CHAPTER II

A FEW PROPERTIES OF DETERMINANTS

7. Some Definitions. We assume that the reader is familiar w the determinant notation, and will merely recall to him that b determinant of the nth order

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

we understand a certain homogeneous polynomial of the nth degr in the n^2 elements a_{ij}. By the side of these determinants it is oft desirable to consider the system of the n^2 elements arranged in t order in which they stand in the determinant, but not combin into a polynomial. Such a square array of n^2 elements we spe of as a *matrix*. In fact, we will lay down the following somewh more general definition of this term:

DEFINITION 1. *A system of mn quantities arranged in a rectang lar array of m rows and n columns is called a matrix. If $m = n$, say that we have a square matrix of order n.*

It is customary to place double bars on each side of this arra thus:

$$\left\Vert \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{matrix} \right\Vert.$$

Sometimes parentheses are used, thus:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Even when a matrix is square, it must be carefully noticed that it is not a determinant. In fact, a matrix is not a quantity at all,* but a system of quantities. This difference between a square ma- trix and a determinant is clearly brought out if we consider the effect of interchanging columns and rows. This interchange has no effect on a determinant, but gives us a wholly new matrix. In fact, we will lay down the definition:

DEFINITION 2. *Two square matrices*

$$\left\Vert \begin{matrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{matrix} \right\Vert, \qquad \left\Vert \begin{matrix} a_{11} & \cdots & a_{n1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{1n} & \cdots & a_{nn} \end{matrix} \right\Vert,$$

of which either is obtained from the other by interchanging rows and columns are called conjugate † *to each other.*

Although, as we have pointed out, square matrices and deter- minants are wholly different things, every determinant determines a square matrix, the *matrix of the determinant*, and conversely every square matrix determines a determinant, the *determinant of the matrix.*

Every matrix contains other matrices obtained from it by strik- ing out certain rows or columns or both. In particular it contains certain square matrices; and the determinants of these square matrices we will call the determinants of the matrix. If the matrix contains m rows and n columns, it will contain determinants of all orders from 1 (the elements themselves) to the smaller of the two integers and m and n inclusive.‡ In many important problems all

* Cf., however, § 21. † Sometimes also *transposed.*

‡ If $m = n$, there is only one of these determinants of highest order, and it was this which we called above *the* determinant of the square matrix.

of these determinants above a certain order are zero, and it is often of great importance to specify the order of the highest non-vanishing determinant of a given matrix. For this purpose we lay down the following definition:

DEFINITION 3. *A matrix is said to be of rank r if it contains at least one r-rowed determinant which is not zero, while all determinants of order higher than r which the matrix may contain are zero.*

A matrix is said to be of rank 0 *if all its elements are zero.*

For brevity, we shall speak also of the rank of a determinant, meaning thereby the rank of the matrix of the determinant.

We turn now to certain definitions concerning the minors of determinants; that is, the determinants obtained from the given determinant by striking out certain rows and columns.

It is a familiar fact that to every element of a determinant corresponds a certain first minor; namely, the one obtained by striking out the row and column of the determinant in which the given element lies. Now the elements of a determinant of the nth order may be regarded as its $(n-1)$th minors. Accordingly we have here a method of pairing off each one-rowed minor of a given determinant with one of its $(n-1)$-rowed minors.

Similarly, if M is a two-rowed minor of a determinant of the nth order D, we may pair it off against the $(n-2)$-rowed minor N obtained by striking out from D the two rows and columns which are represented in M. The two minors M and N we will speak of as *complementary*. Thus, in the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix},$$

the two minors

$$\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \qquad \begin{vmatrix} a_{12} & a_{14} & a_{15} \\ a_{42} & a_{44} & a_{45} \\ a_{52} & a_{54} & a_{55} \end{vmatrix},$$

are complementary.

In the same way we pair off with every three-rowed minor an $(n-3)$-rowed minor; etc. In general we lay down

DEFINITION 4. *If D is a determinant of the nth order and M one of its k-rowed minors, then the $(n-k)$-rowed minor N obtained by striking out from D all the rows and columns represented in M is called the complement of M.*

Conversely, M is clearly the complement of N.

Let us go back now for a moment to the case of the one-rowed minors; that is to the elements themselves. Let a_{ij} be the element of the determinant D which stands in the ith row and the jth column. Let D_{ij} represent the corresponding first minor. It will be recalled that we frequently have occasion to consider not this minor D_{ij} but the *cofactor* A_{ij} of a_{ij} defined by the equation $A_{ij}=(-1)^{i+j}D_{ij}$.

Similarly, it is often convenient to consider not the complement of a given minor but its *algebraic complement*, which in the case just mentioned reduces to the cofactor, and which, in general, we define as follows:

DEFINITION 5. *If M is the m-rowed minor of D in which the rows $k_1, \cdots k_m$ and the columns $l_1, \cdots l_m$ are represented, then the algebraic complement of M is defined by the equation*

$$\text{alg. compl. of } M=(-1)^{k_1+\cdots+k_m+l_1+\cdots+l_m}[\text{compl. of } M].$$

The following special case is important:

DEFINITION 6. *By a principal minor of a determinant D is understood a minor obtained by striking out from D the same rows as columns.*

Since in this case, using the notation of Definition 5, we have

$$k_1+\cdots+k_m=l_1+\cdots+l_m,$$

it follows that *the algebraic complement of any principal minor is equal to its plain complement.*

We have so far assumed tacitly that the orders of the minors we were dealing with were less than the order n of the determinant itself. By the n-rowed minor of a determinant D of the nth order we of course understand this determinant itself. The complement of this minor has, however, by our previous definition no meaning. *We will define the complement in this case to be* 1, and, by Definition 5, this will also be the algebraic complement.

EXERCISE

Prove that, if M and N are complementary minors, either M and N are the algebraic complements of each other, or $-N$ is the algebraic complement of M and $-M$ is the algebraic complement of N.

8. Laplace's Development. Just as the elements of any row or column and their corresponding cofactors may be used to develop a determinant in terms of determinants of lower orders, so the k-rowed minors formed from any k rows or columns may be used, along with their algebraic complements, to obtain a more general development of the determinant, due to Laplace, and which includes as a special case the one just referred to. In order to establish this development, we begin with the following preliminary theorem:

THEOREM 1. *If the rows and columns of a determinant D be shifted in such a way as to bring a certain minor M into the upper left-hand corner without changing the order of the rows and columns either of M or of its complement N, then this shifting will change the sign of D or leave it unchanged according as $-N$ or N is the algebraic complement of M.*

To prove this let us, as usual, number the rows and columns of D, beginning at the upper left-hand corner, and let the numbers of the rows and columns represented in M, arranged in order of increasing magnitude, be $k_1, \cdots k_m$, and $l_1, \cdots l_m$ respectively. In order to effect the rearrangement mentioned in the theorem, we may first shift the row numbered k_1 upward into the first position, thus carrying it over $k_1 - 1$ other rows and therefore changing the sign of the determinant $k_1 - 1$ times. Then shift the row numbered k_2 into the second position. This carries it over $k_2 - 2$ rows and hence changes the sign $k_2 - 2$ times. Proceed in this way until the row numbered k_m has been shifted into the mth position. Then shift the columns in a similar manner. The final result is to multiply D by

$$(-1)^{k_1+\cdots+k_m+l_1+\cdots+l_m-2(1+2+\cdots+m)} = (-1)^{k_1+\cdots+k_m+l_1+\cdots+l_m}.$$

Comparing this with Definition 5, § 7, the truth of our theorem is obvious.

LEMMA. *If M is a minor of a determinant D, the product of M by its algebraic complement is identical, when expanded, with some of the terms of the expansion of D.*

Let
$$D = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix},$$

and call the order of M, m, and its complement N. We will first prove our lemma in the special case in which M stands in the upper left-hand corner of D, so that N, which in this case is the algebraic complement, is in the lower right-hand corner. What we have to show here is that the product of any term of M by a term of N is a term of D, and that this term does not come in twice to the product MN. Any term of M may be written

$$(-1)^{\mu} a_{1l_1} a_{2l_2} \cdots a_{ml_m},$$

where the integers $l_1, l_2, \cdots l_m$ are merely some arrangement of the integers $1, 2, \cdots m$, and μ is the number of inversions of order in this arrangement. Similarly, any term of N may be written

$$(-1)^{\nu} a_{m+1,\, l_{m+1}} a_{m+2,\, l_{m+2}} \cdots a_{n,\, l_n},$$

where $l_{m+1}, \cdots l_n$ is merely some arrangement of the integers $m+1, \cdots n$, and ν is the number of inversions of order in this arrangement. The product of these two terms

$$(-1)^{\mu+\nu} a_{1l_1} a_{2l_2} \cdots a_{nl_n},$$

is a term of D, for the factors a are chosen in succession from the first, second, $\cdots$ nth rows of D, and no two are from the same column, and $\mu+\nu$ is clearly precisely the number of inversions of order in the arrangement $l_1, l_2, \cdots l_n$, as compared to the natural arrangement, $1, 2, \cdots n$, of these integers.

Having thus proved our lemma in the special case in which M lies in the upper left-hand corner of D, we now pass to the general case. Here we may, by shifting rows and columns, bring M into the upper left-hand corner and N into the lower right-hand corner. This has, by Theorem 1, the effect of leaving each term in the expansion of D unchanged, or of reversing the sign of all of them according as N or $-N$ is the algebraic complement of M. Accordingly, since the product MN gives, as we have just seen, terms in the expansion of this rearranged determinant, the product of M by its algebraic complement gives terms in the expansion of D itself. as was to be proved.

Laplace's Development, which may be stated in the form of the following rule, now follows at once:

THEOREM 2. *Pick out any m rows (or columns) from a determinant D, and form all the m-rowed determinants from this matrix. The sum of the products of each of these minors by its algebraic complement is the value of D.*

Since, by our lemma, each of these products when developed consists of terms of D, it remains merely to show that every term of D occurs in one and only one of these products. This is obviously the case; for every term of D contains one element from each of the m rows of D from which our theorem directs us to pick out m-rowed determinants, and, since these elements all lie in different columns, they lie in one and only one of these m-rowed determinants, say M. Since the other elements in this term of D obviously all lie in the complement N of M, this term will be found in the product MN and in none of the other products mentioned in our theorem.

EXERCISES

1. From a square matrix of order n and rank r, s rows (or columns) are selected. Prove that the rank of the matrix thus obtained cannot be less than $r + s - n$.
2. Generalize the theorem of Exercise 1.

9. The Multiplication Theorem. Laplace's Development enables us to write out at once the product of any two determinants as a single determinant whose order is the sum of the orders of the two given determinants

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & \cdots & b_{1m} \\ \cdot & \cdot & \cdot \\ b_{m1} & \cdots & b_{mm} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1n} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} & 0 & \cdots & 0 \\ p_{11} & \cdots & p_{1n} & b_{11} & \cdots & b_{1m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{m1} & \cdots & p_{mn} & b_{m1} & \cdots & b_{mm} \end{vmatrix},$$

whatever the values of the p's may be. For, expanding the large determinant in terms of the n-rowed minors of the first n rows, all the terms of the expansion are zero except the one written in the first member of the equation.

From this formula we will now deduce a far more important one for expressing the product of two determinants of the same order as a determinant of that order. For this purpose let us choose the p's in the last formula as follows:

$$p_{ij} = 0 \quad \text{when } i \neq j, \quad p_{ii} = -1,$$

and let us consider for simplicity the product of two determinants of the third order. We have

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \cdot \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & 0 & 0 & 0 \\ -1 & 0 & 0 & a_1 & a_2 & a_3 \\ 0 & -1 & 0 & b_1 & b_2 & b_3 \\ 0 & 0 & -1 & c_1 & c_2 & c_3 \end{vmatrix}.$$

Let us now reduce this six-rowed determinant by multiplying its first column by a_1 and adding it to the fourth column; then multiply the first column by a_2 and add it to the fifth; then multiply the first column by a_3 and add it to the sixth. In this way we bring zeros into the last three places in the fourth row. Next multiply the second column successively by b_1, b_2, b_3 and add it to the fourth, fifth, and sixth columns respectively. Finally multiply the third column successively by c_1, c_2, c_3 and add it to the fourth, fifth, and sixth columns. The determinant thus takes the form

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_1 a_1 + \alpha_2 b_1 + \alpha_3 c_1 & \alpha_1 a_2 + \alpha_2 b_2 + \alpha_3 c_2 & \alpha_1 a_3 + \alpha_2 b_3 + \alpha_3 c_3 \\ \beta_1 & \beta_2 & \beta_3 & \beta_1 a_1 + \beta_2 b_1 + \beta_3 c_1 & \beta_1 a_2 + \beta_2 b_2 + \beta_3 c_2 & \beta_1 a_3 + \beta_2 b_3 + \beta_3 c_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_1 a_1 + \gamma_2 b_1 + \gamma_3 c_1 & \gamma_1 a_2 + \gamma_2 b_2 + \gamma_3 c_2 & \gamma_1 a_3 + \gamma_2 b_3 + \gamma_3 c_3 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{vmatrix},$$

and this reduces at once to the three-rowed determinant

$$\begin{vmatrix} \alpha_1 a_1 + \alpha_2 b_1 + \alpha_3 c_1 & \alpha_1 a_2 + \alpha_2 b_2 + \alpha_3 c_2 & \alpha_1 a_3 + \alpha_2 b_3 + \alpha_3 c_3 \\ \beta_1 a_1 + \beta_2 b_1 + \beta_3 c_1 & \beta_1 a_2 + \beta_2 b_2 + \beta_3 c_2 & \beta_1 a_3 + \beta_2 b_3 + \beta_3 c_3 \\ \gamma_1 a_1 + \gamma_2 b_1 + \gamma_3 c_1 & \gamma_1 a_2 + \gamma_2 b_2 + \gamma_3 c_2 & \gamma_1 a_3 + \gamma_2 b_3 + \gamma_3 c_3 \end{vmatrix}.$$

We have thus expressed the product of two determinants of the third order as a single determinant of the third order. The method we have used is readily seen to be entirely general, and we thus get the following rule for multiplying together two determinants of the nth order:

THEOREM. *The product of two determinants of the nth order may be expressed as a determinant of the nth order in which the element which lies in the ith row and jth column is obtained by multiplying each element of the ith row of the first factor by the corresponding element of the jth column of the second factor and adding the results.*

It should be noted that changing rows into columns in either or both of the given determinants, while not affecting the value of the product, will alter its form materially. For example,

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} \cdot \begin{vmatrix} 1 & 7 \\ 6 & 9 \end{vmatrix} = \begin{vmatrix} 20 & 41 \\ 34 & 73 \end{vmatrix} = 66,$$

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} \cdot \begin{vmatrix} 1 & 6 \\ 7 & 9 \end{vmatrix} = \begin{vmatrix} 23 & 39 \\ 39 & 69 \end{vmatrix} = 66,$$

$$\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} \cdot \begin{vmatrix} 1 & 7 \\ 6 & 9 \end{vmatrix} = \begin{vmatrix} 26 & 50 \\ 33 & 66 \end{vmatrix} = 66,$$

$$\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} \cdot \begin{vmatrix} 1 & 6 \\ 7 & 9 \end{vmatrix} = \begin{vmatrix} 30 & 48 \\ 38 & 63 \end{vmatrix} = 66;$$

and similarly the product of any two determinants of the same order may be written in four different forms.

10. Bordered Determinants. If to a determinant of the nth order we add one or more rows and the same number of columns of n quantities each and fill in the vacant corner with zeros, the resulting determinant is called a *bordered determinant*. Thus starting from the two-rowed determinant

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}$$

we may form the bordered determinants

$$\begin{vmatrix} \alpha & \beta & u_1 \\ \gamma & \delta & u_2 \\ v_1 & v_2 & 0 \end{vmatrix}, \quad \begin{vmatrix} \alpha & \beta & u_1 & u_1' \\ \gamma & \delta & u_2 & u_2' \\ v_1 & v_2 & 0 & 0 \\ v_1' & v_2' & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} \alpha & \beta & u_1 & u_1' & u_1'' \\ \gamma & \delta & u_2 & u_2' & u_2'' \\ v_1 & v_2 & 0 & 0 & 0 \\ v_1' & v_2' & 0 & 0 & 0 \\ v_1'' & v_2'' & 0 & 0 & 0 \end{vmatrix}, \cdots$$

If in the second of these examples we use Laplace's Development to expand the bordered determinant according to the two-rowed determinants of the last two rows, we see that its value is

$$\begin{vmatrix} u_1 & u_1' \\ u_2 & u_2' \end{vmatrix} \cdot \begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix},$$

a quantity into which the elements α, β, γ, δ of the original determinant do not enter. Similarly expanding the third of the above bordered determinants according to the three-rowed determinants of its last three rows, we see that its value is zero.

The reasoning we have here used is of general application and leads to the following results:

THEOREM 1. *If a determinant of the nth order is bordered with n rows and n columns, the resulting determinant has a value which depends only on the bordering quantities.*

THEOREM 2. *If a determinant of the nth order is bordered with more than n rows and columns, the resulting determinant always has the value zero.*

The cases of interest are therefore those in which the determinant is bordered with less than n rows and columns. Concerning these we will establish the following fact:

THEOREM 3. *If a determinant of the nth order be bordered by p rows and p columns ($p < n$) of independent variables, the resulting determinant is a polynomial of degree 2p in the bordering quantities, whose coefficients are the pth minors of the original determinant; and conversely, every pth minor of the original determinant is the coefficient of at least one term of this polynomial.*

Let us consider the special case where $n = 4$ and $p = 2$.

$$D \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & u_1 & u_1' \\ a_{21} & a_{22} & a_{23} & a_{24} & u_2 & u_2' \\ u_{31} & a_{32} & a_{33} & a_{34} & u_3 & u_3' \\ a_{41} & a_{42} & a_{43} & a_{44} & u_4 & u_4' \\ v_1 & v_2 & v_3 & v_4 & 0 & 0 \\ v_1' & v_2' & v_3' & v_4' & 0 & 0 \end{vmatrix}.$$

Developing this determinant, by Laplace's method (§ 8), in terms of the two-rowed determinants of the last two rows, we have

$$D \equiv \begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix} \begin{vmatrix} a_{13} & a_{14} & u_1 & u_1' \\ a_{23} & a_{24} & u_2 & u_2' \\ a_{33} & a_{34} & u_3 & u_3' \\ a_{43} & a_{44} & u_4 & u_4' \end{vmatrix} + \cdots \text{ to 6 terms.}$$

If now we expand each of these four-rowed determinants, by Laplace's method, in terms of the two-rowed determinants of their last two columns, and then arrange the result as a polynomial in the u's and v's, the truth of the theorem is apparent. We leave it to the reader to fill in the details of the proof here sketched.

11. Adjoint Determinants and their Minors.

DEFINITION. *If, in the determinant*

$$D = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix},$$

A_{ij} is the cofactor of the element a_{ij}, then the determinant

$$D' = \begin{vmatrix} A_{11} & \cdots & A_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ A_{n1} & \cdots & A_{nn} \end{vmatrix}$$

is called the adjoint of D.

By corresponding minors of D and D', or indeed of any two determinants of the same order, we shall naturally understand minors obtained by striking out the same rows and columns from D as from D'. These definitions being premised, the fundamental theorem here is the following:

THEOREM. *If D' is the adjoint of any determinant D, and M and M' are corresponding m-rowed minors of D and D' respectively, then M' is equal to the product of D^{m-1} by the algebraic complement of M.*

We will prove this theorem first for the special case in which the minors M and M' lie at the upper left-hand corners of D and D' respectively. We may then write

$$M' = \begin{vmatrix} A_{11} & \cdots & A_{1m} & \cdots & \cdots & A_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{m1} & \cdots & A_{mm} & \cdots & \cdots & A_{mn} \\ 0 & \cdots & 0 \;\; 1 \;\; 0 & \cdots & & 0 \\ 0 & \cdots & 0 \;\; 0 \;\; 1 & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 \;\; 0 \;\; 0 & \cdots & & 1 \end{vmatrix}.$$

Let us now interchange the columns and rows of D,

$$D = \begin{vmatrix} a_{11} & \cdots & a_{n1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{1n} & \cdots & a_{nn} \end{vmatrix},$$

and then form the product $M'D$ by the theorem of § 9. This gives

$$M'D = \begin{vmatrix} D & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & D & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & D & 0 & \cdots & 0 \\ a_{1,m+1} & a_{2,m+1} & \cdots & a_{m,m+1} & a_{m+1,m+1} & \cdots & a_{n,m+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{mn} & a_{m+1,n} & \cdots & a_{nn} \end{vmatrix}$$

$$= D^m \begin{vmatrix} a_{m+1,m+1} & \cdots & a_{n,m+1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{m+1,n} & \cdots & a_{nn} \end{vmatrix}.$$

Let us here regard $a_{11}, \cdots a_{nn}$ as n^2 independent variables. Then the equation just written becomes an identity, from which D, since it is not identically zero, may be cancelled out, and we get

$$(1) \qquad M' \equiv D^{m-1} \begin{vmatrix} a_{m+1,m+1} & \cdots & a_{n,m+1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{m+1,n} & \cdots & a_{n,n} \end{vmatrix}.$$

Since the determinant which is written out in (1) is precisely the algebraic complement of M, our theorem is proved in the special case we have been considering. It should be noticed that this proof holds even in the case $m=n$; cf. Corollary 2 below.

Turning now to the case in which the minors M and M' do not lie at the upper left-hand corners of D and D', let us denote by a the sum of the numbers which specify the location of the rows and columns in M or M', the numbering running, as usual, from the upper left-hand corner. Then by Definition 5, § 7,

$$(2) \qquad \text{alg. compl. of } M = (-1)^a \,[\text{compl. of } M].$$

Let us now, by shifting rows and columns, bring the determinant M into the upper left-hand corner of D. Calling the determinant D, as thus rearranged, D_1, we have (cf. Theorem 1, § 8)

$$(3) \qquad D_1 = (-1)^a D.$$

The cofactors in D_1 are equal to $(-1)^a A_{ij}$, since the interchange of two adjacent rows or columns of a determinant changes the sign of every one of its cofactors. Accordingly the adjoint of D_1, which we will call D_1', may be obtained from D' by rearranging its rows and columns in the same way as the rows and columns of D were rearranged to give D_1, and then prefixing the factor $(-1)^a$ to each element.

Let us now apply the special case already established of our theorem to the determinant D_1 and its adjoint D_1', the m-rowed minors M_1 and M_1' being those which are situated in the upper left-hand corner of D_1 and D_1' respectively. We thus get

$$(4) \qquad M_1' = D_1^{m-1} \,[\text{alg. compl. of } M_1].$$

Now, since M_1 is a principal minor, its algebraic complement is the same as its ordinary complement, and this in turn is the same as the ordinary complement of the minor M in D. Accordingly, using (2), we may write

$$(5) \qquad \text{alg. compl. of } M_1 = (-1)^a \,[\text{alg. compl. of } M].$$

Since the elements of M_1' differ from those of M' only in having the factor $(-1)^a$ prefixed to each, it follows that

$$(6) \qquad M_1' = (-1)^{ma} M'.$$

We may now reduce (4) by means of (3), (5), and (6). We thus get

$$(-1)^{ma} M' = (-1)^{a(m-1)} D^{m-1} (-1)^a \,[\text{alg. compl. of } M].$$

Cancelling out the factor $(-1)^{ma}$ from both sides of this equation, we see that our theorem is proved.

We proceed now to point out a number of special cases of this theorem which are worth noting on account of their frequent occurrence.

COROLLARY 1. *If a_{ij} is any element of a determinant D of the nth order, and if α_{ij} is the cofactor of the corresponding element A_{ij} in the adjoint of D, then*

$$\alpha_{ij} = D^{n-2} a_{ij}.$$

This is merely the special case of our general theorem in which $m=n-1$, modified, however, slightly in statement by the use of the cofactor α_{ij} in place of the $(n-1)$-rowed minor $(-1)^{i+j}\alpha_{ij}$.

COROLLARY 2. *If D is any determinant of the nth order and D' its adjoint, then*

$$D' = D^{n-1}.$$

This is the special case $m=n$.

COROLLARY 3. *If D is any determinant, and S is the second minor obtained from it by striking out its ith and kth rows and its jth and lth columns, and if we denote by A_{ij} the cofactor of the element which stands in the ith row and the jth column of D, then*

$$\begin{vmatrix} A_{ij} & A_{il} \\ A_{kj} & A_{kl} \end{vmatrix} = (-1)^{i+j+k+l} DS.$$

This is the special case $m=2$.