linearly dependent. If then we regard the independent variables as rectangular coördinates, these equations give us geometric loci in space of as many dimensions as there are independent variables. Thus, in the cases of two and three variables, we have plane curves and surfaces respectively. The case of two loci is of no interest, as they must coincide in order to be linearly dependent. In the case of three linearly dependent loci it is easily shown that any one must meet the other two in all their common points and in no others. The following theorems will serve to illustrate the geometric meaning of linear dependence:

(1) In the plane:

THEOREM 8. *Three circles are linearly dependent when, and only when, they belong to the same coaxial family.*

THEOREM 9. *Four circles are linearly dependent when, and only when, they have a (real or imaginary) common orthogonal circle.*

THEOREM 10. *Four circles are linearly dependent when, and only when, the points of intersection of the first and second, and the points of intersection of the third and fourth, lie on a common circle.*

THEOREM 11. *Five or more circles are always linearly dependent.*

(2) In space (using homogeneous coördinates):

THEOREM 12. *Three planes are linearly dependent when, and only when, they intersect in a line.*

THEOREM 13. *Four planes are linearly dependent when, and only when, they intersect in a point.*

THEOREM 14. *Five or more planes are always linearly dependent.*

CHAPTER IV

LINEAR EQUATIONS

16. Non-homogeneous Linear Equations. In every elementary treatment of determinants, however brief, it is explained how to solve by determinants a system of n equations of the first degree in n unknowns, provided that the determinant of the coefficients of the unknowns is not zero. Cramer's Rule, by which this is done, is this:

CRAMER'S RULE. *If in the equations*

$$\begin{array}{c} a_{11}x_1 + \cdots + a_{1n}x_n = k_1, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1}x_1 + \cdots + a_{nn}x_n = k_n, \end{array}$$

the determinant

$$a = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

is not zero, the equations have one and only one solution, namely:

$$x_1 = \frac{a_1}{a},\ x_2 = \frac{a_2}{a},\ \cdots\ x_n = \frac{a_n}{a},$$

where a_i is the n-rowed determinant obtained from a by replacing the elements of the ith column by the elements $k_1, k_2, \cdots k_n$.

This rule, whose proof we assume to be known,* is of fundamental importance in the general theory of linear equations to which we now proceed.

* The proof as given in most English and American text-books merely establishes the fact that *if the equations have a solution* it is given by Cramer's formulæ. That these formulæ really satisfy the equations in all cases is not commonly proved, but may be easily estabished by direct substitution. We leave it for the reader to do this.

Consider the system of m linear equations in n variables:

$$a_{11}x_1 + \cdots + a_{1n}x_n + b_1 = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n + b_m = 0,$$

where m and n may be any positive integers. Three cases arise:

(1) The equations may have no solution, in which case they are said to be *inconsistent.*

(2) They may have just one solution.

(3) They may have more than one solution, in which case it will presently appear that they necessarily have an infinite number of solutions.

Let us consider the two matrices:

$$\mathbf{a} = \begin{Vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} \end{Vmatrix}, \qquad \mathbf{b} = \begin{Vmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{Vmatrix}.$$

We will call $\mathbf{a}$ the matrix of the system of equations, $\mathbf{b}$ the *augmented matrix.*

It is evident that the rank of the matrix $\mathbf{a}$ cannot be greater than that of the matrix $\mathbf{b}$, since every determinant contained in $\mathbf{a}$ is also contained in $\mathbf{b}$. We have, then, two cases:

I. Rank of $\mathbf{a}$ = Rank of $\mathbf{b}$.

II. Rank of $\mathbf{a}$ < Rank of $\mathbf{b}$.

We will consider Case II first.

Let r be the rank of $\mathbf{b}$. Then $\mathbf{b}$ must contain at least one r-rowed determinant which is not zero. Moreover, this determinant must contain a column of b's, since otherwise it would be contained in $\mathbf{a}$ also, which is contrary to our hypothesis. Suppose for definiteness that this non-vanishing r-rowed determinant is the one situated in the upper right-hand corner of $\mathbf{b}$. There is no loss of generality in assuming this, since by writing the equations in a different order and changing the order of the variables $x_1, \cdots x_n$ we can always bring the determinant into this position. Now for brevity let us represent the polynomials forming the first members of our given equations by $F_1, F_2, \cdots F_m$ respectively, and the homogeneous polynomials obtained by omitting the constant terms in each of these equations by $f_1, f_2, \cdots f_m$. Then we have the identities:

$$F_i \equiv f_i + b_i, \qquad (i = 1, 2, \ldots m).$$

Consider the first r of these identities. Since the rank of $\mathbf{a}$ is less than r, the polynomials $f_1, f_2, \cdots f_r$ are linearly dependent,

$$c_1 f_1 + c_2 f_2 + \cdots + c_r f_r \equiv 0,$$

hence $$c_1 F_1 + \cdots + c_r F_r \equiv c_1 b_1 + \cdots + c_r b_r = C.$$

But since the rank of $\mathbf{b}$ is r, the polynomials $F_1, \cdots F_r$ are linearly independent and therefore $C \neq 0$. Hence *the given equations are inconsistent*, for if they were consistent all the F's would be zero for some suitably chosen values of $x_1, \cdots x_n$, and if we substitute these values in the last written identity we should have

$$0 = C \neq 0.$$

Let us now consider Case I. Let r be the common rank of $\mathbf{a}$ and $\mathbf{b}$, then there is at least one r-rowed determinant in $\mathbf{a}$ which is not zero. This same determinant also occurs in $\mathbf{b}$. Suppose it to be situated in the upper left-hand corner of each matrix. Since all $(r+1)$-rowed determinants of either matrix are zero, the first $(r+1)$ of the F's are linearly dependent, and we have

$$c_1 F_1 + c_2 F_2 + \cdots + c_r F_r + c_{r+1} F_{r+1} \equiv 0\,;$$

and, since $F_1, \cdots F_r$ are linearly independent, c_{r+1} cannot be zero; hence we may divide through by it and express F_{r+1} linearly in terms of $F_1, \cdots F_r$. The same argument holds if instead of F_{r+1} we take F_{r+2} or any other one of the remaining F's. Hence

$$F_{r+l} \equiv k_1^{[l]} F_1 + \cdots + k_r^{[l]} F_r \qquad (l = 1, 2, \cdots m - r).$$

From these identities it is obvious that at any point $(x_1, \cdots x_n)$ where $F_1, \cdots F_r$ all vanish, the remaining F's also vanish. In other words, any solution which the first r equations of the given system may have is necessarily a solution of the whole system.

Now consider the first r of the given equations. Assign to $x_{r+1}, \cdots x_n$ any fixed values $x'_{r+1}, \cdots x'_n$, and transpose all the terms after the rth in each equation to the second member,

$$a_{11}x_1 + \cdots + a_{1r}x_r = -a_{1,r+1}x'_{r+1} - \cdots - a_{1n}x'_n - b_1,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a_{r1}x_1 + \cdots + a_{rr}x_r = -a_{r,r+1}x'_{r+1} - \cdots - a_{rn}x'_n - b_r.$$

Remembering that the right-hand sides of these equations are known constants, and that the determinant of the coefficients on the left is not zero, we see that we have the case to which Cramer's Rule applies, and that this system of equations has therefore just one solution. Hence the given system of equations is consistent, and we have the theorem:

THEOREM 1. *A necessary and sufficient condition for a system of linear equations to be consistent is that the matrix of the system have the same rank as the augmented matrix.*

From the foregoing considerations we have also

THEOREM 2. *If in a system of linear equations the matrix of the system and the augmented matrix have the same rank r, the values of $n-r$ of the unknowns may be assigned at pleasure and the others will then be uniquely determined.*

The $n-r$ unknowns whose values may be assigned at pleasure may be chosen in any way provided that the matrix of the coefficients of the remaining unknowns is of rank r.

EXERCISES

Solve completely the following systems of equations:

1. $$\begin{cases} 2x - y + 3z - 1 = 0, \\ 4x - 2y - z + 3 = 0, \\ 2x - y - 4z + 4 = 0, \\ 10x - 5y - 6z + 10 = 0. \end{cases}$$

2. $$\begin{cases} 4x - y + z + 5 = 0, \\ 2x - 3y + 5z + 1 = 0, \\ x + y - 2z + 2 = 0, \\ 5x \quad\;\; - z + 2 = 0. \end{cases}$$

3. $$\begin{cases} 2x - 3y + 4z - w = 3, \\ x + 2y - z + 2w = 1, \\ 3x - y + 2z - 3w = 4, \\ 3x - y + z - 7w = 4. \end{cases}$$

17. Homogeneous Linear Equations. We will now consider the special case where the equations of the last section are *homogeneous*, *i.e.* where all the b's are zero,

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0, \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

The matrices $\mathfrak{a}$ and $\mathfrak{b}$ of the last section differ here only by a column of zeros; hence they always have the same rank and this is called *the rank of the system of equations*. Theorems 1 and 2 of the last section become

THEOREM 1. *A system of homogeneous linear equations always has one or more solutions.*

THEOREM 2. *If the rank of a system of homogeneous linear equations in n variables is r, the values of $n-r$ of the unknowns may be assigned at pleasure and the others will then be uniquely determined.**

If the rank of the equations is n, there will therefore be only one solution, and this solution is obviously $x_1 = x_2 = \cdots = x_n = 0$. Since the rank can never be greater than n, we have

THEOREM 3. *A necessary and sufficient condition for a system of homogeneous linear equations in the n variables $(x_1, \cdots x_n)$ to have a solution other than $x_1 = x_2 = \cdots = x_n = 0$ is that their rank be less than n.*

COROLLARY 1. *If there are fewer equations than unknowns, the equations always have solutions other than $x_1 = x_2 = \cdots = x_n = 0$.*

COROLLARY 2. *If the number of equations is equal to the number of unknowns, a necessary and sufficient condition for solutions other than $x_1 = x_2 = \cdots = x_n = 0$ is that the determinant of the coefficients be zero.*

In the special case where the number of equations is just one less than the number of unknowns and the equations are linearly independent, we will prove the following:

THEOREM 4. *Every set of values of $x_1, \cdots x_n$ which satisfies a system of $n-1$ linearly independent,† homogeneous linear equations in*

* Cf. also the closing lines of Theorem 2, § 16.

† The theorem is still true if the equations are linearly dependent, but it is then trivial, since the determinants in question are all zero.

n unknowns is proportional to the set of $(n-1)$-rowed determinants taken alternately with plus and minus signs, and obtained by striking out from the matrix of the coefficients first the first column, then the second, etc.

Let us denote by a_i the $(n-1)$-rowed determinant obtained by striking out the ith column from the matrix of the equations. Since the equations are linearly independent, there must be at least one of the determinants $a_1, a_2, \cdots a_n$ which is not zero. Let it be a_i. Now assign to x_i any fixed value, c, and transpose the ith term of each equation to the second member and we have

$$\begin{array}{l} a_{11}x_1 + \cdots + a_{1,i-1}x_{i-1} + a_{1,i+1}x_{i+1} + \cdots + a_{1n}x_n = -a_{1i}c, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{n-1,1}x_1 + \cdots + a_{n-1,i-1}x_{i-1} + a_{n-1,i+1}x_{i+1} + \cdots + a_{n-1,n}x_n = -a_{n-1,i}c. \end{array}$$

Hence: $$x_k = \frac{(-1)^{i-k}\cdot c \cdot a_k}{a_i} \qquad (k=1, 2, \cdots n),$$

from which it is clear that $(x_1, \cdots x_n)$ are proportional to the determinants $(a_1, -a_2, a_3, \cdots (-1)^{n-1}a_n)$, as was to be proved.

The theory of homogeneous linear equations has here been deduced from the theory of linear dependence. It can, however, in turn be used to obtain further results in this last-mentioned theory. As an example of this we will deduce the following theorem, which we shall find useful later:

THEOREM 5. *If a set of points $(x_1, \cdots x_n)$, finite or infinite in number, have the property that k points can be found among them upon which every other point of the set is linearly dependent, then any $k+1$ points of the set will be linearly dependent.*

Let $(x_1', \cdots x_n')$, $(x_1'', \cdots x_n'')$, $\cdots (x_1^{[k]}, \cdots x_n^{[k]})$ be the k points upon which every other point of the set is linearly dependent, and let

$$(X_1', \cdots X_n'), (X_1'', \cdots X_n''), \cdots (X_1^{[k+1]}, \cdots X_n^{[k+1]})$$

be any $k+1$ points of the set. Then we may write

$$(1) \qquad \begin{cases} X_1^{[i]} = c_1^{[i]}x_1' + c_2^{[i]}x_1'' + \cdots + c_k^{[i]}x_1^{[k]}, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ X_n^{[i]} = c_1^{[i]}x_n' + c_2^{[i]}x_n'' + \cdots + c_k^{[i]}x_n^{[k]}. \end{cases} \qquad (i=1, 2, \cdots k+1)$$

This is true by hypothesis if $(X_1^{[i]}, \cdots X_n^{[i]})$ is not one of the first k points, and if it is one of these points, it is obviously true. We have then to prove that $k+1$ constants, $C_1, C_2, \cdots C_{k+1}$, not all zero, can be found such that

$$C_1X_j' + C_2X_j'' + \cdots + C_{k+1}X_j^{[k+1]} = 0 \qquad (j=1, 2, \cdots n).$$

By substituting here the values of the X's from (1), we see that these equations will be fulfilled if

$$\begin{array}{l} C_1c_1' + C_2c_1'' + \cdots + C_{k+1}c_1^{[k+1]} = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ C_1c_k' + C_2c_k'' + \cdots + C_{k+1}c_k^{[k+1]} = 0, \end{array}$$

and this is a system of fewer equations than unknowns, which is therefore satisfied by a set of C's not all zero. (Cf. Theorem 3, Cor. 1.)

EXERCISES

Solve completely the following systems of equations:

$$\textbf{1.} \begin{cases} 11x + 8y - 2z + 3w = 0, \\ 2x + 3y - z + 2w = 0, \\ 7x - y + z - 3w = 0, \\ 4x - 11y + 5z - 12w = 0. \end{cases}$$

$$\textbf{2.} \begin{cases} 2x - 3y + 5z + 3w = 0, \\ 4x - y + z + w = 0, \\ 3x - 2y + 3z + 4w = 0. \end{cases}$$

18. Fundamental Systems of Solutions of Homogeneous Linear Equations. If $(x_1', \cdots x_n')$ is a solution of the system of equations

$$(1) \qquad \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0, \end{cases}$$

then $(cx_1', \cdots cx_n')$ is also a solution, and by giving to c different values we get thus (except in the special case in which the x''s are all zero) an infinite number of solutions. These may include all the solutions of (1) (cf. Theorem 4 of the last section), but in general this will not be the case.

Suppose, again, that $(x_1', \cdots x_n')$ and $(x_1'', \cdots x_n'')$ are two solutions of (1), then $(c_1x_1' + c_2x_1'', \cdots c_1x_n' + c_2x_n'')$ is also a solution. If the two given solutions are proportional to each other, this clearly gives us nothing more than what we had above by starting from a single solution; but if these two solutions are linearly independent, we build up from them, by allowing c_1 and c_2 to take on all values, a doubly infinite system of solutions; but even this system will usually not include all the solutions of (1). Similarly we see that, if we can find three linearly independent solutions, we can build up from them a triply infinite system of solutions, etc. If, proceeding in this way, we succeed in finding a finite number of linearly independent solutions in terms of which all solutions can be expressed, this finite number of solutions is said to form a *fundamental system.*

DEFINITION. *If* $(x_1^{[i]}, \cdots x_n^{[i]})$ $(i = 1, 2, \cdots k)$ *are a system of* k *solutions of* (1) *which satisfy the following two conditions, they are said to form a fundamental system:*

(a) *They shall be linearly independent.*

(b) *Every solution of* (1) *shall be expressible in the form*

$$(c_1x_1' + c_2x_1'' + \cdots + c_kx_1^{[k]}, \cdots\cdots c_1x_n' + c_2x_n'' + \cdots + c_kx_n^{[k]}).$$

THEOREM 1. *If the equations* (1) *are of rank* $r < n$, *they possess an infinite number of fundamental systems each of which consists of* $n - r$ *solutions.*

Suppose the r-rowed determinant which stands in the upper left-hand corner of the matrix of the equations (1) does not vanish, and let us consider the first r of these equations. Any solution of these will be a solution of all the others. Transpose all terms after the rth to the second members, and let $(x_{r+1}, \cdots x_n)$ have any fixed set of values $(x_{r+1}', \cdots x_n')$, not all zero; then these r equations will have just one solution given by Cramer's Rule. Call it $(x_1', \cdots x_n')$. Now let $(x_{r+1}, \cdots x_n)$ have any *other* fixed set of values $(x_{r+1}'', \cdots x_n'')$, not all zero, and we get another solution, $(x_1'', \cdots x_n'')$. Continue in this way until we have $n-r$ solutions

$$\begin{array}{l} x_1', \cdots x_r', x_{r+1}', \cdots x_n', \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_1^{[n-r]}, \cdots x_r^{[n-r]}, x_{r+1}^{[n-r]}, \cdots x_n^{[n-r]}. \end{array}$$

If we have chosen these $n-r$ sets of values for $(x_{r+1}, \cdots x_n)$ so that the determinant

$$(2) \qquad \begin{vmatrix} x_{r+1}' & \cdots & x_n' \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ x_{r+1}^{[n-r]} & \cdots & x_n^{[n-r]} \end{vmatrix}$$

is not zero, — and this may clearly be done in an infinite variety of ways, — these $n-r$ solutions will be linearly independent. That is to say, we may thus obtain an infinite number of sets of $n-r$ solutions each, each of which satisfies condition (a) of our definition for a fundamental system.

To prove that these sets of solutions also satisfy condition (b), let us suppose that $(X_1, \cdots X_n)$ is any solution of the r equations we are considering. The last $n-r$ of these X's are linearly dependent on the $n-r$ sets of values we have chosen for $(x_{r+1}, \cdots x_n)$ since we have here more sets of constants than there are elements in each set (cf. Theorem 2, §13), and the determinant (2) is not zero. Thus

$$(3) \qquad X_i = c_1x_i' + c_2x_i'' + \cdots + c_{n-r}x_i^{[n-r]} \qquad (i = r+1, r+2, \cdots n).$$

Let us now solve the first r equations (1) by Cramer's Rule, regarding $x_{r+1}, \cdots x_n$ as known. We thus get results of the form

$$x_j = A_j'x_{r+1} + A_j''x_{r+2} + \cdots + A_j^{[n-r]}x_n \qquad (j = 1, 2, \cdots r).$$

By assigning special values here to $x_{r+1}, \cdots x_n$, we get

$$(4) \qquad \begin{cases} x_j' = A_j'x_{r+1}' + A_j''x_{r+2}' + \cdots + A_j^{[n-r]}x_n', \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_j^{[n-r]} = A_j'x_{r+1}^{[n-r]} + A_j''x_{r+2}^{[n-r]} + \cdots + A_j^{[n-r]}x_n^{[n-r]}, \\ X_j = A_j'X_{r+1} + A_j''X_{r+2} + \cdots + A_j^{[n-r]}X_n. \end{cases} \qquad (j = 1, 2, \cdots r)$$

If we multiply the first $n-r$ of these equations by $c_1, \cdots c_{n-r}$ respectively and add, we get, by (3),

$$c_1x_j' + \cdots + c_{n-r}x_j^{[n-r]} = A_j'X_{r+1} + \cdots + A_j^{[n-r]}X_n.$$

Consequently, by the last equation (4),

$$(5) \qquad X_j = c_1x_j' + \cdots c_{n-r}x_j^{[n-r]} \qquad (j = 1, 2, \cdots r).$$

Equations (3) and (5) together prove our theorem.

We thus see that the totality of all solutions of the system (1) forms a set of points satisfying the conditions of Theorem 5, § 17. Consequently,

THEOREM 2. *If the rank of a system of homogeneous linear equations in n variables is r, then any $n-r+1$ solutions are linearly dependent.*

Finally we will prove the theorem.

THEOREM 3. *A necessary and sufficient condition that a set of solutions of a system of homogeneous linear equations of rank r in n variables form a fundamental system is that they be*

(a) linearly independent,
(b) $n-r$ in number.

By definition, (*a*) is a necessary condition. To see that (*b*) also is necessary, notice that by Theorem 2 there cannot be more than $n-r$ linearly independent solutions. We have, then, merely to show that l linearly independent solutions never form a fundamental system when $l < n-r$. If they did, then by Theorem 5, § 17, any set of $l+1$ solutions would be linearly dependent, and therefore the same would be true of any set of $n-r$ solutions (since $n-r \geqq l+1$). But by Theorem 1, this is not true.

In order now to prove that conditions (*a*) and (*b*) are also sufficient, let

$$(x_1^{[i]}, x_2^{[i]}, \dots x_n^{[i]}) \qquad (i=1, 2, \dots n-r)$$

be any system of $n-r$ linearly independent solutions of our system of equations, and let $(x_1, \dots x_n)$ be any solution of the system. Then, by Theorem 2, we have $n-r+1$ constants $(c_1, \dots c_{n-r+1})$, not all zero, and such that

$$c_1x_j' + c_2x_j'' + \dots + c_{n-r}x_j^{[n-r]} + c_{n-r+1}x_j = 0 \qquad (j=1, 2, \dots n)$$

But since the $n-r$ given points are linearly independent, $c_{n-r+1} \neq 0$; accordingly these last equations enable us to express the solution $(x_1, \dots x_n)$ linearly in terms of the $n-r$ given solutions, and this shows that these $n-r$ solutions form a fundamental system.

EXERCISES

1. Prove that all the fundamental systems of solutions of a system of homogeneous linear equations are included in the infinite number obtained in the proof of Theorem 1.

2. Given three planes in space by their equations in homogeneous coördinates. What are their relative positions when the rank of the system of equations is 3? when it is 2? when it is 1?

3. Given three planes in space by their equations in non-homogeneous coördinates. What are their relative positions for the different possible pairs of values of the ranks of the matrices and augmented matrices?