea of isomorphism may obviously be extended to the case of any two systems provided merely that there are the same number of rules of combination in both cases. Thus the system of all scalar matrices on the one hand and of all scalars on the other, the rules of combination being in both cases addition and multiplication, are obviously isomorphic. It is for this reason that no confusion arises if no distinction is made between scalar matrices and scalars.

Third Example. (*a*) The group of all real quantities; the rule of combination being addition.

(*b*) The group of all scalar matrices of order k with real elements; the rule of combination being addition.

(*c*) The group of all translations of space parallel to a given line.

Fourth Example. (*a*) The group of all non-singular matrices of order n, with multiplication as the rule of combination.

(*b*) The group of all non-singular homogeneous linear transformations in n variables.

We might be tempted to mention as a group of geometrical transformations isomorphic with the last two groups, the group of all non-singular collineations in space of $n-1$ dimensions. This, however, would be incorrect, for the correspondence we have established between collineations and linear transformations is not one-to-one; to every linear transformation corresponds one collineation, but to every collineation correspond an infinite number of linear transformations, whose coefficients are proportional to one another.* In order to get a group of geometrical transformations isomorphic with the group of non-singular matrices of the nth order it is sufficient to interpret the variables $x_1, \cdots x_n$ as non-homogeneous coördinates in space of n dimensions, and to consider the geometric transformation effected by non-singular homogeneous linear transformations of these x's. These transformations are those affine transformations of space of n dimensions which leave the origin unchanged; cf. the footnote on p. 70. Thus the group of all non-singular matrices of the nth order is isomorphic with a certain subgroup of the group of collineations in space of n dimensions, not with the group of all non-singular collineations in space of $n-1$ dimensions.

An essential difference between these two groups is that one

* This does not really prove that the groups are not isomorphic, since it is conceivable that some other correspondence might be established between their elements which would be one-to-one and of such a sort as to prove isomorphism. Even the fact, to be pointed out presently, that the groups depend on a different number of parameters does not settle the question. A reference to the result stated in Exercise 7, § 25, shows that the groups are not isomorphic; for, according to it, the only non-singular collineation which is commutative with all collineations is the identical transformation, whereas all linear transformations with scalar matrices have this property.

depends on n^2 parameters (the n^2 coefficients of the linear transformation) while the other depends only on $n^2 - 1$ parameters (the ratios of the coefficients of the collineation).

We can, however, by looking at the subject a little differently, obtain a group of matrices isomorphic with the group of all non-singular collineations in space of $n-1$ dimensions. For this purpose we need merely to regard two matrices as equal whenever the elements of one can be obtained from those of the other by multiplying all the elements by the same quantity not zero. When we take this point of view with regard to matrices, it is desirable to indicate it by a new terminology and notation. According to a suggestion of E. H. Moore of Chicago, we will call such matrices *fractional* matrices, and write them

$$\left\|\begin{array}{cc} a_{11} & a_{12} \\ \hline a_{21} & a_{22} \end{array}\right\|, \quad \left\|\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right\|, \text{ etc.}$$

Agreeing that fractional matrices are to be added and multiplied according to the same rules as ordinary matrices, we may now say that the group of all non-singular collineations in space of $n-1$ dimensions is isomorphic with the group of all fractional matrices of the nth order whose determinants are not zero.*

To take another example, the groups in the second example above are isomorphic with the group whose elements are the four fractional matrices

$$\left\|\begin{array}{cc} 1 & 0 \\ \hline 0 & 1 \end{array}\right\|, \quad \left\|\begin{array}{cc} 1 & 0 \\ \hline 0 & -1 \end{array}\right\|, \quad \left\|\begin{array}{cc} 0 & 1 \\ \hline 1 & 0 \end{array}\right\|, \quad \left\|\begin{array}{cc} 0 & 1 \\ \hline -1 & 0 \end{array}\right\|,$$

and where the law of combination is multiplication. These four matrices, if regarded as ordinary matrices, would not even satisfy the first condition for a group.

The reader wishing to get a further insight into the theory of groups of linear transformations will find the following three treatments interesting and instructive. They duplicate each other to only a very slight extent.

* It should be noticed that we cannot speak of the *value* of the determinant of a fractional matrix unless this value is zero, for if we multiply all the elements of the matrix by c we do not change the matrix, but do multiply the determinant by c^n. There is in particular no such thing as a unimodular fractional matrix. We may however, speak of the rank of a fractional matrix.

Weber, *Algebra*, Vol. II.

Klein, *Vorlesungen über das Ikosaeder.*

Lie-Scheffers, *Vorlesungen über continuirliche Gruppen.*

EXERCISES

1. DEFINITION. *A group is said to be of order n if it contains n, and only n, elements.*

If a group of order n has a subgroup, prove that the order of this subgroup is a factor of n.

[SUGGESTION. Denote the elements of the subgroup by $a_1 \cdots a_k$, and let b be any other element of the group. Show that $ba_1, ba_2, \cdots ba_k$ are all elements of the group distinct from each other and distinct from the a's. If there are still other elements, let c be one and consider the elements $ca_1, \cdots ca_k$, etc.]

2. Prove that if a is any element of a group of finite order, it is possible by multiplying a by itself a sufficient number of times to get the identical element.

DEFINITION. *The lowest power to which a can be raised so as to give the identical element is called the period of a.*

3. Prove that every element of a group of order n has as its period a factor of n (1 and n included).

4. DEFINITION. *A group is called cyclic if all its elements are powers of a single element.*

Prove that all cyclic groups of order n are isomorphic with the group of rotations about an axis through angles $0, \omega, 2\omega, \cdots (n-1)\omega$, where $\omega = 2\pi/n$, and that conversely every such group of rotations is a cyclic group.

5. Prove that every group whose order is a prime number is a cyclic group.

6. Prove that all groups of order 4 are either cyclic or isomorphic with the groups of the second example above. A group of this last kind is called a *fours group* (*Vierergruppe*).

7. Obtain groups with regard to one or the other of which all groups of order 6 are isomorphic.

8. Obtain groups with regard to one or the other of which all groups of order 8 are isomorphic.