CHAPTER VI

LINEAR TRANSFORMATIONS AND THE COMBINATION OF MATRICES

21. Matrices as Complex Quantities. We have said in § 7 that a matrix of m rows and n columns is not a quantity, but a set of mn quantities. This statement is true only if we restrict the term *quantity* to the real and complex quantities of ordinary algebra. A moment's reflection, however, will show that the conception of quantity as used in arithmetic and algebra has been gradually enlarged from the primitive conception of the positive integer by using the word *quantity* to denote entities which, at an earlier stage, would not have been regarded as quantities at all, as, for instance, negative quantities. We will consider here only one of these extensions, namely the introduction of complex quantities, as this will lead us to look at our matrices from a broader point of view.

If we have objects of two or more different kinds which can be counted or measured, and if we consider aggregates of such objects, we get concrete examples of complex quantities, as, for instance, 5 horses, 3 cows, and 7 sheep. A convenient way to write such a complex quantity is (5, 3, 7), it being agreed that, in the illustration we are considering, the first place shall always indicate horses, the second cows, and the third sheep. In the abstract theory of complex quantities we do not specify any concrete objects such as horses, cows, etc., but merely consider sets of quantities (couples, triplets, etc.), distinguishing these quantities by the position they occupy in our symbol. Such a complex quantity we often find it convenient to designate by a single letter,

$$\alpha = (a, b, c)$$

just as in ordinary algebra we denote a fraction ($\frac{2}{3}$ for instance), which really involves two numbers, by a single letter. We speak here of the simple quantities a, b, c of which α is composed as its first,

second, third components; and we call two complex quantities equal when and only when the components of one are equal respectively to the corresponding components of the other. Similarly a complex quantity is said to vanish when and only when all of its components are zero.

What makes it worth while to speak of such sets of quantities as complex quantities is that it is found useful to perform certain algebraic operations on them. By the sum and difference of two complex quantities

$$\alpha_1 = (a_1, b_1, c_1), \quad \alpha_2 = (a_2, b_2, c_2)$$

we mean the two new complex quantities

$$\alpha_1 + \alpha_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2), \; \alpha_1 - \alpha_2 = (a_1 - a_2, b_1 - b_2, c_1 - c_2).^*$$

When it comes to the question of defining what we shall understand by the product of two complex quantities, things are by no means so simple. It is necessary here to lay down some rule according to which, when two complex quantities are given, a third, which we call their product, is determined. Such rules may be laid down in an infinite variety of ways, and each such rule gives us a different system of complex quantities.†

We come now to the subject of matrices. A matrix of m rows and n columns being merely a set of mn quantities (which we assume to be either real quantities or the ordinary complex quantities of elementary algebra) arranged in a definite order, is, according to the point of view we have explained, a complex quantity with mn components; and it is only a special application of the theory of complex quantities which we have sketched, when we lay down the following definitions:

Definition 1. *A matrix is said to be zero when and only when all of its elements are zero.*

Definition 2. *Two matrices are said to be equal when and only when they have the same number of rows and of columns, and every element of one is equal to the corresponding element of the other.*

* That this is the natural meaning to be attached to the terms *sum* and *difference* will be seen by reference to the concrete illustration given above.

† If, in particular, we wish to introduce the ordinary system of complex quantities of elementary algebra, we use a system of couples, and define the product of two couples,

$$\alpha_1 = (a_1, b_1), \quad \alpha_2 = (a_2, b_2),$$

by the formula

$$\alpha_1\alpha_2 = (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1).$$

For further details cf. Burkhardt's *Funktionentheorie*, §§ 2, 3.

DEFINITION 3. *By the sum (or difference) of two matrices of m rows and n columns each, we understand a matrix of m rows and n columns, each of whose elements is the sum (or difference) of the corresponding elements of the given matrices.*

In order to distinguish them from matrices, we will call the ordinary quantities of algebra (real quantities and ordinary complex quantities) *scalars*.

Before proceeding, as we shall do in the next section, to the definition of the product of two matrices, we will define the product of a matrix and a scalar.

DEFINITION 4. *If* **a** *is a matrix* * *and k a scalar, then by the product* $k\mathbf{a}$ *or* $\mathbf{a}k$ *we understand the matrix each of whose elements is k times the corresponding element of* **a**.

As an obvious consequence of our definitions we state the theorem:

THEOREM. *All the laws of ordinary algebra hold for the addition or subtraction of matrices and their multiplication by scalars.*

For instance, if **a**, **b**, **c** are matrices, and k, l scalars,

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a},$$
$$\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c},$$
$$k\mathbf{a}+k\mathbf{b}=k(\mathbf{a}+\mathbf{b}),$$
$$k\mathbf{a}+l\mathbf{a}=(k+l)\mathbf{a}.\dagger$$

EXERCISE

If r_1 and r_2 are the ranks of two matrices and R the rank of their sum, prove that

$$R \leqq r_1 + r_2.$$

22. The Multiplication of Matrices. Up to this point we have considered matrices with m rows and n columns. For the sake of simplicity of statement, we shall confine our attention from now on to square matrices, that is to the case $m = n$. This involves no real loss

* The notation here used, matrices being denoted by heavy-faced type, will be systematically followed in this book.

† We add that, as a matter of notation, we shall write

$$(-1)\,\mathbf{a} = -\mathbf{a}.$$

of generality provided we agree to consider a matrix of m rows and n columns, where $m \neq n$, as equivalent to a square matrix of order equal to the larger of the two integers m, n and obtained from the given matrix by filling in the lacking rows or columns with zeros.

The question now presents itself: How shall we define the product of two square matrices of the same order? It must be clearly understood that we are logically free to lay down here such definition as we please, and that the definition we select is preferable to others not on any *a priori* grounds, but only because it turns out to be more useful. We select the following definition, which is suggested * by the multiplication theorem for determinants:

DEFINITION 1. *The product* **ab** *of two square matrices of the nth order is a square matrix of the nth order in which the element which lies in the ith row and jth column is obtained by multiplying each element of the ith row of* **a** *by the corresponding element of the jth column of* **b** *and adding the results.*

Let us denote by a_{ij} and b_{ij} the elements in the ith row and jth column of **a** and **b** respectively, or, as we will say for brevity, the element (i, j) of these matrices. Then, according to our definition, the element (i, j) of the product **ab** is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}, \tag{1}$$

while the element (i, j) in the matrix **ba** is

$$a_{1j}b_{i1} + a_{2j}b_{i2} + \cdots + a_{nj}b_{in}. \tag{2}$$

Since the two quantities (1) and (2) are not in general equal, we obtain

THEOREM 1. *The multiplication of matrices is not in general commutative, that is, in general*

$$\mathbf{ab} \neq \mathbf{ba}.$$

Let us now consider a third matrix **c** whose element (i, j) is c_{ij}, and form the product $(\mathbf{ab})\mathbf{c}$. The element (i, j) of this matrix is

$$\begin{aligned} &(a_{i1}b_{11} + a_{i2}b_{21} + \cdots + a_{in}b_{n1})c_{1j} \\ &+(a_{i1}b_{12} + a_{i2}b_{22} + \cdots + a_{in}b_{n2})c_{2j} \\ &+ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ &+(a_{i1}b_{1n} + a_{i2}b_{2n} + \cdots + a_{in}b_{nn})c_{nj}. \end{aligned} \tag{3}$$

* Historically this definition was suggested to Cayley by the consideration of the composition of linear transformations; cf. § 23.

On the other hand, the element (i, j) of the matrix a(bc) is

$$\begin{aligned} (4) \qquad & a_{i1}(b_{11}c_{1j} + b_{12}c_{2j} + \cdots + b_{1n}c_{nj}) \\ & + a_{i2}(b_{21}c_{1j} + b_{22}c_{2j} + \cdots + b_{2n}c_{nj}) \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + a_{in}(b_{n1}c_{1j} + b_{n2}c_{2j} + \cdots + b_{nn}c_{nj}). \end{aligned}$$

Since the two quantities (3) and (4) are equal, we have established

THEOREM 2. *The multiplication of matrices is associative, that is,*

$$(\mathrm{ab})\mathrm{c} = \mathrm{a}(\mathrm{bc}).$$

Finally, since the element (i, j) of the matrix a (b + c) is clearly equal to the sum of the elements (i, j) of the matrices ab and ac, we have the result

THEOREM 3. *The multiplication of matrices is distributive, that is,*

$$\mathrm{a}(\mathrm{b} + \mathrm{c}) = \mathrm{ab} + \mathrm{ac}.$$

Besides the commutative, associative, and distributive laws, there is one other principle of elementary algebra which is of constant use, namely, the principle that a product cannot vanish unless at least one of the factors is zero. Simple examples show that this is not true in the algebra of matrices. We have, for instance,

$$(5) \qquad \begin{Vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} \end{Vmatrix} = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} = 0,$$

whatever the values of the a's and b's may be. Hence

THEOREM 4. *From the vanishing of the product of two or more matrices, we cannot infer that one of the factors is zero.*

The process of cancelling out non-vanishing factors which enter throughout an equation will, therefore, be inadmissible in the algebra of matrices.

We next state a result which follows at once from the similarity between the theorem for the multiplication of determinants and our definition of the product of two matrices:

THEOREM 5. *The determinant of a matrix which is obtained by multiplying together two or more matrices is equal to the product of the determinants of these matrices.*

The conception of the conjugate of a matrix, as defined in § 7, Definition 2, is an important one, and the following theorem concerning it is often useful:

THEOREM 6. *The conjugate of the product of any number of matrices is the product of their conjugates taken in the reverse order.*

In order to prove this theorem we first notice that its truth in the case of two matrices follows at once from the definition of the product of two matrices. Its truth will therefore follow in all cases if, assuming the theorem to be true for the product of $n-1$ matrices, we can prove that it is true for the product of n matrices. Let us write

$$\mathrm{b} = \mathrm{a}_2\mathrm{a}_3 \cdots \mathrm{a}_n.$$

Then, from what we have assumed,

$$\mathrm{b}' = \mathrm{a}_n' \cdots \mathrm{a}_3' \, \mathrm{a}_2',$$

where we use accents to denote conjugates. Accordingly,

$$(\mathrm{a}_1\mathrm{a}_2 \cdots \mathrm{a}_n)' = (\mathrm{a}_1\mathrm{b})' = \mathrm{b}'\mathrm{a}_1' = \mathrm{a}_n' \cdots \mathrm{a}_2'\mathrm{a}_1',$$

and our theorem is proved.

In conclusion we lay down the following:

DEFINITION 2. *A square matrix is said to be singular if its determinant is zero.*

According to the convention made at the beginning of this section, it will be seen that *all* matrices which are not square are singular.

EXERCISES

1. DEFINITION. *A matrix* a *is called a divisor of zero if a matrix* b *different from zero exists such that either* ab = 0 *or* ba = 0.

Prove that every matrix one of whose rows or columns is composed wholly of zeros is a divisor of zero.

2. If it is possible to pass from a to b by means of an elementary transformation (cf. § 19, Definition 1), prove that there either exists a non-singular matrix c such that

$$\mathrm{ac} = \mathrm{b},$$

or a non-singular matrix d such that

$$\mathrm{da} = \mathrm{b}.$$

3. If all the elements of a matrix are real, and if the product of this matrix and its conjugate is zero, prove that the matrix itself is zero.

4. If the corresponding elements of two matrices a and b are conjugate imaginaries, and, b′ being the matrix conjugate to b, if

$$\mathbf{ab}'=0, \text{ then } \mathbf{a}=\mathbf{b}=0.$$

23. Linear Transformation. Before going farther with the theory of matrices we will take up, in this section and the next, the closely allied subject of linear transformation, which may be regarded as one of the most important applications of the theory of matrices.

In algebra and analysis we frequently have occasion to introduce, in place of the unknowns, or variables, we had originally to deal with, certain functions of these quantities which we regard as new unknowns or variables. Such a transformation, or change of variables, is particularly simple, and for many purposes particularly important, if the functions in question are homogeneous linear polynomials. It is then called a homogeneous linear transformation, or, as we shall say for brevity, simply a *linear transformation*. If $x_1, \cdots x_n$ are the original variables, and $x_1', \cdots x_n'$ the new ones, we have, as the formulæ for the transformation,

$$\begin{cases} x_1' = a_{11}x_1 + \cdots + a_{1n}x_n, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_n' = a_{n1}x_1 + \cdots + a_{nn}x_n. \end{cases}$$

The square matrix

$$\mathbf{a} = \begin{Vmatrix} a_{11} \cdots a_{1n} \\ \cdot \quad \cdot \quad \cdot \\ a_{n1} \cdots a_{nn} \end{Vmatrix}$$

is called the matrix of the transformation, and the determinant of this matrix, which we will represent by a, is called the determinant of the transformation. Inasmuch as the transformation is completely determined by its matrix, no confusion will arise if we speak of the transformation **a**.

In most cases where we have occasion to use a transformation it is important for us to be able, in the course of our work, to pass back to the original variables, and for this purpose it must be possible, not merely to express $x_1', \cdots x_n'$ as functions of $x_1, \cdots x_n$, but also to express $x_1, \cdots x_n$ as functions of $x_1', \cdots x_n'$. In the case of linear transformations this can in general be done. For the equations of the transformation may be regarded as non-homogeneous linear equations in $x_1, \cdots x_n$, and if the determinant a of the transformation is not zero, they can be solved and give

$$\mathbf{A} \begin{cases} x_1 = \dfrac{A_{11}}{a}x_1' + \cdots + \dfrac{A_{n1}}{a}x_n', \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_n = \dfrac{A_{1n}}{a}x_1' + \cdots + \dfrac{A_{nn}}{a}x_n', \end{cases}$$

where $A_{11}, \cdots A_{nn}$ are the cofactors of $a_{11}, \cdots a_{nn}$ in a.

This transformation **A** is called the *inverse* of the transformation **a**, but it must be remembered that it exists only if $a \neq 0$. A linear transformation for which $a=0$ is called a *singular transformation.* If **a** is non-singular, its inverse **A** is also non-singular, since the determinant of **A** is a^{-1} (cf. Corollary 2, § 11).

DEFINITION. *The special linear transformation*

$$x_1'=x_1,\ x_2'=x_2, \cdots\ x_n'=x_n,$$

whose matrix is

$$\mathbf{I} = \begin{Vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 \end{Vmatrix},$$

is called the identical transformation.

The determinant of this transformation is 1.

We turn now to the subject of the composition of linear transformations. If we introduce a new set of variables x' as functions of the original variables x, and then make a second transformation by introducing a third set of variables x'' as functions of the variables x', these two transformations can obviously be combined and the variables x'' expressed directly in terms of the x's. If the two transformations which we combine are linear transformations, it is readily seen that the resulting transformation will also be linear. The precise formulæ are important here, and for the sake of simplicity we will write them in the case of three variables, a case which will be seen to be perfectly typical of the general case.

Let

$$\mathbf{a}\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ x_2' = a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ x_3' = a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \end{cases} \qquad \mathbf{b}\begin{cases} x_1'' = b_{11}x_1' + b_{12}x_2' + b_{13}x_3', \\ x_2'' = b_{21}x_1' + b_{22}x_2' + b_{23}x_3', \\ x_3'' = b_{31}x_1' + b_{32}x_2' + b_{33}x_3', \end{cases}$$

be two linear transformations. Replacing the x''s in $\mathbf{b}$ by their values from $\mathbf{a}$, we get

$$\begin{cases} x_1'' = (a_{11}b_{11} + a_{21}b_{12} + a_{31}b_{13})x_1 \\ \quad + (a_{12}b_{11} + a_{22}b_{12} + a_{32}b_{13})x_2 \\ \quad + (a_{13}b_{11} + a_{23}b_{12} + a_{33}b_{13})x_3, \\ x_2'' = (a_{11}b_{21} + a_{21}b_{22} + a_{31}b_{23})x_1 \\ \quad + (a_{12}b_{21} + a_{22}b_{22} + a_{32}b_{23})x_2 \\ \quad + (a_{13}b_{21} + a_{23}b_{22} + a_{33}b_{23})x_3, \\ x_3'' = (a_{11}b_{31} + a_{21}b_{32} + a_{31}b_{33})x_1 \\ \quad + (a_{12}b_{31} + a_{22}b_{32} + a_{32}b_{33})x_2 \\ \quad + (a_{13}b_{31} + a_{23}b_{32} + a_{33}b_{33})x_3. \end{cases}$$

It will be seen that the matrix of this transformation is $\mathbf{ba}$. Hence,

THEOREM. *If we pass from the variables x to the variables x' by a linear transformation of matrix* $\mathbf{a}$, *and from the variables x' to the variables x'' by another linear transformation of matrix* $\mathbf{b}$, *then the linear transformation of matrix* $\mathbf{ba}$ *will carry us directly from the variables x to the variables x''.**

24. Collineation. We come now to an important geometrical application of the subject of linear transformation. For the sake of simplicity we begin with the case of three variables, which we will regard as the homogeneous coördinates of points in a plane.

The equations

$$(1) \qquad \begin{cases} x' = a_1x + b_1y + c_1t, \\ y' = a_2x + b_2y + c_2t, \\ t' = a_3x + b_3y + c_3t \end{cases}$$

* This result may be remembered conveniently by means of the following symbolic notation, which is often convenient. Let us denote the transformation $\mathbf{a}$ by the symbolic equation $x' = \mathbf{a}(x)$, and the transformation $\mathbf{b}$ by $x'' = \mathbf{b}(x')$. The result of combining these two transformations is then $x'' = \mathbf{b}(\mathbf{a}(x))$ or simply $x'' = \mathbf{ba}(x)$.

may be regarded as defining a transformation of the points of the plane; that is, if (x, y, t) is an arbitrarily given point, we can compute, by means of (1), the coördinates (x', y', t') of a second point into which we regard the first point as being transformed. The only exception is when the computed values of x', y', t' are all three zero, in which case there is no point into which the given point is transformed. This exceptional case can clearly occur only when the determinant of the transformation (1) is zero. Let us then confine our attention to non-singular linear transformations. In this case, not only does every point (x, y, t) correspond to a definite point (x', y', t'), but conversely, every point (x', y', t') corresponds to a definite point (x, y, t), since the transformation (1) now has an inverse

$$(2) \qquad \begin{cases} x = \dfrac{A_1}{D}x' + \dfrac{A_2}{D}y' + \dfrac{A_3}{D}t', \\ y = \dfrac{B_1}{D}x' + \dfrac{B_2}{D}y' + \dfrac{B_3}{D}t', \\ t = \dfrac{C_1}{D}x' + \dfrac{C_2}{D}y' + \dfrac{C_3}{D}t', \end{cases}$$

where D is the determinant of (1), and A_i, B_i, C_i are the cofactors in D.

The points (x, y, t) of the line

$$(3) \qquad \alpha x + \beta y + \gamma t = 0$$

are transformed by means of the non-singular transformation (1) into points of another line,

$$(4) \quad \frac{\alpha A_1 + \beta B_1 + \gamma C_1}{D}x' + \frac{\alpha A_2 + \beta B_2 + \gamma C_2}{D}y' + \frac{\alpha A_3 + \beta B_3 + \gamma C_3}{D}t' = 0,$$

as we see by using formulæ (2). Conversely every point of the line (4) corresponds, as we see by using (1), to a point on (3). That is, the transformation establishes a one-to-one correspondence between the points on the two lines (3) and (4), or, as we say, it transforms the line (3) into the line (4). On account of this property of transforming straight lines into straight lines, the transformation is called a *collineation*. The transformation is also known as a *projective transformation*, for it may be shown that it can be effected by projecting one plane on to another by means of straight lines radiating from a point in space.

What we have here said in the case of two dimensions applies with no essential change to three dimensions. The transformation

$$(5)\qquad \begin{cases} x' = a_1x + b_1y + c_1z + d_1t, \\ y' = a_2x + b_2y + c_2z + d_2t, \\ z' = a_3x + b_3y + c_3z + d_3t, \\ t' = a_4x + b_4y + c_4z + d_4t \end{cases}$$

gives us, provided its determinant is not zero, a one-to-one transformation of the points of space, which carries over planes into planes, and therefore also straight lines into straight lines, and is called a collineation or projective transformation of space. The same idea can be extended to spaces of higher dimensions.

Quite as important is the case of one dimension. The transformation

$$(6)\qquad \begin{cases} x' = a_1x + b_1t, \\ t' = a_2x + b_2t \end{cases}$$

gives us, provided its determinant is not zero, a one-to-one transformation of the points on a line. This we call a projective transformation of the line, the term *collineation* being in this case obviously inadequate.

It is possible, although for most purposes not desirable, to express the projective transformations (6), (1), (5) in one, two, and three dimensions in terms of non-homogeneous, instead of homogeneous coördinates. We thus get the formulæ

$$(7)\qquad X' = \frac{a_1X + b_1}{a_2X + b_2},$$

$$(8)\qquad \begin{cases} X' = \dfrac{a_1X + b_1Y + c_1}{a_3X + b_3Y + c_3}, \\ Y' = \dfrac{a_2X + b_2Y + c_2}{a_3X + b_3Y + c_3}, \end{cases}$$

$$(9)\qquad \begin{cases} X' = \dfrac{a_1X + b_1Y + c_1Z + d_1}{a_4X + b_4Y + c_4Z + d_4}, \\ Y' = \dfrac{a_2X + b_2Y + c_2Z + d_2}{a_4X + b_4Y + c_4Z + d_4}, \\ Z' = \dfrac{a_3X + b_3Y + c_3Z + d_3}{a_4X + b_4Y + c_4Z + d_4}. \end{cases}$$

These fractional forms may, in particular, be used to advantage in case their denominators reduce to mere constants. This special case, which is known as an *affine transformation*, may clearly be characterized by saying that all finite points go into finite points.*

* If we consider the still more special case in which the constant terms in the numerators of (8) and (9) are zero, that is, affine transformations in which the origin is transformed into itself, we see that our formulæ (8) and (9) have the form (6) and (1) respectively. Thus (6) may be regarded either as the general projective transformation of a line (if x, t are regarded as homogeneous coördinates) or as a special affine transformation of the plane (if x, t are regarded as non-homogeneous coördinates). Similarly (1) may be regarded either as the general projective transformation of a plane, or as a special affine transformation of space.

These affine transformations are of much importance in mechanics, where they are known as *homogeneous strains;* cf., for instance, Webster's *Dynamics* (Leipzig, Teubner), pp. 427–444.

Although we propose to leave the detailed discussion of singular transformations to the reader (see Exercise 1 at the end of this section), we will give one theorem concerning them.

THEOREM 1. *If the points $P_1, P_2, \cdots$ are carried over by a singular projective transformation into the points $P_1', P_2', \cdots$, then, if our transformation is in one dimension, the points P' will all coincide; if in two dimensions, they will all be collinear; if in three dimensions, they will all be complanar, etc.*

Suppose, for instance, that we have to deal with two dimensions. Since the determinant of the collineation (1) is supposed to be zero, the three polynomials in the second members of (1) are linearly dependent; that is, there exist three constants, k_1, k_2, k_3, not all zero, and such that for all values of x, y, t,

$$(10)\qquad k_1x' + k_2y' + k_3t' = 0.$$

Accordingly all points (x', y', t') obtained by this transformation lie on the line (10).

Similar proofs apply to the cases of one dimension and of three or more dimensions.

THEOREM 2. *Any three distinct points on a line may be carried over respectively into any three distinct points on the line by one, and only one, projective transformation.*

Let the three initial points be P_1, P_2, P_3, with homogeneous coördinates (x_1, t_1), (x_2, t_2), (x_3, t_3) respectively, and let the points into which we wish them transformed be P_1', P_2', P_3' with coördinates (x_1', t_1'), (x_2', t_2'), (x_3', t_3'). The projective transformation

$$x' = \alpha x + \beta t,$$
$$t' = \gamma x + \delta t$$

carries over any given point (x, t) into a point (x', t') whose position depends on the values of the constants $\alpha, \beta, \gamma, \delta$. Our theorem is true if it is possible to find one, and, except for a constant factor only one, set of seven constants — four, $\alpha, \beta, \gamma, \delta$, and three others, ρ_1, ρ_2, ρ_3, none of which is zero — which satisfy the six equations

$$\begin{cases}\rho_1 x_1' = \alpha x_1 + \beta t_1, \\ \rho_1 t_1' = \gamma x_1 + \delta t_1,\end{cases} \quad \begin{cases}\rho_2 x_2' = \alpha x_2 + \beta t_2, \\ \rho_2 t_2' = \gamma x_2 + \delta t_2,\end{cases} \quad \begin{cases}\rho_3 x_3' = \alpha x_3 + \beta t_3, \\ \rho_3 t_3' = \gamma x_3 + \delta t_3.\end{cases}$$

Since the x's and t's are all known, we have here six homogeneous linear equations in seven unknowns. Hence there are always solutions other than zeros, the number of independent ones depending on the rank of the matrix of the coefficients. Transposing and rearranging the equations, we have

$$\begin{aligned}
x_1\alpha + t_1\beta \qquad\qquad - x_1'\rho_1 \qquad\qquad &= 0,\\
x_1\gamma + t_1\delta - t_1'\rho_1 \qquad\qquad &= 0,\\
x_2\alpha + t_2\beta \qquad\qquad - x_2'\rho_2 \qquad &= 0,\\
x_2\gamma + t_2\delta \quad - t_2'\rho_2 \qquad &= 0,\\
x_3\alpha + t_3\beta \qquad\qquad\qquad - x_3'\rho_3 &= 0,\\
x_3\gamma + t_3\delta \qquad\qquad - t_3'\rho_3 &= 0.
\end{aligned}$$

The matrix of these equations is of rank six. For consider the determinant of the first six columns with its sign reversed,

$$D = \begin{vmatrix} x_1 & t_1 & 0 & 0 & x_1' & 0 \\ x_2 & t_2 & 0 & 0 & 0 & x_2' \\ 0 & 0 & x_1 & t_1 & t_1' & 0 \\ 0 & 0 & x_2 & t_2 & 0 & t_2' \\ x_3 & t_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & t_3 & 0 & 0 \end{vmatrix}.$$

Since P_1, P_2, P_3 are distinct, there exist two constants c_1, c_2 neither of which is zero, such that

$$c_1x_1 + c_2x_2 + x_3 = 0,$$
$$c_1t_1 + c_2t_2 + t_3 = 0.$$

Hence, adding to the fifth row of D c_1 times the first row and c_2 times the second, and to the sixth row c_1 times the third row and c_2 times the fourth, we have

$$D = \begin{vmatrix} x_1 & t_1 & 0 & 0 & x_1' & 0 \\ x_2 & t_2 & 0 & 0 & 0 & x_2' \\ 0 & 0 & x_1 & t_1 & t_1' & 0 \\ 0 & 0 & x_2 & t_2 & 0 & t_2' \\ 0 & 0 & 0 & 0 & c_1x_1' & c_2x_2' \\ 0 & 0 & 0 & 0 & c_1t_1' & c_2t_2' \end{vmatrix} = c_1c_2 \begin{vmatrix} x_1 & t_1 \\ x_2 & t_2 \end{vmatrix}^2 \cdot \begin{vmatrix} x_1' & x_2' \\ t_1' & t_2' \end{vmatrix},$$

and this is not zero, since P_1' and P_2' are distinct as well as P_1 and P_2.

In the same way we see that the determinants obtained by striking out the sixth and the fifth columns respectively of the matrix are not zero. Accordingly, by Theorem 4, § 17, we see that the equations have a solution in which none of the quantities ρ_1, ρ_2, ρ_3 are zero, and that every solution is proportional to this one. All these solutions obviously yield the same projective transformation of the line.

COROLLARY. *The transformation just determined is non-singular.*

This follows, by a reference to Theorem 1, from the fact that it does not carry P_1, P_2, P_3 into a single point.

EXERCISES

1. Discuss singular projective transformations in one, two, and three dimensions; noting, in particular, the effect of the rank of the matrix of the transformation, first, on the distribution of the points which have no corresponding points after the transformation, and secondly, on the distribution of the points into which no points are carried over by the transformation.

2. Prove that any four complanar points no three of which are collinear may be carried over into any four points in the plane, no three of which are collinear, by one and only one collineation.

3. State and prove the corresponding theorem in n dimensions.

4. Prove that the transformation from a first system of homogeneous coördinates to a second is effected by a non-singular linear transformation. Consider the case of one, two, and three dimensions.