CHAPTER III

THE THEORY OF LINEAR DEPENDENCE

12. Definitions and Preliminary Theorems. Two sets of constants (a_1, b_1, c_1, d_1) and (a_2, b_2, c_2, d_2) are usually said to be proportional to one another if every element of one set may be obtained from the corresponding element of the other by multiplying by the same constant factor. For example, (1, 2, 3, 4) and (2, 4, 6, 8) are proportional. It is ordinarily assumed that *either* set may be thus obtained from the other, and in most cases this is true; but in the case of the two sets (1, 2, 3, 4) and (0, 0, 0, 0) we can pass from the first to the second by multiplying by 0, but we cannot pass from the second to the first.

A more convenient definition, for many purposes, and one which is easily seen to be equivalent to the above-mentioned one, is the following:

DEFINITION 1. *The two sets of constants*

$$x_1', x_2', \cdots x_n',$$
$$x_1'', x_2'', \cdots x_n'',$$

are said to be proportional to each other if two constants c_1 and c_2, not both zero, exist such that

$$c_1 x_i' + c_2 x_i'' = 0 \qquad (i = 1, 2, \cdots n).$$

If $c_1 \neq 0$, we have

$$x_1' = -\frac{c_2}{c_1}x_1'', \; x_2' = -\frac{c_2}{c_1}x_2'', \cdots x_n' = -\frac{c_2}{c_1}x_n'',$$

and if $c_2 \neq 0$, we have

$$x_1'' = -\frac{c_1}{c_2}x_1', \; x_2'' = -\frac{c_1}{c_2}x_2', \cdots x_n'' = -\frac{c_1}{c_2}x_n'.$$

The two sets of constants
$$x_1', x_2', \cdots x_n',$$
$$0, 0, \cdots 0,$$

are evidently proportional, since if we take $c_1 = 0$ and c_2 = any constant not zero, we have a pair of c's which fulfill the requirements of our definition.

Linear dependence may be regarded as a generalization of the conception of proportionality. Instead of two sets of constants we now consider m sets, and give the following:

DEFINITION 2. *The m sets of n constants each,*

$$x_1^{[i]}, x_2^{[i]}, \cdots x_n^{[i]} \qquad (i = 1, 2, \cdots m),$$

are said to be linearly dependent if m constants $c_1, c_2, \cdots c_m$, not all zero, exist such that

$$c_1 x_j' + c_2 x_j'' + \cdots + c_m x_j^{[m]} = 0 \qquad (j = 1, 2, \cdots n).$$

If this is not the case, the sets of quantities are said to be linearly independent.

In the same way we generalize the familiar conception of the proportionality of two polynomials as follows:

DEFINITION 3. *The m polynomials (in any number of independent variables) $f_1, f_2, \cdots f_m$ are said to be linearly dependent if m constants $c_1, c_2, \cdots c_m$, not all zero, exist such that*

$$c_1 f_1 + c_2 f_2 + \cdots + c_m f_m \equiv 0.$$

*If this is not the case, the polynomials are said to be linearly independent.**

The following theorems about linear dependence, while almost self-evident, are of sufficient importance to deserve explicit statement:

THEOREM 1. *If m sets of constants (or if m polynomials) are linearly dependent, it is always possible to express one — but not necessarily any one — of them linearly in terms of the others. This set of constants (or this polynomial) is then said to be linearly dependent on the others.*

This is seen at once if we remember that at least one of the c's is not zero. The relations (or relation) in which the c's occur can, then, be divided through by this c.

* We might clearly go farther and consider the linear dependence of m sets of n polynomials each. The two cases of the text would be merely special cases from this general point of view.

THEOREM 2. *If there exist among the sets of constants (or among the polynomials) a smaller number of sets (or of polynomials) which are linearly dependent, then the m sets (or the m polynomials) are linearly dependent.*

For suppose there are l sets of constants (or l polynomials) which are linearly dependent ($l<m$), then we may take for our set of m c's, the l c's which must exist for the l sets (or polynomials) and $(m-l)$ zeros.

THEOREM 3. *If any one of the m sets of constants consists exclusively of zeros (or if any one of the polynomials is identically zero), the m sets (or the m polynomials) are linearly dependent.*

For we may take for the c corresponding to this particular set (or polynomial) any constant whatever, except zero, and for the other $(m-1)$ c's, $(m-1)$ zeros.

13. The Condition for Linear Dependence of Sets of Constants. In considering m sets of n constants each,

(1) $$x_1^{[i]},\ x_2^{[i]},\ \cdots\ x_n^{[i]} \qquad (i=1,\ 2,\ \cdots\ m),$$

it will be convenient to distinguish between the two cases $m \leqq n$ and $m>n$.

(*a*) $m \leqq n$. We wish here to prove the following fundamental theorem:

THEOREM 1. *A necessary and sufficient condition for the linear dependence of the m sets (1) of n constants each, when $m \leqq n$, is that all the m-rowed determinants of the matrix*

$$\left\|\begin{array}{cccc} x_1' & x_2' & \cdots & x_n' \\ x_1'' & x_2'' & \cdots & x_n'' \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{[m]} & x_2^{[m]} & \cdots & x_n^{[m]} \end{array}\right\|$$

should vanish.

That this is a necessary condition is at once obvious; for if the m sets of constants are linearly dependent, one of the rows can be expressed as a linear combination of the others. Accordingly if in any of the m-rowed determinants we subtract from the elements of this row the corresponding elements of the other rows after each row has been multiplied by a suitable constant, the elements of this row will reduce to zero. The determinant therefore vanishes.

We come now to the proof that the vanishing of these determinants is also a sufficient condition. We assume, therefore, that all the m-rowed determinants of the above matrix vanish. Let us also assume that the rank of the matrix is $r>0$* (cf. Definition 3, § 7). Without any real loss of generality we may (and will) assume that *the r-rowed determinant which stands in the upper left-hand corner of the matrix does not vanish;* for by changing the order of the sets of constants and the order of the constants in each set (and these orders are clearly quite immaterial) we can bring one of the non-vanishing r-rowed determinants into this position.

We will now prove that the first $(r+1)$ sets of constants are linearly dependent. From this the linear dependence of the m sets follows by Theorem 2, § 12.

Let us denote by $c_1,\ c_2,\ \cdots\ c_{r+1}$ the cofactors in the $(r+1)$-rowed determinant which stands in the upper left-hand corner of the matrix, and which correspond to the elements of its last column. If we remember that all the $(r+1)$-rowed determinants vanish, we get the relations

$$c_1x_j' + c_2x_j'' + \cdots + c_{r+1}x_j^{[r+1]} = 0 \qquad (j=r+1,\ r+2,\ \cdots\ n).$$

Since the sum of the products of the elements of any column of a determinant by the cofactors of the corresponding elements of *another* column is zero, this equation is also true when $j=1,\ 2,\ \cdots\ r$.

This establishes the linear dependence of the first $(r+1)$ sets of constants, since c_{r+1}, being the r-rowed determinant which stands in the upper left-hand corner of the matrix, is not zero.

(*b*) $m>n$. This case can be reduced to the one already considered by the following simple device. Add to each set of n constants $m-n$ zeros. We then have m sets of m constants each. Their matrix contains only one m-rowed determinant, and this vanishes since one, at least, of its columns is composed of zeros. Therefore these m sets of m constants each are linearly dependent; and hence the original m sets of n constants each were linearly dependent. Thus we get the theorem:

THEOREM 2. *m sets of n constants each are always linearly dependent if $m>n$.*

* In general we shall have $r=m-1$, but r may have any value less than m. The only case which we here exclude is that in which all the elements of the matrix are zero, a case in which the linear dependence is at once obvious.

EXERCISES

Determine whether the following sets of constants are linearly dependent or not:

$$\textbf{1.}\ \begin{cases} 3a, & -2b, & -3c, & 6d, \\ a, & 0, & -c, & 4d, \\ 0, & -b, & 0, & -3d. \end{cases}$$

$$\textbf{2.}\ \begin{cases} 1, & 0, & 0, & 5, \\ 1, & 2, & 6, & 7, \\ 3, & 1, & 3, & 16. \end{cases}$$

$$\textbf{3.}\ \begin{cases} 5, & 2, & 1, & 3, & 4, \\ 0, & 3, & 0, & 0, & 8, \\ 15, & 7, & 3, & 9, & 7. \end{cases}$$

$$\textbf{4.}\ \begin{cases} 5, & -7, & 0, & 1, & -1, \\ 1, & -3, & -2, & 3, & -1, \\ 4, & 0, & 7, & -9, & 2. \end{cases}$$

14. The Linear Dependence of Polynomials. Suppose we have m polynomials,

$$f_1, f_2, \cdots f_m,$$

in any number of independent variables. A necessary and sufficient condition for the linear dependence of these polynomials is evidently the linear dependence of their m sets of coefficients. Thus the conditions deduced in the last section can be applied at once to the case of polynomials.

EXERCISES

Determine whether the following polynomials are linearly dependent or not:

$$\textbf{1.}\ \begin{cases} 16x + 30z, \\ 6x + 2y + 5z - 4, \\ 15x + 9y - 18. \end{cases}$$

$$\textbf{2.}\ \begin{cases} 3x_1 + 4x_2 - 4x_3 + 6x_4, \\ 7x_1 + 3x_3 + 7x_4, \\ 2x_1 - x_2 - 3, \\ -5x_1 + 9x_2 - x_3 + 4x_4 + 8 \end{cases}$$

$$\textbf{3.}\ \begin{cases} 2x^2 + 8xy + 6y^2 + 14x + 12y - 4, \\ 7x^2 + y^2 + 6x - 4y, \\ 3x^2 - 6xy + 3y^2 - 5x + 7, \\ 5x^2 + 20xy + 15y^2 + 35x + 30y - 10. \end{cases}$$

15. Geometric Illustrations. The sets of n constants with which we had to deal in §§ 12, 13 may, provided that not all the constants in any one set are zero, advantageously be regarded as the homogeneous coördinates of points in space of $n-1$ dimensions. It will then be convenient to speak of the linear dependence or independence of these points. The geometric meaning of linear dependence will be at once evident from the following theorems for the case $n = 4$.

Two points will here be represented by two sets of four constants each,

$$x_1, y_1, z_1, t_1,$$
$$x_2, y_2, z_2, t_2,$$

which will be linearly dependent when, and only when, they are proportional, that is, when the points coincide. Hence:

THEOREM 1. *Two points are linearly dependent when, and only when, they coincide.*

If we have three points in space, P_1, P_2, P_3, whose coördinates are (x_1, y_1, z_1, t_1), (x_2, y_2, z_2, t_2), (x_3, y_3, z_3, t_3), respectively, and which are linearly dependent, there must exist three constants c_1, c_2, c_3, not all zero, such that

$$\begin{aligned} c_1x_1 + c_2x_2 + c_3x_3 &= 0, \\ c_1y_1 + c_2y_2 + c_3y_3 &= 0, \\ c_1z_1 + c_2z_2 + c_3z_3 &= 0, \\ c_1t_1 + c_2t_2 + c_3t_3 &= 0. \end{aligned}$$

Let us suppose the order of the points to be so taken that $c_3 \neq 0$, and solve for x_3, y_3, z_3, t_3:

$$(1)\qquad \begin{cases} x_3 = k_1x_1 + k_2x_2, \\ y_3 = k_1y_1 + k_2y_2, \\ z_3 = k_1z_1 + k_2z_2, \\ t_3 = k_1t_1 + k_2t_2, \end{cases}$$

where $k_1 = -c_1/c_3$, $k_2 = -c_2/c_3$. Now if

$$Ax + By + Cz + Dt = 0$$

is the equation of any plane through the points P_1 and P_2, we have

$$Ax_1 + By_1 + Cz_1 + Dt_1 = 0,$$
$$Ax_2 + By_2 + Cz_2 + Dt_2 = 0.$$

Multiplying the first of these equations by k_1, the second by k_2, and adding, we have, by means of the equations (1),

$$Ax_3 + By_3 + Cz_3 + Dt_3 = 0.$$

Hence every plane through P_1 and P_2 passes through P_3 also, and the three points are collinear.

Now, in order to prove conversely that any three collinear points are linearly dependent, let us suppose the three points P_1, P_2, P_3 collinear. We may assume that these three points are distinct, as otherwise their linear dependence would follow from Theorem 1. We have seen that when three points are linearly dependent, the line through two of them contains the third. Hence if we let

$$x' = k_1x_1 + k_2x_2,$$
$$y' = k_1y_1 + k_2y_2,$$
$$z' = k_1z_1 + k_2z_2,$$
$$t' = k_1t_1 + k_2t_2,$$

where k_1 and k_2 are two constants, not both zero, the point (x', y', z', t') or P' lies on the line P_1P_2, and our theorem will be established if we can show that the constants k_1 and k_2 can be so chosen that the points P' and P_3 coincide. Now let $ax + by + cz + dt = 0$ be the equation of any plane through the point P_3 but *not* through P_1 or P_2. Thus P_3 is determined as the intersection of this plane with the line P_1P_2, so that if P', which we know lies on P_1P_2, can be made to lie in this plane, it must coincide with P_3 and the proof is complete. The condition for P' to lie in this plane is $ax' + by' + cz' + dt' = 0$. Substituting for x', y', z', t' their values given above, we have

$$k_1(ax_1 + by_1 + cz_1 + dt_1) + k_2(ax_2 + by_2 + cz_2 + dt_2) = 0.$$

But neither of these parentheses is zero, since the plane does not pass through P_1 or P_2, hence we may give to k_1 and k_2 values different from zero for which this equation is satisfied. We have thus proved

THEOREM 2. *Three points are linearly dependent when, and only when, they are collinear.*

The proofs of the following theorems are left to be supplied by the reader. It will be found that some of them are readily proved

from the definition of linear dependence, as above, while for others it is more convenient to use the condition for linear dependence obtained in § 13.

THEOREM 3. *Four points are linearly dependent when, and only when, they are complanar.*

THEOREM 4. *Five or more points are always linearly dependent.*

Another geometric application is suggested by the following considerations:

A set of n ordinary * quantities is nothing more nor less than a complex quantity with n components (cf. § 21). Our first definition of linear dependence is therefore precisely equivalent to the following:

The m complex quantities

$$a_1, a_2, \cdots a_m$$

are said to be linearly dependent if m ordinary quantities $c_1, c_2, \cdots c_m$, not all zero, exist such that:

$$c_1a_1 + c_2a_2 + \cdots + c_ma_m = 0.$$

Now the simplest geometric interpretation for a complex quantity with n components is as a vector in space of n dimensions,† and we are thus led to the conception of linear dependence of vectors. The geometric meaning of this linear dependence will be seen from the following theorems for the case $n = 3$:

THEOREM 5. *Two vectors are linearly dependent when, and only when, they are collinear.*

THEOREM 6. *Three vectors are linearly dependent when, and only when, they are complanar.*

THEOREM 7. *Four or more vectors are always linearly dependent.*

In order to get a geometric interpretation of the linear dependence of polynomials, we must consider, not the polynomials themselves, but the equations obtained by equating them to zero. We speak of these equations as being linearly dependent if the polynomials are

* Two different standpoints are here possible according as we understand the term *ordinary quantity* to mean *real quantity*, or *ordinary complex quantity*.

† There are of course other possible geometric interpretations. Thus in the case $n = 4$ we may regard our complex quantities as quaternions, and consider the meaning of linear dependence of two, three, or four quaternions.

linearly dependent. If then we regard the independent variables as rectangular coördinates, these equations give us geometric loci in space of as many dimensions as there are independent variables. Thus, in the cases of two and three variables, we have plane curves and surfaces respectively. The case of two loci is of no interest, as they must coincide in order to be linearly dependent. In the case of three linearly dependent loci it is easily shown that any one must meet the other two in all their common points and in no others. The following theorems will serve to illustrate the geometric meaning of linear dependence:

(1) In the plane:

Theorem 8. *Three circles are linearly dependent when, and only when, they belong to the same coaxial family.*

Theorem 9. *Four circles are linearly dependent when, and only when, they have a (real or imaginary) common orthogonal circle.*

Theorem 10. *Four circles are linearly dependent when, and only when, the points of intersection of the first and second, and the points of intersection of the third and fourth, lie on a common circle.*

Theorem 11. *Five or more circles are always linearly dependent.*

(2) In space (using homogeneous coördinates):

Theorem 12. *Three planes are linearly dependent when, and only when, they intersect in a line.*

Theorem 13. *Four planes are linearly dependent when, and only when, they intersect in a point.*

Theorem 14. *Five or more planes are always linearly dependent.*

CHAPTER IV

LINEAR EQUATIONS

16. Non-homogeneous Linear Equations. In every elementary treatment of determinants, however brief, it is explained how to solve by determinants a system of n equations of the first degree in n unknowns, provided that the determinant of the coefficients of the unknowns is not zero. Cramer's Rule, by which this is done, is this:

Cramer's Rule. *If in the equations*

$$a_{11}x_1 + \cdots + a_{1n}x_n = k_1,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a_{n1}x_1 + \cdots + a_{nn}x_n = k_n,$$

the determinant

$$a = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

is not zero, the equations have one and only one solution, namely:

$$x_1 = \frac{a_1}{a},\ x_2 = \frac{a_2}{a},\ \cdots\ x_n = \frac{a_n}{a},$$

where a_i is the n-rowed determinant obtained from a by replacing the elements of the ith column by the elements $k_1, k_2, \cdots k_n$.

This rule, whose proof we assume to be known,* is of fundamental importance in the general theory of linear equations to which we now proceed.

* The proof as given in most English and American text-books merely establishes the fact that *if the equations have a solution* it is given by Cramer's formulæ. That these formulæ really satisfy the equations in all cases is not commonly proved, but may be easily estabished by direct substitution. We leave it for the reader to do this.