CHAPTER V

SOME THEOREMS CONCERNING THE RANK OF A MATRIX

19. General Matrices. In order to show that a given matrix is of rank r, we have first to show that at least one r-rowed determinant of the matrix is not zero, and secondly that all $(r+1)$-rowed determinants are zero. This latter work may be considerably shortened by the following theorem:

THEOREM 1. *If in a given matrix a certain r-rowed determinant is not zero, and all the $(r+1)$-rowed determinants of which this r-rowed determinant is a first minor are zero, then all the $(r+1)$-rowed determinants of the matrix are zero.*

We will assume, as we may do without loss of generality, that the non-vanishing r-rowed determinant stands in the upper left-hand corner of the matrix. Let the matrix be

$$\begin{Vmatrix} a_{11} \cdots a_{1n} \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ a_{m1} \cdots a_{mn} \end{Vmatrix},$$

and consider the $r+1$ sets of n quantities each which lie in the first $r+1$ rows of this matrix. These $r+1$ sets of quantities are linearly dependent, as will be seen by reference to the proof of Theorem 1, §13, for although we knew there that *all* the $(r+1)$-rowed determinants were zero, we made use of this fact only for those $(r+1)$-rowed determinants which we now assume to be zero. Moreover, since the r sets of constants which stand in the first r rows of our matrix are linearly independent, it follows that the $(r+1)$th row is linearly dependent on the first r. Precisely the same reasoning shows that each of the subsequent rows is linearly dependent on the first r rows. Accordingly, by Theorem 5, §17, any $r+1$ rows are linearly dependent; and therefore, by Theorem 1, §13, all the $(r+1)$-rowed determinants of our matrix are zero, as was to be proved.

Still another method of facilitating the determination of the rank of a matrix is by changing the form of the matrix in certain ways which do not change its rank. In order to explain this method, we begin by laying down the following definition:

DEFINITION 1. *By an elementary transformation of a matrix we understand a transformation of any one of the following forms:*

(*a*) *the interchange of two rows or of two columns;*

(*b*) *the multiplication of each element of a row (or column) by the same constant not zero;*

(*c*) *the addition to the elements of one row (or column) of the products of the corresponding elements of another row (or column) by one and the same constant.*

It is clear that if we can pass from a matrix **a** to a matrix **b** by one of these transformations, we can pass back from **b** to **a** by an elementary transformation.

DEFINITION 2. *Two matrices are said to be equivalent if it is possible to pass from one to the other by a finite number of elementary transformations.*

THEOREM 2. *If two matrices are equivalent, they have the same rank.*

It is evident that the transformations (*a*) and (*b*) of Definition 1 do not change the rank of a matrix, since they do not affect the vanishing or non-vanishing of any determinant of the matrix. In order to prove our theorem, it is therefore sufficient to prove that the rank of a matrix is not changed by a transformation (*c*).

Suppose this transformation consists in adding to the elements of the pth row of a matrix **a** k times the elements of the qth row, thus giving the matrix **b**. Let r be the rank of the matrix **a**. We will first show that this rank cannot be increased by the transformation, that is, that all $(r+1)$-rowed determinants of the matrix **b** are zero. By hypothesis all the $(r+1)$-rowed determinants of the matrix **a** are zero, and some of these determinants are clearly not changed by the transformation, namely, those which do not contain the pth row, or which contain both the pth and the qth row. The other determinants, which contain the pth row but not the qth, take on after the transformation the form $A \pm kB$ where A and B are $(r+1)$-rowed determinants of **a**, and are therefore zero. Thus we see that the transformation (*c*) never increases the rank of a matrix.

Moreover, the rank of b cannot be less than that of a, for then the transformation (c) which carries b into a would increase the rank of b, and this we have just seen is impossible.

This theorem can often be used to advantage in determining the rank of a matrix, for by means of elementary transformations it is often easy to simplify the matrix very materially.

EXERCISES

Determine the ranks of the following matrices:

1.
$$\begin{Vmatrix} 14 & 12 & 6 & 8 & 2 \\ 6 & 104 & 21 & 9 & 17 \\ 7 & 6 & 3 & 4 & 1 \\ 35 & 30 & 15 & 20 & 5 \end{Vmatrix}$$

2.
$$\begin{Vmatrix} 75 & 0 & 116 & -39 & 0 \\ 171 & -69 & 402 & 123 & 45 \\ 301 & 0 & 87 & -417 & -169 \\ 114 & -46 & 268 & 82 & 30 \end{Vmatrix}$$

3. Prove that any matrix of rank r can be reduced by means of elementary transformations to a form where the element in the ith row and ith column is 1 when $i \leqq r$, while all the other elements of the matrix are zero.

4. Hence prove that two matrices with m rows and n columns each are always equivalent when they have the same rank.

5. Prove that a necessary and sufficient condition that the matrix

$$\begin{Vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} \end{Vmatrix}$$

be of rank 0 or 1 is that there exist $m+n$ constants $\alpha_1, \cdots \alpha_m, \beta_1, \cdots \beta_n$ such that $a_{ij} = \alpha_i \beta_j$.

20. Symmetrical Matrices.

DEFINITION. *The square matrix*

$$\mathbf{a} = \begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & & a_{nn} \end{Vmatrix}$$

(and also its determinant) is said to be symmetrical if the pairs of terms which are situated symmetrically with respect to the principal diagonal are equal. That is, if $a_{ij} = a_{ji}$.

We will denote by M_i an i-rowed principal minor of a. It is our main object in this section to show how the rank of the symmetrical matrix may be determined by an examination of the principal minors only. This may be done by means of the following three theorems.

THEOREM 1. *If an r-rowed principal minor M_r of the symmetrical matrix* a *is not zero, while all the principal minors obtained by adding one row and the same column, and also all those obtained by adding two rows and the same two columns, to M_r are zero, then the rank of* a *is r.*

Let the non-vanishing minor be the one which stands in the upper left-hand corner of a, and let $B_{\alpha\beta}$ denote the determinant obtained by adding the αth row and the βth column to M_r. If we can show that $B_{\alpha\beta} = 0$ for all unequal values of α and β our theorem will be proved. Cf. Theorem 1, § 19. Give to the integers α and β any two unequal values, and let C denote the determinant obtained by adding to M_r the αth and βth rows and the αth and βth columns of a. Then we have, by hypothesis, $M_r \neq 0$, $B_{\alpha\alpha} = 0$, $B_{\beta\beta} = 0$, $C = 0$. Let M_2' be the two-rowed principal minor of the adjoint of C which corresponds to the complement of M_r in C. Then by Corollary 3, § 11, we have

$$M_2' = CM_r = 0.$$

But $\quad M_2' = B_{\alpha\alpha}B_{\beta\beta} - B_{\alpha\beta}^2.$

Therefore $\quad B_{\alpha\beta} = 0.$

THEOREM 2. *If all the $(r+1)$-rowed principal minors of the symmetrical matrix* a *are zero, and also all the $(r+2)$-rowed principal minors, then the rank of* a *is r or less.*

If $r = 0$, all the elements in the principal diagonal are zero and all the two-rowed principal minors are zero.

That is, $\quad a_{ii} \cdot a_{jj} - a_{ij}^2 = 0,$

and therefore, since $a_{ii} = a_{jj} = 0$, $a_{ij} = 0$. That is, every element is zero and hence the rank is zero, and the theorem is true in this special case.

Now, assume it true when $r = k$; that is, we assume that when all $(k+1)$-rowed principal minors are zero and all $(k+2)$-rowed principal minors are zero, the rank of a is less than $k+1$. Then it follows that when all $(k+2)$-rowed, and all $(k+3)$-rowed principal minors are zero.

the rank of a is less than $k+2$. For in this case, if all $(k+1)$-rowed principal minors are zero, the rank is less than $k+1$, by hypothesis, and if some $(k+1)$-rowed principal minor is *not* zero, the rank is exactly $k+1$, by the last theorem. We see then that if the theorem is true for $r=k$ it is true for $r=k+1$. But we have proved it true for $r=0$, hence it is true for all values of r.

THEOREM 3. *If the rank of the symmetrical matrix* a *is* $r>0$, *there is at least one r-rowed principal minor of* a *which is not zero.*

For all $(r+1)$-rowed principal minors are zero, and, if all r-rowed principal minors were zero also, the rank of a would be $r-1$ or less, by the last theorem.

We close with a theorem of a somewhat special character which will be found useful later (cf. Exercises 4–6, § 50).

THEOREM 4. *If the rank of the symmetrical matrix* a *is* $r>0$, *we may shift the rows (at the same time shifting the columns in the same way, thus keeping* a *symmetrical) in such a way that no consecutive two of the set of quantities*

$$M_0, M_1, M_2, \dots M_r$$

shall be zero and $M_r \neq 0$; M_0 *being unity, and the other M's being the principal minors of* a *of orders indicated by their subscripts, which stand in the upper left-hand corner of* a *after the shifting.*

By definition we have $M_0 \neq 0$. Leaving aside for the moment the special case in which all the elements of the principal diagonal are zero, let us suppose the element a_{ii} is not zero. Then by shifting the ith row and column to the first place, we have $M_1 \neq 0$. We have thus fixed the first row and column, but we are still at liberty to shift all the others. Now consider the two-rowed principal minor obtained by adding to M_1 one row and the same column. Leaving aside still the special case in which these are all zero, let us suppose that the two-rowed determinant obtained by striking out all the rows and columns except those numbered 1 and i_1 is not zero. Then, by shifting the i_1th row and column into the second place, we have $M_2 \neq 0$. We next have to consider the three-rowed principal minors of which M_2 is a first minor. We can evidently proceed in this way until we have so shifted our rows and columns that none of the quantities $M_0, M_1, \dots M_r$ are zero, unless at a certain stage we find that all the principal minors of a certain order which we have to consider are zero. In this case we should have so shifted our first k rows and columns that none of the quantities $M_0, M_1, \dots M_k$ are zero. but we

should then find that all $(k+1)$-rowed principal minors of which M_k is a first minor vanish, so that, however we may shift the last $n-k$ rows and columns, we have $M_{k+1}=0$. Let us then examine the $(k+2)$-rowed principal minors of which M_k is a second minor.* These can (by Theorem 1) not all be zero as otherwise the rank of a would be $k<r$. That is, if $M_{k+1}=0$, we can so arrange the rows and columns that $M_{k+2} \neq 0$. Thus we see that the rows and columns of a may be so shifted that no consecutive two of the M's are zero. Now, if $M_{r-1}=0$, the above proof shows that we can make $M_r \neq 0$. But even though $M_{r-1} \neq 0$ we can still make $M_r \neq 0$, for by hypothesis † all the determinants obtained by adding to M_{r-1} two rows and the same two columns vanish, and if all those obtained by adding one row and the same column were zero also, the rank of a would be $r-1$, by Theorem 1.

A symmetrical matrix is said to be arranged in *normal form* when no consecutive two of the M's of Theorem 4 are zero and $M_r \neq 0$.

EXERCISES

1. Determine the ranks of the following matrices:

$$\left\|\begin{matrix} 2 & 1 & 11 & 2 \\ 1 & 0 & 4 & -1 \\ 11 & 4 & 56 & 5 \\ 2 & -1 & 5 & -6 \end{matrix}\right\|, \quad \left\|\begin{matrix} 0 & 4 & 10 & 1 \\ 4 & 8 & 18 & 7 \\ 10 & 18 & 40 & 17 \\ 1 & 7 & 17 & 3 \end{matrix}\right\|,$$

$$\left\|\begin{matrix} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 & 6 \\ 1 & 2 & 3 & 14 & 32 \\ 4 & 5 & 6 & 32 & 77 \end{matrix}\right\|, \quad \left\|\begin{matrix} 0 & 1 & b & d \\ 1 & 0 & c & e \\ b & c & 2bc & cd+be \\ d & e & cd+be & 2de \end{matrix}\right\|.$$

2. By a skew-symmetric determinant, or matrix, is meant one in which $a_{ij} = -a_{ji}$ (and therefore $a_{ii}=0$).

Establish for such matrices theorems similar to Theorems 1, 2, 3 of this section.

3. By considering the effect of changing rows into columns, prove that a skew-symmetric determinant of odd order is always zero.

4. Prove that the rank of a skew-symmetric matrix is always even.

* The tacit assumption is here made that when $k=r-1$, $r<n$, as otherwise M_{k+2} would have no meaning. The case $r=n$ can, however, obviously not occur here, for then we should have $M_{k+1}=a \neq 0$.

† Here again we assume that $r<n$, for if $r=n$, $M_r = a \neq 0$