RATIO AND PROPORTION

RATIO

381. The relation of two numbers that is expressed by the quotient of the first divided by the second is called their **ratio**.

382. The **sign of ratio** is a colon (:).

A ratio is expressed also in the form of a fraction.

The ratio of a to b is written $a : b$ or $\frac{a}{b}$.

The colon is sometimes regarded as derived from the sign of division by omitting the line.

383. To compare two quantities they must be expressed in *terms of a common unit.*

Thus, to indicate the ratio of 20¢ to \$1, both quantities must be expressed either in cents or in dollars, as 20¢ : 100¢ or \$$\frac{1}{5}$: \$1.

There can be no ratio between 2 pounds and 3 feet.

The ratio of two quantities is the *ratio* of their *numerical measures.*

Thus, the ratio of 4 rods to 5 rods is the ratio of 4 to 5.

384. The first term of a ratio is called the **antecedent**, and the second, the **consequent**. Both terms form a **couplet**.

The antecedent corresponds to a dividend or numerator; the consequent, to a divisor or denominator.

In the ratio $a : b$, or $\frac{a}{b}$, a is the antecedent, b the consequent, and the terms a and b form a couplet.

385. The ratio of the reciprocals of two numbers is called the **reciprocal**, or **inverse, ratio** of the numbers.

It may be expressed by interchanging the terms of the couplet.

The inverse ratio of a to b is $\frac{1}{a} : \frac{1}{b}$. Since $\frac{1}{a} \div \frac{1}{b} = \frac{b}{a}$, the inverse ratio of a to b may be written $\frac{b}{a}$ or $b : a$.

Properties of Ratios

386. It is evident from the definition of a ratio that ratios have the same properties as fractions; that is, they may be *reduced to higher or lower terms, added, subtracted,* etc. Hence,

PRINCIPLES. — 1. *Multiplying or dividing both terms of a ratio by the same number does not change the value of the ratio.*

2. *Multiplying the antecedent or dividing the consequent of a ratio by any number multiplies the ratio by that number.*

3. *Dividing the antecedent or multiplying the consequent by any number divides the ratio by that number.*

EXERCISES

387. 1. What is the ratio of $8m$ to $4m$? of $4m$ to $8m$?

2. Express the ratio of 6 : 9 in its lowest terms; the ratio $12x : 16y$; $am : bm$; $20ab : 10bc$; $(m+n) : (m^2 - n^2)$.

3. Which is the greater ratio, 2 : 3 or 3 : 4? 4 : 9 or 2 : 5?

4. What is the ratio of $\frac{1}{2}$ to $\frac{1}{4}$? $\frac{1}{2}$ to $\frac{1}{3}$? $\frac{2}{3}$ to $\frac{3}{4}$?

SUGGESTION. — When fractions have a common denominator, they have the ratio of their numerators.

5. What is the inverse ratio of 3 : 10? of 12 : 7?

Reduce to lowest terms the ratios expressed by:

6. 10 : 2.	8. 3 : 27.	10. $\frac{12}{24}$.	12. $75 \div 100$.
7. 12 : 6.	9. 4 : 40.	11. $\frac{16}{48}$.	13. $60 \div 120$.

14. What is the ratio of 15 days to 30 days? of 21 days to 1 week? of 1 rod to 1 mile?

Find the value of each of the following ratios:

15. $\frac{4}{5}x : \frac{2}{5}x^2$. 17. $2\frac{1}{2} : 7\frac{1}{2}$. 19. $a^2b^3x^4 : a^4b^2x^2$.

16. $\frac{3}{4}ab : \frac{1}{2}ac$. 18. $.7m : .8n$. 20. $(x^2 - y^2) : (x - y)^2$.

21. If 9 is subtracted from 4 and then from 5, find the ratio of the first remainder to the second.

22. Change each to a ratio whose antecedent shall be 1:

$$5:20;\ 3x:12x;\ \tfrac{3}{2}:\tfrac{5}{6};\ .4:1.2.$$

23. When the antecedent is $6x$ and the ratio is $\frac{1}{3}$, what is the consequent?

PROPORTION

388. An equality of ratios is called a **proportion.**

$3:10 = 6:20$ and $a:x = b:y$ are proportions.

The double colon (::) is often used instead of the sign of equality.

The double colon has been supposed to represent the extremities of the lines that form the sign of equality.

The proportion $a:b = c:d$, or $a:b::c:d$, is read, 'the ratio of a to b is equal to the ratio of c to d,' or 'a is to b as c is to d.'

389. In a proportion, the first and fourth terms are called the **extremes**, and the second and third terms, the **means.**

In $a:b = c:d$, a and d are the extremes, b and c are the means.

390. Since a proportion is an equality of ratios each of which may be expressed as a fraction, a proportion may be expressed as an equation each member of which is a fraction.

Hence, it follows that:

General Principle. — *The changes that may be made in a proportion without destroying the equality of its ratios correspond to the changes that may be made in the members of an equation without destroying their equality and in the terms of a fraction without altering the value of the fraction.*

Properties of Proportions

391. Principle 1. — *In any proportion the product of the extremes is equal to the product of the means.*

For, given $a:b = c:d$,

or $\frac{a}{b} = \frac{c}{d}$.

Clearing of fractions, $ad = bc$.

Test the following by principle 1 to find whether they are true proportions:

1. $6:16 = 3:8$. 2. $\frac{12}{15} = \frac{16}{20}$. 3. $7:8 = 10:12$

392. In the proportion $a:m = m:b$, m is called a **mean proportional** between a and b.

By Prin. 1, $m^2 = ab$;

$\therefore\ m = \sqrt{ab}$.

Hence, *a mean proportional between two numbers is equal to the square root of their product.*

1. Show that the mean proportional between 3 and 12 is either 6 or -6. Write both proportions.

2. Find two mean proportionals between 4 and 25.

393. Principle 2. — *Either extreme of a proportion is equal to the product of the means divided by the other extreme.*

Either mean is equal to the product of the extremes divided by the other mean.

For, given $a:b = c:d$.

By Prin. 1, $ad = bc$.

Solving for a, d, b, and c, in succession, Ax. 4,

$$a = \frac{bc}{d},\ d = \frac{bc}{a},\ b = \frac{ad}{c},\ c = \frac{ad}{b}.$$

1. Solve the proportion $3:4 = x:20$, for x.

2. Solve the proportion $x:a = 2m:n$, for x.

3. If $a:b=b:c$, the term c is called a **third proportional** to a and b. Find a third proportional to 6 and 2.

4. In the proportion $a:b=c:d$, the term d is called a **fourth proportional** to a, b, and c. Find a fourth proportional to $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$.

394. Principle 3. — *If the product of two numbers is equal to the product of two other numbers, one pair of them may be made the extremes and the other pair the means of a proportion.*

For, given $ad=bc$.

Dividing by bd, Ax. 4, $\frac{a}{b}=\frac{c}{d}$;

that is, $a:b=c:d$.

By dividing both members of the given equation, or of $bc=ad$, by the proper numbers, various proportions may be obtained; but in all of them a and d will be the extremes and b and c the means, or *vice versa*, as illustrated in the proofs of principles 4 and 5.

1. If a men can do a piece of work in x days, and if b men can do the same work in y days, the number of days' work for one man may be expressed by either ax or by. Form a proportion between a, b, x, and y.

2. The formula $pd=WD$ (See p. 166)

expresses the physical law that, when a lever just balances, the product of the numerical measures of the power and its distance from the fulcrum is equal to the product of the numerical measures of the weight and its distance from the fulcrum. Express this law by means of a proportion.

395. Principle 4. — *If four numbers are in proportion, the ratio of the antecedents is equal to the ratio of the consequents;* that is, *the numbers are in proportion by* **alternation**.

For, given $a:b=c:d$.

Then, Prin. 1, $ad=bc$.

Dividing by cd, Ax. 4, $\frac{a}{c}=\frac{b}{d}$;

that is, $a:c=b:d$.

396. Principle 5. — *If four numbers are in proportion, the ratio of the second to the first is equal to the ratio of the fourth to the third;* that is, *the numbers are in proportion by* **inversion**.

For, given $a:b=c:d$.

Then, Prin. 1, $ad=bc$.

$\therefore bc=ad$.

Dividing by ac, Ax. 4, $\frac{b}{a}=\frac{d}{c}$;

that is, $b:a=d:c$.

397. Principle 6. — *If four numbers are in proportion, the sum of the terms of the first ratio is to either term of the first ratio as the sum of the terms of the second ratio is to the corresponding term of the second ratio;* that is, *the numbers are in proportion by* **composition**.

For, given $a:b=c:d$,

or $\frac{a}{b}=\frac{c}{d}$.

Then, Ax. 1, $\frac{a}{b}+1=\frac{c}{d}+1$,

or $\frac{a+b}{b}=\frac{c+d}{d}$;

that is, $a+b:b=c+d:d$.

Similarly, taking the given proportion by inversion (Prin. 5), and adding 1 to both members, we obtain

$$a+b:a=c+d:c.$$

398. Principle 7. — *If four numbers are in proportion, the difference between the terms of the first ratio is to either term of the first ratio as the difference between the terms of the second ratio is to the corresponding term of the second ratio;* that is, *the numbers are in proportion by* **division**.

For, in the proof of Prin. 6, if 1 is *subtracted* instead of added, the following proportions are obtained:

$$a-b:b=c-d:d,$$

and

$$a-b:a=c-d:c.$$

399. Principle 8. — *If four numbers are in proportion, the sum of the terms of the first ratio is to their difference as the sum of the terms of the second ratio is to their difference;* that is, *the numbers are in proportion by* **composition and division.**

For, given $a : b = c : d.$

By Prin. 6, $\dfrac{a+b}{b} = \dfrac{c+d}{d}.$ (1)

By Prin. 7, $\dfrac{a-b}{b} = \dfrac{c-d}{d}.$ (2)

Dividing (1) by (2), Ax. 4, $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d};$

that is, $a+b : a-b = c+d : c-d.$

400. Principle 9. — *In a proportion, if both terms of a couplet or both antecedents, or both consequents are multiplied or divided by the same number, the resulting four numbers form a proportion.*

For, given $a : b = c : d,$

or $\dfrac{a}{b} = \dfrac{c}{d}.$

Then, § 181, $\dfrac{ma}{mb} = \dfrac{nc}{nd}$, or $ma : mb = nc : nd.$

Also, Ax. 3, $\dfrac{a}{b} \cdot \dfrac{m}{n} = \dfrac{c}{d} \cdot \dfrac{m}{n}$, or $ma : nb = mc : nd.$

401. Principle 10. — *In a series of equal ratios, the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.*

For, given $a : b = c : d = e : f,$

or $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = r$, the value of each ratio.

Then, Ax. 3, $a = br,\ c = dr,\ e = fr;$

whence, Ax. 1, $a + c + e = (b + d + f)r,$

$\therefore \dfrac{a+c+e}{b+d+f} = r = \dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f};$

that is, $a + c + e : b + d + f = a : b$ or $c : d$ or $e : f.$

EXERCISES

402. 1. Find the value of x in the proportion $3 : 5 = x : 55.$

Solution. $3 : 5 = x : 55.$

Prin. 2, $x = \dfrac{3 \cdot 55}{5} = 33.$

Find the value of x in each of the following proportions:

2. $2 : 3 = 6 : x.$
3. $5 : x = 4 : 3.$
4. $1 : x = x : 9.$
5. $x + 2 : x = 10 : 6.$
6. $x : x - 1 = 15 : 12.$
7. $x + 2 : x - 2 = 3 : 1.$
8. Show that a mean proportional between any two numbers having like signs has the sign $\pm$.
9. Find two mean proportionals between $\sqrt{2}$ and $\sqrt{8}$.
10. Find a third proportional to 4 and 6.
11. Find a fourth proportional to 3, 8, and $7\frac{1}{2}$.

Test to see whether the following are true proportions:

12. $5\frac{1}{2} : 3 = 4 : 1\frac{1}{2}.$
13. $4 : 13 = 2 : 6\frac{1}{2}.$
14. $5 : 7x = 10 : 14x.$
15. $2.4a : .8a = 6a : 2a.$
16. Given $a : b = c : d,$

to prove that $2a + 3c : 2a - 3c = 8b + 12d : 8b - 12d.$

Proof. — Given $a : b = c : d.$

By alternation, Prin. 4, $a : c = b : d.$

Expressing as a fractional equation, $\dfrac{a}{c} = \dfrac{b}{d}.$

Multiplying the first member by $\frac{2}{3}$ and the second by $\frac{8}{12}$, the equal of $\frac{2}{3}$,

$$\frac{2a}{3c} = \frac{8b}{12d};$$

that is, $2a : 3c = 8b : 12d.$

By composition and division, Prin. 8,

$$2a + 3c : 2a - 3c = 8b + 12d : 8b - 12d.$$

When $a : b = c : d$, prove that:

17. $d : b = c : a.$
18. $c : d = \dfrac{1}{b} : \dfrac{1}{a}.$
19. $b^3 : d^3 = a^3 : c^3.$
20. $a^2 : b^2c^2 = 1 : d^2.$
21. $ma : \dfrac{b}{2} = mc : \dfrac{d}{2}.$
22. $ac : bd = c^2 : d^2.$

When $a:b = c:d$, prove that:

23. $a + b:c + d = a - b:c - d.$

24. $2a + 5b:2a = 2c + 5d:2c.$

25. $4a - 3b:4c - 3d = a:c.$

26. $a:a + b = a + c:a + b + c + d.$

27. $a + b:c + d = \sqrt{a^2 + b^2}:\sqrt{c^2 + d^2}.$

Problems

403. 1. The ratio of the rate of a local train in the New York subway to an express train is 1:2. If the local train runs 15 miles an hour, find the rate of the express train.

2. The consumption of gas in New York City one year was to that in Chicago as 7:4. If 12 billion cubic feet were consumed in Chicago, what was the consumption in New York?

3. Michigan produces yearly 25% of the iron ore of the United States. The ratio of Michigan's output to Minnesota's is 5:8. What per cent of the country's output does Minnesota produce?

4. The United States manufactured 285 million pens one year. The ratio of the steel pens to the whole number of pens was 18:19. How many steel pens were manufactured?

5. A diver descended 210 feet into a lake. The ratio of this distance to the distance that is usually considered the limit for divers is 7:5. Find the usual limit for divers.

6. How many pounds of tea are made from 4200 pounds of the green leaf, if the ratio of the weight of the manufactured tea to that of the green leaf is 5:21?

7. Two machines, one old and one modern, turn out 960 pins per minute. The ratio of the number turned out by the old machine to the number turned out by the modern one is 1:15. How many were turned out by each machine?

8. Find a number that added to each of the numbers 1, 2, 4, and 7 will give four numbers in proportion.

9. The United States published 20,000 newspapers recently. The relation of this number to those published in the whole world was 2:5. How many were published in the world?

10. The ratio of the greatest length of Lake Erie to the greatest length of Lake Michigan is 5 : 6. What is the length of each, if Lake Michigan is 50 miles longer than Lake Erie?

11. The ratio of the loss of life in the Lisbon earthquake to that in the Messina earthquake is 12:23. If 55,000 more lives were lost in the latter than in the former, find the loss of life in each earthquake.

12. The length of a giant candle was to that of a Christmas candle as 40:1. If 8 times the length of the latter was 96 inches less than that of the former, find the length of each.

13. The wool sales for one week in New York amounted to 555,000 pounds. The ratio of the domestic sales to the foreign was 14:23. What were the foreign sales?

14. Out of a lot of shell caps, 100 times the number rejected by the government inspector for imperfections was to the total number as 3 : 11. If 1097 were accepted, how many were rejected?

15. In one year Egypt and Russia together sent $9\frac{1}{2}$ million pounds of eggs to Paris. If Egypt had sent twice as many, the ratio of this number to those sent by Russia would have been 1 : 18. How many pounds were sent by each country?

16. The sum of the three dimensions of a block of ice is 77 inches, and the width, 22 inches, is a mean proportional between the other two dimensions. Find the length and thickness.

17. The ratio of the length of a gold nugget to its width was 11: 6, but if its length had been $\frac{1}{2}$ of an inch more, the ratio would have been 2 : 1. Find its length and width.

18. The area of the right triangle shown in Fig. 1 may be expressed either as $\frac{1}{2}ab$ or as $\frac{1}{2}ch$. Form a proportion whose terms shall be a, b, c, h.

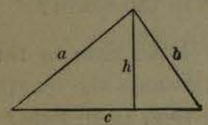

FIG. 1.

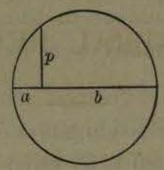

FIG. 2.

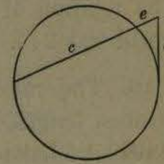

FIG. 3.

19. In Fig. 2, the perpendicular p, which is 20 feet long, is a mean proportional between a and b, the parts of the diameter, which is 50 feet long. Find the length of each part.

20. In Fig. 3, the tangent t is a mean proportional between the whole secant $c+e$, and its external part e. Find the length of t, if $e=9\frac{3}{5}$ and $c=50\frac{2}{5}$.

21. The strings of a musical instrument produce sound by vibrating. The relation between the number of vibrations N and N' of two strings, different only in their lengths l and l', is expressed by the proportion

$$N:N'=l':l.$$

A c string and a g string, exactly alike except in length, vibrate 256 and 384 times per second, respectively. If the c string is 42 inches long, find the length of the g string.

22. If L and l are the lengths of two pendulums and T and t the times they take for an oscillation, then

$$T^2:t^2=L:l.$$

A pendulum that makes one oscillation per second is approximately 39.1 inches long. How often does a pendulum 156.4 inches long oscillate?

23. Using the proportion of exercise 22, find how many feet long a pendulum would have to be to oscillate once a minute.

GENERAL REVIEW

404. 1. Distinguish between known and unknown numbers.

2. When $\times$, $\div$, or both occur in connection with $+$, $-$, or both in an expression, what is the sequence of operations?

Illustrate by finding the value of: $7-3\times 2+6\div 2$.

3. Name and illustrate three ways of indicating multiplication; two ways of indicating division.

4. When is x^n-y^n exactly divisible by $x+y$? by $x-y$?

5. When is a trinomial a perfect square? When is a fraction in its lowest terms? What are similar fractions?

6. By what principle may cancellation be used in reducing fractions to lowest terms?

7. Factor the following by three different methods:

$$(a^2-2)^2-a^2.$$

8. Define power; root; like terms; transposition; simultaneous equations; surd.

9. Express the following without parentheses:

$$(a^2x^m)^n;\ -[-(a^2)^2]^2;\ (a^3)^4;\ (a^4)^3;\ (-a^2)^5.$$

10. What is the sign of any power of a positive number? of any even power of a negative number? of any odd power of a negative number?

11. How may the involution of a trinomial be performed by the use of the binomial formula?

Illustrate by raising $x+2y-z$ to the third power.

12. Explain the meaning of a negative integral exponent; of a positive fractional exponent; of a zero exponent.

13. Define evolution; binomial surd; similar surds; conjugate surds; symmetrical equation.

14. Is $\sqrt{2+\sqrt{4}}$ a surd? State reasons for your answer.

15. Represent $\sqrt{10}$ inches by a line.

16. Why is it specially important to test the values of the unknown number found in the solution of radical equations?

17. Define coördinate axes; imaginary number; axiom; coefficient; elimination.

18. How is the degree of an equation determined? What is the degree of $x+b=c$? of $x^2+3x=7$? of $5x+xy=11$?

19. What name is given to an equation of the first degree? of the second degree? of any higher degree?

20. What is a pure quadratic equation? a complete quadratic equation? Illustrate each.

21. What is the root of an equation? What is the principle relating to the roots of a pure quadratic equation?

Illustrate by solving the following:

$$7x^2-5=23.$$

22. Give two methods of completing the square in the solution of affected quadratic equations. When is it advantageous to use the Hindoo method?

Solve the following equation by each method:

$$3x^2+5x=22.$$

23. Outline the method of solving quadratic equations by factoring.

Illustrate by solving the following:

$$2x^2-5x=12.$$

24. When is an equation in the quadratic form? Illustrate.

25. What roots should be associated when the roots of a system of equations are given thus: $x=\pm 2$, $y=\mp 3$?

26. Explain how, in the solution of problems, negative roots of quadratic equations, while mathematically correct, are often inadmissible.

27. What is the advantage of letting $x^2=y$ in the graphic solution of a quadratic equation of the form $ax^2+bx+c=0$?

28. How does the graph of a quadratic equation show the fact, if the roots are real and equal? real and unequal? imaginary?

29. What is the form of the graph of a simple equation? of two inconsistent equations? of two indeterminate equations?

30. What is the form of the graph of an equation like $ax^2+bx+c=0$? like $x^2+y^2=r^2$?

31. What is meant by the minimum point of a graph?

Solve graphically the following equation and indicate the minimum point of the graph:

$$x^2+6x+5=0.$$

32. How many roots has a simple equation? a quadratic equation? a system of two independent simultaneous equations, one simple and the other quadratic? a system of two independent simultaneous quadratic equations?

33. Define homogeneous equation; antecedent; consequent; inverse ratio; proportion.

34. Give and illustrate two principles relating to ratios. Upon what do these principles depend?

35. Find the ratio of $(x-y)^3$ to $x^2-2xy+y^2$.

36. In the following proportion indicate the means; the extremes; the mean proportional; the third proportional:

$$x:y=y:z.$$

37. Find a fourth proportional to $3a$, $9a$, and $5a$.

38. Add $x\sqrt{y}+y\sqrt{x}+\sqrt{xy}$, $x^{\frac{1}{2}}y^{\frac{1}{2}}-\sqrt{x^2y}-\sqrt{xy^2}$, $\sqrt{x^2y}-\sqrt{xy^2}-\sqrt{xy}$, and $y\sqrt{x}-x\sqrt{4y}-\sqrt{9xy}$.

39. Simplify $a-\{b-a-[a-b-(2a+b)+(2a-b)-a]-b\}$.

40. Divide x^4-y^4 by $x-y$ by inspection. Test.

41. Separate $a^{12}-1$ into six rational factors.

42. Reduce $\dfrac{a^2-b^2-c^2-2bc}{a^2-b^2+c^2+2ac}$ to its lowest terms.

43. Simplify $\dfrac{x}{x+1}-\dfrac{x}{1-x}+\dfrac{x^2}{x^2-1}$.

44. Simplify $\dfrac{1}{(a-b)(b-c)}-\dfrac{1}{(c-b)(c-a)}+\dfrac{1}{(c-a)(a-b)}$.

45. Reduce to the simplest form: $\sqrt{\frac{2}{3}}$; $\sqrt[4]{25a^4}$; $\dfrac{\sqrt{3}+3\sqrt{2}}{\sqrt{6}+2}$.

Solve the following equations for x:

46. $3x^2-2x=65$.

47. $x^4+\dfrac{1}{2}=\dfrac{3x^2}{2}$.

48. $\sqrt{x-9}=\sqrt{x}-1$.

49. $x^2+\sqrt{x^2+16}=14$.

Solve the following for the letters involved:

50. $\begin{cases}\dfrac{1}{x}+\dfrac{1}{y}=10,\\ \dfrac{3}{x}+\dfrac{2}{y}=10.\end{cases}$

51. $\begin{cases}x^2+y^2=25,\\ x+y=7.\end{cases}$

52. $\begin{cases}2x+3y+z=9,\\ x+2y+3z=13,\\ 3x+y+2z=11.\end{cases}$

53. $\begin{cases}x^2+xy=24,\\ y^2+xy=12.\end{cases}$

Solve graphically:

54. $\begin{cases}x-y=1,\\ x^2+y^2=16.\end{cases}$

55. $\begin{cases}4x^2+9y^2=36,\\ y=3.\end{cases}$

Find the value of x in the following proportions:

56. $7\frac{2}{3}:6=x:4\frac{1}{2}$.

57. $x:7.2=3.9:117$.

58. The railways of the United States use annually 150 million tons of coal. If the amount used in drawing trains is $\frac{1}{19}$ as much as goes up the smokestacks, how much is used to draw trains?

59. In one year about 30,000 vessels passed a lighthouse in Massachusetts. The number that used steam was to the number of the remainder as $1:5$. How many used steam?

60. Out of 63 bakeries inspected in a certain city, the number of 'absolutely clean' ones was 3 more than that of the 'fairly clean' ones, and the number of 'unsanitary' ones was 2 less than twice that of the 'absolutely clean' ones. Find the number of bakeries in each class.

61. The number of parts in a certain manufacturer's mower is twice that in his horse rake and $\frac{3}{19}$ that in his binder. If the binder has 3500 parts more than the rake, how many parts has each machine?

62. Two men earned \$3.50 one day for picking pine needles. They were paid 25 cents per 100 pounds. How many pounds did each pick, if one picked $\frac{3}{4}$ as many as the other?

63. One of the largest rugs ever made in this country contains 3180 square feet. Its length is 7 feet greater than its width. What are its dimensions?

64. Alfred the Great measured time by candles lighted in succession. The number used in a day was $\frac{1}{2}$ the number of inches in the length of each candle, and each burned at the rate of 3 inches per hour. How many candles were used per day and how long was each?

65. A target used in practice by the United States fleet was 1 foot longer than it was wide and 18 feet longer than the square bull's-eye. The area of the target exclusive of the bull's-eye was 411 square feet. Find the dimensions of each.

66. A good operator usually earns \$1.80 a day by binding derby hats. If she bound 1 dozen more hats and received 5¢ less per dozen, she would earn 5¢ less a day. How many hats does she bind a day and how much does she receive per dozen?

67. How far down a river whose current runs 3 miles an hour can a steamboat go and return in 8 hours, if its rate of sailing in still water is 12 miles an hour?

68. A person who can walk n miles an hour has a hours at his disposal. How far may he ride in a coach that travels m miles an hour and return on foot within the allotted time?

69. The first copy of *The Sun* was printed on a sheet $5\frac{1}{2}$ inches longer than it was wide. If the length lacked 6 inches of being twice the width, find the dimensions of the sheet.

70. The *Lusitania* is 26 feet less than 6 times as long as the *Clermont*, and $\frac{1}{10}$ of the length of the *Lusitania* is 11 feet more than $\frac{1}{2}$ of the length of the *Clermont*. Find the length of each.

71. A woman has 13 square feet to add to the area of the rug she is weaving. She therefore increases the length $\frac{1}{8}$ and the width $\frac{1}{4}$, which makes the perimeter 4 feet greater. Find the dimensions of the finished rug.

72. The inventor of toothpicks sold 16,250,000 during his first year of business. Had there been 75,000 more toothpicks in each box, the number of boxes sold would have been 15 fewer. How many boxes did he sell?

73. Two passengers together have 400 pounds of baggage and are charged, for the excess above the weight allowed free, 40 cents and 60 cents, respectively. If the baggage had belonged to one of them, he would have been charged \$1.50. How much baggage is one passenger allowed without charge?

74. A railway train, after traveling 2 hours at its usual rate, was detained 1 hour by an accident. It then proceeded at $\frac{3}{5}$ of its former rate, and arrived $7\frac{2}{3}$ hours behind time. If the accident had occurred 50 miles farther on, the train would have arrived $6\frac{1}{3}$ hours behind time. What was the whole distance traveled by the train?

405. This page contains the questions given in the Elementary Algebra examination of the Regents of the University of the State of New York for June, 1909.

References show where the text provides instruction necessary to answer these questions.

1. Divide $6x^3 + 11x^2 - 1$ by $3x - 1 + 2x^2$ (§§ 38, 108).

2. Find the prime factors of $1 - \frac{x^2}{4}$ (§§ 134, 155), $9a^4 - 90a^3 + 189a^2$ (§§ 133, 142, 155), $a^3 + b^3$ (§ 136), $4x^4 + 3x^2y^2 + 9y^4$ (§ 154), $ax + 4a - 4x - 16$ (§ 145).

3. Solve $\begin{cases} 3x + 8 = 4y + 2, \\ \dfrac{4x}{3} + \dfrac{y}{9} = 3 \end{cases}$ (§§ 231, 232).

Give an axiom justifying each step in the solution (§ 224).

4. Find a number such that if it is added to 1, 4, 9, 16, respectively, the results will form a proportion (§ 403).

5. Solve $\sqrt{x+1} + \sqrt{x-2} = \sqrt{2x+3}$ (§§ 352, 353).

6. Find the square root of $\frac{a^2}{9} + \frac{2ab}{15} - \frac{2a}{3} + \frac{b^2}{25} - \frac{2b}{5} + 1$ (§ 280).

7. Simplify *three* of the following: $\sqrt[3]{-125x^6}$, $\sqrt[5]{\frac{-y^{10}}{32x^{15m}}}$, $\sqrt[3]{108r^4}$ (§ 301), $\sqrt[4]{\frac{4}{9}} \times \sqrt[4]{\frac{16}{27}}$ (§§ 314, 315), $\sqrt{75} - 4\sqrt{243} + 2\sqrt{108}$ (§§ 311–313).

8. Solve $\begin{cases} x^2 + 2xy = 55, \\ 2x^2 - xy = 35. \end{cases}$ (Elimination of similar terms, page 270.)

9. If the speed of a railway train should be lessened 4 miles an hour, the train would be half an hour longer in going 180 miles. Find the rate of the train (§§ 205, 215).

10. If the greater of two numbers is divided by the less, the quotient is 2 and the remainder is 3. The square of the greater number exceeds 6 times the square of the less by 25. Find the numbers (§ 374).

State of New York Regents' examination for June, 1910:

1. Simplify $\dfrac{1+\dfrac{a}{a+1}}{a+\dfrac{1}{1+a}} \div \dfrac{(a+1)^2-a^2}{a^3-1}$ (§ 200).

2. Find the prime factors of *four* of the following: a^4-16 (§§ 134, 149); $20x^2-60xy+45y^2$ (§§ 133, 138); $6ax+10ay-21bx-35by$ (§ 145); $y^2-y+\frac{1}{4}$ (§ 138); $1-\dfrac{x^3}{8}$ (§ 136).

3. Solve $\begin{cases} ax+by=m, \\ bx+ay=n \end{cases}$ (§ 233).

4. Reduce *each* of the following to its simplest form: $(\sqrt{3}+5\sqrt{2})(2\sqrt{3}+3\sqrt{2})$ [§ 315]; $\sqrt{108}\div\sqrt{432}$ (§ 317); $\sqrt{\frac{2}{3}}$ (§ 302); $\sqrt{18}+\sqrt{50}-\sqrt{72}$ (§ 313); $\sqrt[3]{2}\times\sqrt[3]{4}$ (§ 315).

5. It required as many days for a number of men to dig a trench as there were men; if there had been 6 men more the work would have been done in 8 days. Find the number of men (§§ 166, 354).

6. Solve $x^2+5ax=14a^2$ (§§ 350, 351).

7. Solve $\begin{cases} 2x^2+xy=24, \\ x^2-y^2=5 \end{cases}$ (§ 368).

8. The length of a picture at the inner edge of the frame is twice its width; the frame is 4 inches wide and has an area of 328 square inches. Find the dimensions of the picture (§§ 125 and 215 or § 234).

9. Solve $\sqrt{x+7}+\sqrt{x}=7$ (§§ 333, 334).

10. The area of a certain square is $\frac{4}{9}$ of the area of a certain other square and its side is $\frac{1}{4}$ of a yard less. Find the side of each square (§ 374).

11. Two quarts of alcohol are mixed with 5 quarts of water. Find the number of quarts of alcohol that must be added to make the mixture three fourths alcohol (§ 215).

12. Define exponent, surd, term, reciprocal, elimination (§§ 9, 298, 3, 196, 224, respectively, and the glossary).

406. The following questions were asked in the Minnesota High School Board examination for Elementary Algebra in June, 1910.

1. Simplify $a-[-\{-(-3a-\overline{2a-b})\}]$ (§ 116).

2. A's age exceeds B's by 20 years. Ten years ago A was twice as old as B. Find the age of each (§§ 125, 215; also § 234).

3. Factor:

(1) $6bx-15ab-4dx+10ad$ (§ 145).
(2) x^4-16y^4 (§§ 134, 149).
(3) $1-18x-63x^2$ (§ 144).
(4) $64a^6+8$ (§§ 133, 136).
(5) x^3-27 (§ 136).

4. Find the algebraic sum of $\dfrac{x-y}{xy}-\dfrac{x-a}{ax+bx}+\dfrac{y-b}{ay+by}$ (§§ 191, 192).

5. Find the product of $\dfrac{a^3+27}{a^2-16}\times\dfrac{a+4}{a^2-3a+9}$ (§§ 193, 194).

6. Find the quotient of $\dfrac{b-a}{b}\div(a-b)$ (§§ 172, 195–198).

7. Seven men and 5 boys earn \$11.25 per day and at the same wages 4 men and 12 boys earn \$11 per day. Find the wages of each per day (§§ 104, 234).

8. Simplify $\sqrt{16a}+\sqrt{81a}+\sqrt{144a^2b^2}$ (§ 313).

9. Multiply $4\sqrt{8}-\sqrt{32}+2\sqrt{50}$ by $\sqrt{2}$ (§ 315).

10. The distance from Chicago to Minneapolis is 420 miles. By increasing the speed of a certain train 7 miles per hour, the running time is decreased 2 hours. Find the speed of the train (§ 354).

11. Find two numbers in the proportion of 3 to 5 whose sum is 160 (§ 403).

12. Simplify $\dfrac{2m-3+\dfrac{1}{m}}{\dfrac{2m-1}{m}}$ (§ 200).

Minnesota High School Board examination for May, 1907:

1. Define any five: Algebra, term, root, power, binomial, reciprocal (glossary).

2. From the sum of $a^3+2\,a^2b+5\,ab^2+3\,b^3$ and $2\,a^3-3\,a^2b+3\,ab^2-b^3$, take their difference (§§ 62, 63, 66).

3. Express in simplest form:

$$-(4\,x-\overline{7\,y+4})-\{-2\,x-(3\,y-\overline{4\,y-x+4})\}\ (\S\ 116).$$

4. Divide $3\,x^{2n}+13\,x^{2n-1}+15\,x^{2n-2}+9\,x^{2n-3}$ by $x^n+3\,x^{n-1}$ (§ 108).

5. Factor any five: (*a*) $x^{2n}-10\,x^n+16$ (§§ 142, 159); (*b*) $x^{12}-y^{12}$ (§§ 134, 136, 149); (*c*) $12\,x^2+2\,xy-2\,y^2$ (§§ 133, 144); (*d*) $x^3+15\,x^2-x-15$ (§§ 145, 134); (*e*) $a^2-16+b^2+2\,ab$ (§ 151); (*f*) $x^4+4\,y^4$ (§§ 153, 154).

6. Expand by the binomial formula, showing all the steps:

$$(2\,a-x)^5\ [\S\S\ 265,\ 266].$$

7. Solve for x:

$$\begin{cases}\dfrac{x+8}{2}=2-\dfrac{3\,y-x}{6},\\[2ex]\dfrac{2\,x+y}{2}-\dfrac{9\,x-7}{8}=\dfrac{7\,y-4\,x+36}{16}\end{cases}\ (\S\S\ 231,\ 232).$$

8. One half of A's money is equal to B's, and five eighths of B's is equal to C's; together they have \$1450. How much has each? (§§ 47, 75, 125, 205, 215).

9. Simplify $\dfrac{\dfrac{1}{1-a}+\dfrac{1}{1+x}}{\dfrac{1}{1-a}-\dfrac{1}{1+x}}$ (§ 200).

10. A man bought a suit of clothes for \$24 and paid for it in two-dollar bills and fifty-cent pieces, giving twice as many coins as bills. How many bills did he give? (§§ 125, 215, 234).

11. Five years ago the sum of the ages of A and B was 40 years. B is now four times as old as A. What is the present age of each? (§§ 125, 215; also § 234).

FACTORS AND MULTIPLES

407. This chapter gives a brief treatment of highest common factor (§ 183) and lowest common multiple (§ 189) for the benefit of any who may desire a little more work in these topics than their application affords in fractions, the only place in elementary algebra where they are applied.

HIGHEST COMMON FACTOR

408. An expression that is a factor of each of two or more expressions is called a **common factor** of them.

409. The common factor of two or more expressions that has the largest numerical coefficient and is of the highest degree is called their **highest common factor.**

The common factors of $4\,a^3b^2$ and $6\,a^2b$ are 2, a, b, a^2, $2\,a$, $2\,b$, $2\,a^2$, ab, $2\,ab$, a^2b, and $2\,a^2b$ with sign $+$ or $-$. Of these, $2\,a^2b$ (or $-2\,a^2b$) has the largest numerical coefficient and is of the highest degree, and is therefore the highest common factor.

The highest common factor may be positive or negative, but usually only the positive sign is taken.

The highest common factor of two or more expressions is equal to the product of all their common prime factors.

410. Expressions that have no common prime factor, except 1, are said to be **prime to each other.**

EXERCISES

411. 1. Find the h. c. f. of $12\,a^4b^2c$ and $32\,a^2b^3c^3$.

SOLUTION

The arithmetical greatest common divisor or highest common factor of 12 and 32 is 4. The highest common factor of a^4b^2c and of $a^2b^3c^3$ is a^2b^2c. Hence, h. c. f. $=4\,a^2b^2c$.

RULE. — *To the greatest common divisor of the numerical coefficients annex each common literal factor with the least exponent it has in any of the expressions.*

Find the highest common factor of:

2. $10\,x^3y^2$, $10\,x^2y^3$, and $15\,xy^4z$.
3. $70\,a^6b^3$, $21\,a^4b^4$, and $35\,a^4b^5$.
4. $8\,m^7n^3$, $28\,m^6n^4$, and $56\,m^5n^2$.
5. $4\,b^3cd$, $6\,b^2c^2$, and $24\,abc^3$.
6. $3(a+b)^2$ and $6(a+b)^3$.
7. $6(a+b)^2$ and $4(a+b)(a-b)$.
8. $12(a-x)^3$, $6(a-x)^2$, and $(a-x)^4$.
9. $10(x-y)^4z^3$ and $15(z-y)(x-y)^3$.
10. What is the h.c.f. of $3\,x^3-3\,xy^2$ and $6\,x^3-12\,x^2y+6\,xy^2$?

PROCESS

$$\begin{aligned} 3\,x^3-3\,xy^2 &= 3\,x(x+y)(x-y) \\ 6\,x^3-12\,x^2y+6\,xy^2 &= 2\cdot 3\,x(x-y)(x-y) \\ \hline \therefore \text{h.c.f.} &= 3\,x(x-y) \end{aligned}$$

EXPLANATION. — For convenience in selecting the common factors, the expressions are resolved into their simplest factors.

Since the only common prime factors are 3, x, and $(x-y)$, the highest common factor sought (§ 409) is their product $3\,x(x-y)$.

Find the highest common factor of:

11. a^2-x^2 and $a^2-2\,ax+x^2$.
12. a^2-b^2 and $a^2+2\,ab+b^2$.
13. x^3+y^3 and $x^2+2\,xy+y^2$.
14. $x^2-2\,x-15$ and x^2-x-20.
15. $a^2+7\,a+12$ and $a^2+5\,a+6$.

Find the highest common factor of:

16. $x^4+x^2y^2+y^4$ and x^2+xy+y^2.
17. x^3+y^3, x^5+y^5, and x^2y+xy^2.
18. $a^4+a^2b^4+b^8$ and $3\,a^2-3\,ab^2+3\,b^4$.
19. $ax-y+xy-a$ and ax^2+x^2y-a-y.
20. a^2b-b-a^2c+c and $ab-ac-b+c$.
21. $1-4\,x^2$, $1+2\,x$, and $4\,a-16\,ax^2$.
22. $24\,x^3y^3+8\,x^2y^3$ and $8\,x^3y^3-8\,x^2y^3$.
23. $6\,x^2+x-2$ and $2\,x^2-11\,x+5$.
24. $17\,abc^3d^5-51\,a^3bc^4d^4$ and $abc^2d^2-3\,a^3bc^3d$.
25. $38\,xyz-95\,x^3yz^2$ and $34\,xy^2z-85\,x^2yz^2$.
26. $6\,r^7+10\,r^6s-4\,r^5s^2$ and $2\,r^7+2\,r^6s-4\,r^5s^2$.
27. $x^4-x^3-2\,x^2$, $x^4-2\,x^3-3\,x^2$, and $x^4-3\,x^3-4\,x^2$.
28. $7\,l^3t^3+35\,l^2t^3+42\,lt^3$ and $7\,l^4t^3+21\,l^3t^3-28\,l^2t^3-84\,lt^3$.
29. $x^2+a^2-b^2+2\,ax$, $x^2-a^2+b^2+2\,bx$, and $x^2-a^2-b^2-2\,ab$
30. $x^2-6\,x+5$ and $x^3-5\,x^2+7\,x-3$.

SUGGESTION. — Apply the factor theorem to the second expression.

31. $x^3-4\,x+3$ and $x^3+x^2-37\,x+35$.
32. $9-n^2$ and n^2-n-6.

SUGGESTION. — Change $9-n^2$ to $-(n^2-9)=-(n+3)(n-3)$.

33. $1-x^2$ and $x^3-6\,x^2-9\,x+14$.
34. $(9-x^2)^2$ and $x^3+2\,x^2-9\,x-18$.

SUGGESTION. $(9-x^2)^2=(x^2-9)^2$.

35. $(4-c^2)^2$ and $c^3+9\,c^2+26\,c+24$.
36. $xy-y^2$, $-(y^3-x^2y)$, and x^2y-xy^2.
37. $16-s^4$, $2\,s-s^2$, and $s^2-4\,s+4$.
38. y^4-x^4, x^5+y^5, and $y^2+2\,yx+x^2$.
39. $(y-x)^2(n-m)^3$ and $(x^2y-y^3)(m^2n-2\,mn^2+n^3)$.

LOWEST COMMON MULTIPLE

412. An expression that exactly contains each of two or more given expressions is called a **common multiple** of them.

$6abx$ is a common multiple of a, $3b$, $2x$, and $6abx$. These numbers may have other common multiples, as $12abx$, $6a^2b^2x$, $18a^3bx^2$, etc.

413. The expression having the smallest numerical coefficient and of *lowest degree* that will exactly contain each of two or more given expressions is called their **lowest common multiple.**

$6abx$ is the lowest common multiple of a, $3b$, $2x$, and $6abx$.

The lowest common multiple may have either sign + or −, though usually only the positive sign is taken.

The lowest common multiple of two or more expressions is equal to the product of all their different prime factors, each factor being used the greatest number of times it occurs in any of the expressions.

EXERCISES

414. 1. What is the l. c. m. of $12x^2yz^4$, $6a^2xy^2$, and $8axyz^2$?

SOLUTION

The lowest common multiple of the numerical coefficients is found as in arithmetic. It is 24.

The literal factors of the lowest common multiple are each letter with the highest exponent it has in any of the given expressions. They are, therefore, a^2, x^2, y^2, and z^4.

The product of the numerical and literal factors, $24a^2x^2y^2z^4$, is the lowest common multiple of the given expressions.

Find the lowest common multiple of:

2. a^3x^2y, a^2xy^3, and ax^2y.
3. $10a^2b^2c^2$, $5ab^2c$, and $25b^3c^3d^3$.
4. $16a^2b^3c$, $24c^3de$, and $36a^4b^2d^2e^3$.
5. $18a^2br^2$, $12p^2q^2r$, and $54ab^2p^3q$.

6. What is the l. c. m. of $x^2-2xy+y^2$, y^2-x^2, and x^3+y^3?

PROCESS

$$\begin{aligned}
x^2-2xy+y^2 &= (x-y)(x-y)\\
y^2-x^2 = -(x^2-y^2) &= -(x+y)(x-y)\\
x^3+y^3 &= (x+y)(x^2-xy+y^2)\\
\hline
\therefore \text{l. c. m.} &= (x-y)^2(x+y)(x^2-xy+y^2)\\
&= (x-y)^2(x^3+y^3)
\end{aligned}$$

RULE. — *Factor the expressions into their prime factors.*

Find the product of all their different prime factors, using each factor the greatest number of times it occurs in any of the given expressions.

The factors of the l. c. m. may often be selected without separating the expressions into their prime factors.

Find the lowest common multiple of:

7. x^2-y^2 and $x^2+2xy+y^2$.
8. x^2-y^2 and $x^2-2xy+y^2$.
9. x^2-y^2, $x^2+2xy+y^2$, and $x^2-2xy+y^2$.
10. a^2-n^2 and $3a^3+6a^2n+3an^2$.
11. x^4-1 and $a^2x^2+a^2-b^2x^2-b^2$.
12. a^2+1, $ab-b$, a^2+a, and $1-a^2$.
13. $2x+y$, $2xy-y^2$, and $4x^2-y^2$.
14. $1+x$, $x-x^2$, $1+x^2$, and $x^2(1-x)$.
15. $2x+2$, $5x-5$, $3x-3$, and x^2-1.
16. $16b^2-1$, $12b^2+3b$, $20b-5$, and $2b$.
17. $1-2x^2+x^4$, $(1-x)^2$, and $1+2x+x^2$.
18. b^2-5b+6, $b^2-7b+10$, and $b^2-10b+16$.

Find the lowest common multiple of:

19. x^2+7x-8, x^2-1, $x+x^2$, and $3ax^2-6ax+3a$.

20. x^2-a^2, $a-2x$, a^2+2ax, and $a^3-3a^2x+2ax^2$.

21. m^3-x^3, m^2+mx, m^2+mx+x^2, and $(m+x)x^2$.

22. $2-3x+x^2$, x^2+4x+4, x^2+3x+2, and $1-x^2$.

23. x^2-y^2, $x^4+x^2y^2+y^4$, x^3+y^3, and x^2+xy+y^2.

24. $x^3+x^2y+xy^2+y^3$ and $x^3-x^2y+xy^2-y^3$.

25. a^2+4a+4, a^2-4, $4-a^2$, and a^4-16.

26. $a^2-(b+c)^2$, $b^2-(c+a)^2$, and $c^2-(a+b)^2$.

27. $2(ax^2-x^3)^2$, $3x(a^2x-x^3)^3$, and $6(a^2x^2-a^4)$.

28. $(yz^2-xyz)^2$, $y^2(xz^2-x^3)$, and $x^2z^2+2xz^3+z^4$.

Suggestion. — In solving the following, use the factor theorem.

29. $x^3-6x^2+11x-6$ and $x^3-9x^2+26x-24$.

30. $x^3-5x^2-4x+20$ and $x^3+2x^2-25x-50$.

31. x^3-4x^2+5x-2 and $x^3-8x^2+21x-18$.

32. x^3+5x^2+7x+3 and $x^3-7x^2-5x+75$.

33. x^3+2x^2-4x-8, $x^3-x^2-8x+12$, $x^3+4x^2-3x-18$.

GLOSSARY

Abscissa. A distance measured along or parallel to the x-axis.

Absolute Term. A term that does not contain an unknown number.

Absolute Value. The value of a number without regard to its sign.

Addends. Numbers to be added.

Addition. The process of finding a simple expression for the algebraic sum of two or more numbers.

Affected Quadratic. A quadratic equation that contains both the second and first powers of one unknown number.

Algebra. That branch of mathematics which treats of general numbers and the nature and use of equations. It is an extension of arithmetic and it uses both figures and letters to express numbers.

Algebraic Expression. A number represented by algebraic symbols.

Algebraic Numbers. Positive and negative numbers, whether integers or fractions.

Algebraic Sum. The result of adding two or more algebraic numbers.

Antecedent. The first term of a ratio.

Arrangement. When a polynomial is arranged so that in passing from left to right the several powers of some letter are successively *higher* or *lower*, the polynomial is said to be **arranged** according to the **ascending** or **descending powers**, respectively, of that letter.

Axes of Reference. Two straight lines that intersect, usually at right angles, used to locate a point or points in a plane.

Axiom. A principle so simple as to be self-evident.

Binomial. An algebraic expression of two terms.

Binomial Formula. The formula or principle by means of which any indicated power of a binomial may be expanded.

Binomial Quadratic Surd. A binomial surd whose surd or surds are of the second order.