EQUATIONS IN QUADRATIC FORM

358. An equation that contains but two powers of an unknown number or expression, the exponent of one power being twice that of the other, as $ax^{2n} + bx^n + c = 0$, in which n represents any number, is in the **quadratic form**.

EXERCISES

359. 1. Given $x^4 - 10x^2 + 24 = 0$, to find the value of x.

SOLUTION

$$x^4 - 10x^2 + 24 = 0.$$

Factoring, $(x^2 - 4)(x^2 - 6) = 0.$

Hence, $x^2 - 4 = 0$ or $x^2 - 6 = 0$;

whence, $x = \pm 2$ or $x = \pm\sqrt{6}.$

Each of these values substituted in the given equation is found to verify; hence, ± 2 and $\pm\sqrt{6}$ are roots of the equation.

2. Given $x^{\frac{1}{2}} - x^{\frac{1}{4}} - 6 = 0$, to find the values of x.

FIRST SOLUTION

$$x^{\frac{1}{2}} - x^{\frac{1}{4}} - 6 = 0.$$

Let $x^{\frac{1}{4}} = m$, then $x^{\frac{1}{2}} = m^2$ and the equation becomes

$$m^2 - m - 6 = 0.$$

Solving by factoring, $m = 3$ or -2;

that is, $x^{\frac{1}{4}} = 3$ or $-2.$

Raising to the fourth power, $x = 81$ or $16.$

SECOND SOLUTION

Transposing, $x^{\frac{1}{2}} - x^{\frac{1}{4}} = 6.$

Completing the square, $x^{\frac{1}{2}} - x^{\frac{1}{4}} + (\frac{1}{2})^2 = \frac{25}{4}.$

Taking the square root, $x^{\frac{1}{4}} - \frac{1}{2} = \pm\frac{5}{2}.$

$\therefore x^{\frac{1}{4}} = 3$ or $-2.$

Raising to the fourth power, $x = 81$ or $16.$

Since $x = 16$ does not satisfy the given equation, 16 is not a root and should be rejected.

Solve, and verify each result:

3. $x^4 - 13x^2 + 36 = 0.$
4. $x^4 - 18x^2 + 32 = 0.$
5. $x^4 - 14x^2 + 45 = 0.$
6. $2x^4 - 10x^2 + 8 = 0.$
7. $3x^4 - 11x^2 + 8 = 0.$
8. $x^{\frac{1}{2}} - 5x^{\frac{1}{4}} + 6 = 0.$
9. $x^{\frac{2}{3}} - x^{\frac{1}{3}} = 6.$
10. $x + 2\sqrt{x} = 3.$
11. $x^{\frac{2}{3}} - x^{\frac{1}{3}} - 12 = 0.$
12. $(x - 3)^2 + 2(x - 3) = 3.$
13. $(x^2 - 1)^2 - 4(x^2 - 1) = 5.$
14. $(x^2 - 4)^2 - 3(x^2 - 4) = 10.$

15. Solve the equation $x^2 - 7x + \sqrt{x^2 - 7x + 18} = 24.$

SOLUTION

$$x^2 - 7x + \sqrt{x^2 - 7x + 18} = 24. \quad (1)$$

Adding 18 to both members, we have

$$x^2 - 7x + 18 + \sqrt{x^2 - 7x + 18} = 42. \quad (2)$$

Put m for $\sqrt{x^2 - 7x + 18}$ and m^2 for $x^2 - 7x + 18$.

Then, transposing, $m^2 + m - 42 = 0.$ (3)

Solving by factoring, $m = 6$ or $-7.$ (4)

That is, $\sqrt{x^2 - 7x + 18} = 6,$ (5)

or $\sqrt{x^2 - 7x + 18} = -7.$ (6)

Squaring (5), $x^2 - 7x + 18 = 36.$

Solving, $x = 9$ or $-2.$

Since, in accordance with § 293, the radical in (6) cannot equal a negative number, $\sqrt{x^2 - 7x + 18} = -7$ is an impossible equation.

Hence, the only roots of (1) are 9 and -2.

16. Solve the equation $x + 2\sqrt{x+3} = 21$, and verify results.

17. Solve $x^2 - 3x + 2\sqrt{x^2 - 3x + 6} = 18$, and verify results.

18. Solve the equation $x^6 - 9x^3 + 8 = 0$.

SOLUTION

$$x^6 - 9x^3 + 8 = 0. \quad (1)$$

Factoring, $(x^3 - 1)(x^3 - 8) = 0.$ (2)

Therefore, $x^3 - 1 = 0,$ (3)

or $x^3 - 8 = 0.$ (4)

If the values of x are found by transposing the known terms in (3) and (4) and then taking the cube root of each member, only *one* value of x will be obtained from each equation. But if the equations are factored, *three* values of x are obtained for each.

Factoring (3), $(x-1)(x^2 + x + 1) = 0,$ (5)

and (4), $(x - 2)(x^2 + 2x + 4) = 0.$ (6)

Writing each factor equal to zero, and solving, we have:

From (5), $x = 1, \frac{1}{2}(-1 + \sqrt{-3}), \frac{1}{2}(-1 - \sqrt{-3}).$ (7)

From (6), $x = 2, -1 + \sqrt{-3}, -1 - \sqrt{-3}.$ (8)

NOTE. — Since the values of x in (7) are obtained by factoring $x^3 - 1 = 0$, they may be regarded as the *three cube roots of the number* 1. Also, the values of x in (8) may be regarded as the *three cube roots of the number* 8 (§ 271).

Solve:

19. $x^6 - 28x^3 + 27 = 0.$

20. $x^4 - 16 = 0.$

21. Find the three cube roots of -1.

22. Find the three cube roots of -8.

23. Solve the equation $x^4 + 4x^3 - 8x + 3 = 0$.

SOLUTION

By applying the factor theorem (§ 146), the factors of the first member are found to be $x - 1$, $x + 3$, and $x^2 + 2x - 1$; that is,

$$(x - 1)(x + 3)(x^2 + 2x - 1) = 0.$$

Solving, $x = 1, -3, -1 \pm \sqrt{2}.$

Solve, and verify each result:

24. $x^3 + x^2 - 4x = -2.$

25. $x^4 - 4x^3 + 8x = -3.$

26. $x^4 - 8x^2 + 5x + 6 = 0.$

27. $x^4 - 6x^3 + 27x = 10.$

28. $x^4 + 6x^3 + 7x^2 - 6x - 8 = 0.$

29. $x^4 + 2x^3 - 10x^2 - 11x + 30 = 0.$

MISCELLANEOUS EXERCISES

360. Solve, and verify each result:

1. $x^4 - 25x^2 + 144 = 0.$
2. $x^4 - 45x^2 + 500 = 0.$
3. $2x^4 - 11x^2 + 12 = 0.$
4. $5x^4 - 24x^2 + 16 = 0.$
5. $x^{\frac{1}{2}} - 2x^{\frac{1}{4}} - 3 = 0.$
6. $x^{\frac{1}{3}} - 3x^{\frac{1}{6}} = -2.$
7. $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} = 3.$
8. $x^{\frac{4}{3}} - 10x^{\frac{2}{3}} + 9 = 0.$
9. $(x^2 - 1)^2 - 4(x^2 - 1) + 3 = 0.$
10. $(x^2 - 6)^2 - 7(x^2 - 6) - 30 = 0.$
11. $(x^2 - 2x)^2 - 2(x^2 - 2x) = 3.$
12. $x - 6x^{\frac{1}{2}} + 8 = 0.$
13. $x + 20 - 9\sqrt{x} = 0.$
14. $2x^{\frac{1}{2}} - 3x^{\frac{1}{4}} + 1 = 0.$
15. $x^{\frac{2}{3}} - 7x^{\frac{1}{3}} + 10 = 0.$
16. $x - 5 + 2\sqrt{x - 5} = 8.$
17. $2x - 3\sqrt{2x + 5} = -5.$
18. $2x - 6\sqrt{2x - 1} = 8.$
19. $x - 3 = 21 - 4\sqrt{x - 3}.$
20. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$
21. $4x^4 - 4x^3 - 7x^2 + 4x + 3 = 0.$
22. $16x^4 - 8x^3 - 31x^2 + 8x + 15 = 0.$
23. $x^2 - x - \sqrt{x^2 - x + 4} - 8 = 0.$
24. $x^2 - 5x + 2\sqrt{x^2 - 5x - 2} = 10.$

SIMULTANEOUS QUADRATIC EQUATIONS

361. Two simultaneous *quadratic* equations involving two unknown numbers generally lead to equations of the fourth degree, and therefore they cannot usually be solved by quadratic methods.

However, there are some simultaneous equations involving quadratics that may be solved by quadratic methods, as shown in the following cases.

362. When one equation is simple and the other of higher degree.

Equations of this class may be solved by finding the value of one unknown number in terms of the other in the simple equation, and then substituting that value in the other equation.

EXERCISES

363. 1. Solve the equations $\begin{cases} x+y=7, \\ 3x^2+y^2=43. \end{cases}$

SOLUTION

	$x+y=7.$	(1)
	$3x^2+y^2=43.$	(2)
From (1),	$y=7-x.$	(3)
Substituting in (2),	$3x^2+(7-x)^2=43.$	(4)
Solving,	$x=3$ or $\frac{1}{2}$.	(5)
Substituting 3 for x in (3),	$y=4.$	(6)
Substituting $\frac{1}{2}$ for x in (3),	$y=\frac{13}{2}.$	(7)

That is, x and y each have two values $\begin{cases} \text{when } x=3,\ y=4, \\ \text{when } x=\frac{1}{2},\ y=\frac{13}{2}. \end{cases}$

Solve, and verify results:

2. $\begin{cases} x^2+y^2=20, \\ x=2y. \end{cases}$

3. $\begin{cases} 10x+y=3xy, \\ y-x=2. \end{cases}$

4. $\begin{cases} x=6-y, \\ x^3+y^3=72. \end{cases}$

5. $\begin{cases} xy(x-2y)=10, \\ xy=10. \end{cases}$

6. $\begin{cases} x^2+xy=12, \\ x-y=2. \end{cases}$

7. $\begin{cases} m^2-3n^2=13, \\ m-2n=1. \end{cases}$

8. $\begin{cases} 3x(y+1)=12, \\ 3x=2y. \end{cases}$

9. $\begin{cases} 3rs-10r=s, \\ 2-s=-r. \end{cases}$

364. An equation that is not affected by interchanging the unknown numbers involved is called a **symmetrical equation.**

$2x^2+xy+2y^2=4$ and $x^2+y^2=9$ are symmetrical equations.

365. When both equations are symmetrical.

Though equations of this class may usually be solved by substitution, as in §§ 362, 363, it is preferable to find values for $x+y$ and $x-y$ and then solve these simple equations for x and y.

EXERCISES

366. 1. Solve the equations $\begin{cases} x+y=7, \\ xy=10. \end{cases}$

SOLUTION

	$x+y=7.$	(1)
	$xy=10.$	(2)
Squaring (1),	$x^2+2xy+y^2=49.$	(3)
Multiplying (2) by 4,	$4xy=40.$	(4)
Subtracting (4) from (3),	$x^2-2xy+y^2=9.$	(5)
Taking the square root,	$x-y=\pm 3.$	(6)
From (1) + (6),	$x=5$ or $2.$	
From (1) − (6),	$y=2$ or $5.$	

2. Solve the equations $\begin{cases} x^2+y^2=25, \\ x+y=7. \end{cases}$

SOLUTION

	$x^2+y^2=25.$	(1)
	$x+y=7.$	(2)
Squaring (2),	$x^2+2xy+y^2=49.$	(3)
Subtracting (1) from (3),	$2xy=24.$	(4)
Subtracting (4) from (1),	$x^2-2xy+y^2=1.$	(5)
Taking the square root,	$x-y=\pm 1.$	(6)
From (2) + (6),	$x=4$ or $3.$	
From (2) − (6),	$y=3$ or $4.$	

Solve, and verify each result:

3. $\begin{cases} x^2+y^2=50, \\ xy=7. \end{cases}$

4. $\begin{cases} x+y=8, \\ x^2+y^2=34. \end{cases}$

5. $\begin{cases} x+y=9, \\ x^3+y^3=243. \end{cases}$

6. $\begin{cases} x+y=8, \\ x^2+xy+y^2=49. \end{cases}$

7. $\begin{cases} x^2+xy+y^2=31, \\ x^2+y^2=26. \end{cases}$

8. $\begin{cases} x^2+y^2=8, \\ x^2-xy+y^2=4. \end{cases}$

9. $\begin{cases} xy=12, \\ x-xy+y=-5. \end{cases}$

10. $\begin{cases} x^2+3xy+y^2=31, \\ xy=6. \end{cases}$

11. $\begin{cases} x^2+y^2=100, \\ (x+y)^2=196. \end{cases}$

12. $\begin{cases} x^2+y^2=13, \\ x^2+y^2+xy=19. \end{cases}$

367. An equation *all* of whose terms are of the same degree with respect to the unknown numbers is called a **homogeneous equation**.

$x^2-xy=y^2$ and $3x^3+y^3=0$ are homogeneous equations.

An equation like $x^2-xy+y^2=21$ is said to be **homogeneous in the unknown terms**.

368. When both equations are quadratic and homogeneous in the unknown terms.

Substitute vy for x, solve for y^2 in each equation, and compare the values of y^2 thus found, forming a quadratic in v.

EXERCISES

369. 1. Solve the equations $\begin{cases} x^2-xy+y^2=21, \\ y^2-2xy=-15. \end{cases}$

SOLUTION

	$x^2-xy+y^2=21.$	(1)
	$y^2-2xy=-15.$	(2)
Assume	$x=vy.$	(3)
Substituting in (1),	$v^2y^2-vy^2+y^2=21.$	(4)
Substituting in (2),	$y^2-2vy^2=-15.$	(5)
Solving (4) for y^2,	$y^2=\frac{21}{v^2-v+1}.$	(6)
Solving (5) for y^2,	$y^2=\frac{15}{2v-1}.$	(7)
Comparing the values of y^2,	$\frac{15}{2v-1}=\frac{21}{v^2-v+1}.$	(8)
Clearing, etc.,	$5v^2-19v+12=0.$	(9)
Factoring,	$(v-3)(5v-4)=0.$	(10)
	$\therefore v=3$ or $\frac{4}{5}.$	(11)
Substituting 3 for v in (7) or in (6), and since $x=vy$,	$y=\pm\sqrt{3}$, $x=\pm 3\sqrt{3}.$	(12)
Substituting $\frac{4}{5}$ for v in (7) or in (6), and since $x=vy$,	$y=\pm 5$, $x=\pm 4.$	(13)

When the double sign is used, as in (12) and in (13), it is understood that the roots shall be associated by taking the *upper* signs together and the *lower* signs together.

Hence, $\begin{cases} x=3\sqrt{3}; & -3\sqrt{3}; & 4; -4; \\ y=\sqrt{3}; & -\sqrt{3}; & 5; -5. \end{cases}$

Solve, and verify results:

2. $\begin{cases} x^2+3y^2=84, \\ xy-y^2=8. \end{cases}$

3. $\begin{cases} 2x^2-3xy+2y^2=100, \\ x^2+y^2=125. \end{cases}$

4. $\begin{cases} x^2 - xy + y^2 = 21, \\ x^2 + 2y^2 = 27. \end{cases}$

5. $\begin{cases} x(x-y) = 6, \\ x^2 + y^2 = 5. \end{cases}$

6. $\begin{cases} xy + 3y^2 = 20, \\ x^2 - 3xy = -8. \end{cases}$

7. $\begin{cases} x^2 + xy = 12, \\ xy + 2y^2 = 5. \end{cases}$

8. $\begin{cases} x^2 + 2y^2 = 44, \\ xy - y^2 = 8. \end{cases}$

9. $\begin{cases} x^2 + xy = 77, \\ xy - y^2 = 12. \end{cases}$

370. Special devices.

Many systems of simultaneous equations that belong to one or more of the preceding classes, and many that belong to none of them, may be solved readily by *special devices*. It is impossible to lay down any fixed line of procedure, but the object often aimed at is to find values for *any two* of the expressions, $x+y$, $x-y$, and xy, from which the values of x and y may be obtained. Various manipulations are resorted to in attaining this object, according to the forms of the given equations.

EXERCISES

371. 1. Solve the equations $\begin{cases} x^2 + xy = 12, \\ xy + y^2 = 4. \end{cases}$

SOLUTION

$$x^2 + xy = 12. \quad (1)$$

$$xy + y^2 = 4. \quad (2)$$

Adding (1) and (2), $x^2 + 2xy + y^2 = 16.$ (3)

$\therefore x + y = +4$ or $-4.$ (4)

Subtracting (2) from (1), $x^2 - y^2 = 8.$ (5)

Dividing (5) by (4), $x - y = +2$ or $-2.$ (6)

Combining (4) and (6), $x = 3$ or -3; $y = 1$ or -1.

NOTE. — The first value of $x - y$ corresponds only to the first value of $x + y$, and the second value only to the second value.

Consequently, there are only two pairs of values of x and y.

Observe that the given equations belong to the class treated in § 368. The special device adopted here, however, gives a much neater and simpler solution than the method presented in that case.

2. Solve the equations $\begin{cases} x^2 + y^2 + x + y = 14, \\ xy = 3. \end{cases}$

SOLUTION

$$x^2 + y^2 + x + y = 14. \quad (1)$$

$$xy = 3. \quad (2)$$

Adding twice the second equation to the first,

$$x^2 + 2xy + y^2 + x + y = 20.$$

Completing the square, $(x+y)^2 + (x+y) + (\tfrac{1}{2})^2 = 20\tfrac{1}{4}.$

Taking the square root, $x + y + \tfrac{1}{2} = \pm \tfrac{9}{2}.$

$\therefore x + y = 4$ or $-5.$ (3)

Equations (2) and (3) give two pairs of simultaneous equations,

$$\begin{cases} x + y = 4 \\ xy = 3 \end{cases} \text{ and } \begin{cases} x + y = -5 \\ xy = 3 \end{cases}$$

Solving, the corresponding values of x and y are found to be

$$\begin{cases} x = 3;\ 1;\ \tfrac{1}{2}(-5+\sqrt{13});\ \tfrac{1}{2}(-5-\sqrt{13}); \\ y = 1;\ 3;\ \tfrac{1}{2}(-5-\sqrt{13});\ \tfrac{1}{2}(-5+\sqrt{13}). \end{cases}$$

Symmetrical except as to sign. — When one of the equations is symmetrical and the other would be symmetrical if one or more of its signs were changed, or when both equations are of the latter type, the system may be solved by the methods used for symmetrical equations (§ 365).

3. Solve the equations $\begin{cases} x^2 + y^2 = 53, \\ x - y = 5. \end{cases}$

SOLUTION

$$x^2 + y^2 = 53. \quad (1)$$

$$x - y = 5. \quad (2)$$

Squaring (2), $x^2 - 2xy + y^2 = 25.$ (3)

Subtracting (3) from (1), $2xy = 28.$ (4)

Adding (4) and (1), $x^2 + 2xy + y^2 = 81.$ (5)

Taking the square root, $x + y = \pm 9.$ (6)

From (6) and (2), $x = 7$ or -2;

and $y = 2$ or -7.

Division of one equation by the other. — The reduction of equations of higher degree to quadratics is often effected by dividing one of the given equations by the other, *member by member*.

4. Solve the equations $\begin{cases} x^3 - y^3 = 26, \\ x - y = 2. \end{cases}$

SUGGESTION. — Dividing the first equation by the second,

$$x^2 + xy + y^2 = 13.$$

Therefore, solve the system,

$$\begin{cases} x^2 + xy + y^2 = 13, \\ x - y = 2, \end{cases}$$

instead of the given system.

Elimination of similar terms. — It is often advantageous to eliminate similar terms by addition or subtraction, just as in simultaneous simple equations.

5. Solve the equations $\begin{cases} xy + x = 35, \\ xy + y = 32. \end{cases}$

SOLUTION

$$xy + x = 35. \quad (1)$$

$$xy + y = 32. \quad (2)$$

Subtracting (2) from (1), $x - y = 3$;

whence, $y = x - 3. \quad (3)$

Substituting (3) in (1), $x(x - 3) + x = 35$,

or $x^2 - 2x = 35$.

Solving, $x = 7$ or $-5. \quad (4)$

Substituting (4) in (3), $y = 4$ or -8.

Solve, using the methods illustrated in exercises 1–5; verify:

6. $\begin{cases} m^2 + mn = 2, \\ mn + n^2 = -1. \end{cases}$

7. $\begin{cases} p^2 + q^2 + p + q = 14, \\ pq = -6. \end{cases}$

8. $\begin{cases} a^2 + b^2 = 130, \\ a - b = 2. \end{cases}$

9. $\begin{cases} r^3 - s^3 = 54, \\ r - s = 6. \end{cases}$

10. $\begin{cases} xy + x = 32, \\ xy + y = 27. \end{cases}$

11. $\begin{cases} 2x^2 - 3y^2 = 5, \\ 3x^2 - 2y^2 = 30. \end{cases}$

372. All the solutions in §§ 362–371 are but illustrations of methods that are important because they are often applicable. The student is urged to use his ingenuity in devising other methods or modifications of these whenever the given system does not yield readily to the devices illustrated, or whenever a simpler solution would result.

MISCELLANEOUS EXERCISES

373. Solve, and verify each result:

1. $\begin{cases} x + y = 3, \\ xy = 2. \end{cases}$

2. $\begin{cases} 5x^2 - 4y^2 = 44, \\ 4x^2 - 5y^2 = 19. \end{cases}$

3. $\begin{cases} 1 + x = y, \\ x^2 + y^2 = 61. \end{cases}$

4. $\begin{cases} x^2 - xy = 48, \\ xy - y^2 = 12. \end{cases}$

5. $\begin{cases} a + ab + 28 = 0, \\ b + ab + 40 = 0. \end{cases}$

6. $\begin{cases} x^2 + y^2 + x + y = 26, \\ xy = -12. \end{cases}$

7. $\begin{cases} x^2 + y^2 = 40, \\ xy = 12. \end{cases}$

8. $\begin{cases} x^3 + y^3 = 28, \\ x + y = 4. \end{cases}$

9. $\begin{cases} xy + x^2 = 44, \\ xy + y^2 = -28. \end{cases}$

10. $\begin{cases} x^2 + 3xy - y^2 = 9, \\ x + 2y = 4. \end{cases}$

11. $\begin{cases} 4xy = 96 - x^2y^2, \\ x + y = 6. \end{cases}$

12. $\begin{cases} x^2 - xy = 8, \\ xy + y^2 = 12. \end{cases}$

13. $\begin{cases} x(x + y) = x, \\ y(x - y) = -1. \end{cases}$

14. $\begin{cases} x^2 + xy + y^2 = 151, \\ x^2 + y^2 = 106. \end{cases}$

15. $\begin{cases} 1 + x = y, \\ 4 + 4x^3 = y^3. \end{cases}$

16. $\begin{cases} x^4 - y^4 = 369, \\ x^2 - y^2 = 9. \end{cases}$

17. $\begin{cases} x^2 - xy = 6, \\ x^2 + y^2 = 61. \end{cases}$

18. $\begin{cases} x + y = 25, \\ \sqrt{x} + \sqrt{y} = 7. \end{cases}$

19. $\begin{cases} x^2 + xy + y^2 = 19, \\ x^3 - y^3 = 19. \end{cases}$

20. $\begin{cases} x^2 + 3xy = y^2 + 9, \\ x + 3y = 5. \end{cases}$

Problems

374. 1. The sum of two numbers is 12, and their product is 32. What are the numbers?

2. The sum of two numbers is 17, and the sum of their squares is 157. What are the numbers?

3. The difference of two numbers is 1, and the difference of their cubes is 91. What are the numbers?

4. The area of a rectangular mirror is 88 square feet and its perimeter is 38 feet. Find its dimensions.

5. The perimeter of a rectangular ginseng bed is 18 rods and its area is $20\frac{1}{4}$ square rods. What are its demensions?

6. It takes 52 rods of fence to inclose a rectangular garden containing 1 acre. How long and how wide is the garden?

7. The product of two numbers is 59 greater than their sum, and the sum of their squares is 170. What are the numbers?

8. If 63 is subtracted from a certain number expressed by two digits, its digits will be transposed; and if the number is multiplied by the sum of its digits, the product will be 729. What is the number?

9. The smallest of the printed muslin flags made in this country has an area of $5\frac{1}{4}$ square inches. If the width were $\frac{1}{4}$ of an inch less, the length would be twice the width. Find the length and width.

10. A man's ice bill was \$18. If ice had cost \$1 less per ton, he would have received $\frac{3}{5}$ of a ton more for the same money. How many tons did he use and what was the price per ton?

11. The car of the airship *America* is 107 feet longer than it is wide. If its length were $\frac{1}{5}$ as great, the area of the floor would be 184 square feet. Find its dimensions.

12. A Chinaman received \$1.60 for rolling joss sticks. Had he been paid 8 cents more for each lot of ten thousand, he would have had to roll one lot less for the same amount of money. How many such lots did he roll and how much was he paid per lot?

13. The area of a wall painting in a restaurant in Philadelphia is 340 square feet. If its length were 12 feet less and its width 12 feet greater, it would be square. Find its length and width.

14. Before the reduction in letter postage between the United States and Great Britain, it cost 10¢ to send a certain letter that now would just go for 4¢. What is the weight of the letter, if the postage was reduced 3¢ an ounce?

15. If each of a farmer's maple trees had yielded 2 pounds more of sugar, he could have made 750 pounds. If he had had 50 trees more, he could have made 600 pounds. Find the number of trees he had and the yield per tree.

16. A woman in Saxony received 1¢ an hour more for making chiffon hats than for weaving straw hats. If she received 21¢ for her work on the former and 20¢ for her work on the latter, working 14 hours in all, how much did she receive per hour for each?

17. An Illinois farmer raised broom corn and pressed the 6120 pounds of brush into bales. If he had made each bale 20 pounds heavier, he would have had 1 bale less. How many bales did he press and what was the weight of each?

18. If the length of the platform of an elevator were 9 feet less and its width 9 feet more, its area would be 361 square feet. If its length were 9 feet more and its width 9 feet less, its area would be 37 square feet. Find its dimensions.

19. A man bought 4 more loads of sand than of gravel, paying \$.50 less per load for sand than for gravel. The sand cost him \$9 and the gravel \$10. What quantities of each did he buy? What prices did he pay?

20. The capital stock of a creamery company is \$8850. If there had been 10 times as many stockholders, each man's share would have been \$45 less. How many stockholders were there and what was each man's share?

21. Mr. Fuller paid \$2.25 for some Italian olive oil, and \$2 for $\frac{1}{2}$ gallon less of French olive oil, which cost \$.50 more per gallon. How much of each kind did he buy and at what price?

22. In papering a room, 18 yards of border were required, while 40 yards of paper $\frac{1}{2}$ yard wide were needed to cover the ceiling exactly. Find the length and breadth of the room.

23. The water surface area of a tank at Washington, D. C., used in testing ship models is 20,210 square feet. If the length were 3 feet greater, it would be 11 times the width. What are the dimensions of the water surface?

24. If the elm beetle had killed 500 trees less one year in Albany, New York, the total estimated loss would have been \$10,000; if the value of each tree had been $\frac{1}{4}$ as much, the total loss would have been \$3750. How many elm trees were killed?

25. The total area of a window screen whose length is 4 inches greater than its width is 10 square feet. The area inside the wooden frame is 8 square feet. Find the width of the frame.

26. A rectangular skating rink together with a platform around it 25 feet wide covered 37,500 square feet of ground. The area of the platform was $\frac{7}{8}$ of the area of the rink. What were the dimensions of the rink?

27. The course for a 36-mile yacht race is the perimeter of a right triangle, one leg of which is 3 miles longer than the other. How long is each side of the course?

28. At simple interest a sum of money amounted to \$2472 in 9 months and to \$2528 in 16 months. Find the amount of money at interest and the rate.

29. Two men working together can complete a piece of work in $6\frac{2}{3}$ days. If it would take one man 3 days longer than the other to do the work alone, in how many days can each man do the work alone?

GRAPHIC SOLUTIONS

QUADRATIC EQUATIONS — TWO UNKNOWN NUMBERS

EXERCISES

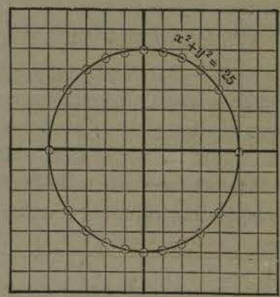
375. 1. Construct the graph of the equation $x^2 + y^2 = 25$.

SOLUTION. — Solving for y, $y = \pm\sqrt{25 - x^2}$.

Since any value numerically greater than 5 substituted for x will make the value of y imaginary, we substitute only values of x between and including -5 and $+5$. The corresponding values of y, or of $\pm\sqrt{25-x^2}$, are recorded in the table below.

It will be observed that each value substituted for x, except ± 5, gives two values of y, and that values of x numerically equal give the same values of y; thus, when $x = 2$, $y = \pm 4.6$, and also when $x = -2$, $y = \pm 4.6$.

x	y
0	± 5
± 1	± 4.9
± 2	± 4.6
± 3	± 4
± 4	± 3
± 5	0



The values given in the table serve to locate twenty points of the graph of $x^2 + y^2 = 25$. Plotting these points and drawing a smooth curve through them, the graph is apparently a *circle*. It may be proved by geometry that this graph is a circle whose radius is 5.

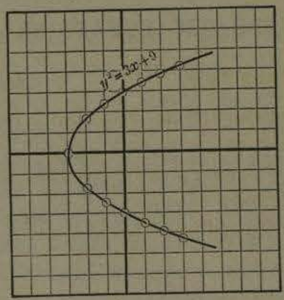
The graph of any equation of the form $x^2 + y^2 = r^2$ is a circle whose radius is r and whose center is at the origin.

2. Construct the graph of the equation $y^2 = 3x + 9$.

SOLUTION. — Solving for y, $y = \pm\sqrt{3x + 9}$.

It will be observed that any value smaller than -3 substituted for x will make y imaginary; consequently, no point of the graph lies to the left of $x = -3$. Beginning with $x = -3$, we substitute values for x and determine the corresponding values of y, as recorded in the table:

x	y
-3	0
-2	± 1.7
-1	± 2.4
0	± 3
1	± 3.5
2	± 3.9
3	± 4.2



Plotting these points and drawing a smooth curve through them, the graph obtained is apparently a *parabola* (§ 356).

The graph of any equation of the form $y^2 = ax + c$ is a **parabola**.

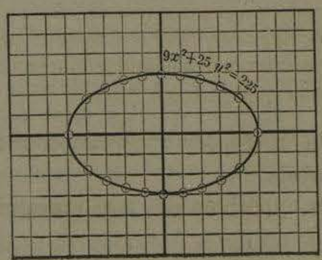
3. Construct the graph of the equation $9x^2 + 25y^2 = 225$.

SOLUTION. — Solving for y, $y = \pm\frac{3}{5}\sqrt{25 - x^2}$.

Since any value numerically greater than 5 substituted for x will make the value of y imaginary, no point of the graph lies farther to the right or to the left of the origin than 5 units; consequently, we substitute for x only values between and including -5 and $+5$.

Corresponding values of x and y are given in the table:

x	y
0	± 3
± 1	± 2.9
± 2	± 2.7
± 3	± 2.4
± 4	± 1.8
± 5	0



Plotting the twenty points tabulated on the preceding page, and drawing a smooth curve through them, we have the graph of $9x^2 + 25y^2 = 225$, which is called an *ellipse*.

The graph of any equation of the form $b^2x^2 + a^2y^2 = a^2b^2$ is an **ellipse**.

4. Construct the graph of the equation $4x^2 - 9y^2 = 36$.

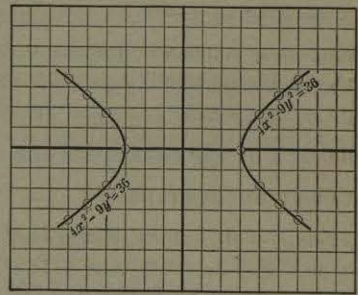
SOLUTION

Solving for y, $\quad y = \pm\frac{2}{3}\sqrt{x^2 - 9}$.

Since any value numerically less than 3 substituted for x will make the value of y imaginary, no point of the graph lies between $x = +3$ and $x = -3$; consequently, we substitute for x only ± 3 and values numerically greater than 3.

Corresponding values of x and y are given in the table:

x	y
± 3	0
± 4	± 1.8
± 5	± 2.7
± 6	± 3.5



Plotting these fourteen points, it is found that half of them are on one side of the y-axis and half on the other side, and since there are no points of the curve between $x = +3$ and $x = -3$, the graph has two separate *branches*, that is, it is *discontinuous*.

Drawing a smooth curve through each group of points, the two branches thus constructed constitute the graph of the equation $4x^2 - 9y^2 = 36$, which is an *hyperbola*.

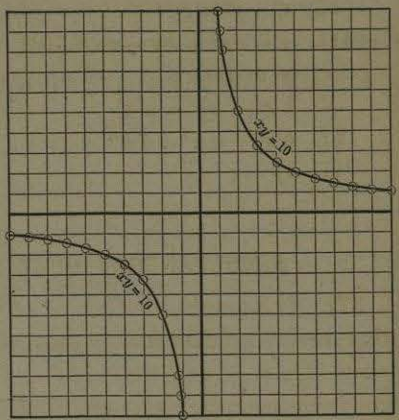
The graph of any equation of the form $b^2x^2 - a^2y^2 = a^2b^2$ is an **hyperbola**.

An hyperbola has two **branches** and is called a **discontinuous** curve.

5. Construct the graph of the equation $xy = 10$.

SOLUTION. — Substituting values for x and solving for y, the corresponding values found are as given in the table:

x	y	x	y
1	10	-1	-10
2	5	-2	-5
3	$3\frac{1}{3}$	-3	$-3\frac{1}{3}$
4	$2\frac{1}{2}$	-4	$-2\frac{1}{2}$
5	2	-5	-2
6	$1\frac{2}{3}$	-6	$-1\frac{2}{3}$
7	$1\frac{3}{7}$	-7	$-1\frac{3}{7}$
8	$1\frac{1}{4}$	-8	$-1\frac{1}{4}$
9	$1\frac{1}{9}$	-9	$-1\frac{1}{9}$
10	1	-10	-1



Plotting these points and drawing a smooth curve through each group of points, the two branches of the curve found constitute the graph of the equation $xy = 10$, which is an *hyperbola*.

The graph of any equation of the form $xy = c$ is an **hyperbola**.

Construct the graph of:

6. $x^2 + y^2 = 9$.
7. $y^2 = 5x + 8$.
8. $9x^2 + 16y^2 = 144$.
9. $9x^2 - 16y^2 = 144$.
10. $xy = 12$.
11. $xy = -6$.

376. Summary. — The types of equations and their respective graphs, here given, will aid the student in plotting graphs, but he will meet other forms of equations that will have some of the same kinds of graphs, the varieties in equations giving rise to varieties in form, size, or location of the graphs.

For example, § 375, exercises 4 and 5 are both equations of the hyperbola, but they are differently located and of different size and shape.

It is possible to determine many characteristics of the various graphs from their equations alone, but a discussion of this is beyond the province of algebra. In the study of graphs, therefore, the student will rely principally on plotting a sufficient number of points to determine their form accurately. The following *types* have been studied:

I.	$ax + by = c$ (§ 248)	**Straight line**
II.	$x^2 + y^2 = r^2$	**Circle**
III.	$y = ax^2 + bx + c$ (§ 356)	**Parabola**
IV.	$y^2 = ax + c$	**Parabola**
V.	$b^2x^2 + a^2y^2 = a^2b^2$	**Ellipse**
VI.	$b^2x^2 - a^2y^2 = a^2b^2$	**Hyperbola**
VII.	$xy = c$	**Hyperbola**

SIMULTANEOUS QUADRATIC EQUATIONS

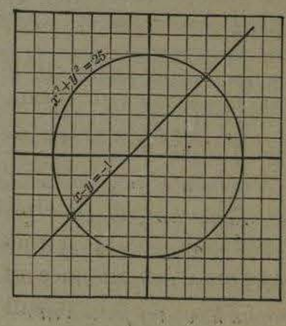
377. The graphic method of solving simultaneous equations that involve quadratics is precisely the same as for simultaneous linear equations (§ 252). Construct the graph of each equation, both graphs being referred to the same axes, and determine the coördinates of the points where the graphs intersect. If they do not intersect, interpret this fact.

NOTE. — The student should construct the following graphs for himself. Roots are expected to the nearest tenth of a unit. To obtain this degree of accuracy, numerous points should be plotted and a scale of about $\frac{1}{2}$ inch to 1 unit should be used.

EXERCISES

378. 1. Solve graphically $\begin{cases} x^2 + y^2 = 25, \\ x - y = -1. \end{cases}$

SOLUTION. — Constructing the graphs of these equations, we find the first, as in § 375, exercise 1, to be a circle; and the second, as in § 248, a straight line.



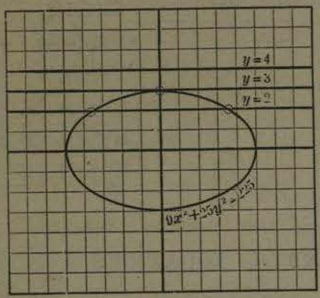
The line intersects the circle in two points $(-4, -3)$ and $(3, 4)$.

Hence, there are two solutions

$x = -4,\ y = -3$ and $x = 3,\ y = 4$.

TEST. — The student may test the roots found graphically by performing the algebraic solution.

2. Solve graphically $\begin{cases} 9x^2 + 25y^2 = 225, \\ y = 2. \end{cases}$



SOLUTION.—On constructing the graphs (for the first, see exercise 3, § 375), it is found that they intersect at the points $x = 3.7$, $y = 2$ and $x = -3.7$, $y = 2$.

Since the graphs have these two points in common, and no others, their coördinates are the only values of x and y that satisfy both equations and are the roots sought.

Observe that the pairs of values $x = 3.7$, $y = 2$ and $x = -3.7, y = 2$ are *real*, and different, or *unequal*.

NOTE.—The roots are estimated to the nearest tenth; their accuracy may be tested by performing the algebraic solution.

3. Solve graphically $\begin{cases} 9x^2 + 25y^2 = 225, \\ y = 3. \end{cases}$

SOLUTION.—Imagine the straight line $y = 2$ in the figure for exercise 2 to move upward until it coincides with the line $y = 3$. The real unequal roots represented by the coördinates of the points of intersection approach equality, and when the line becomes the tangent line $y = 3$, they coincide.

Hence, the given system of equations has *two real equal roots* $x = 0$, $y = 3$ and $x = 0$, $y = 3$.

4. Find the nature of the roots of $\begin{cases} 9x^2 + 25y^2 = 225, \\ y = 4. \end{cases}$

SOLUTION.—Imagine the straight line $y = 2$ in the figure for exercise 2 to move upward until it coincides with the line $y = 4$. The graphs will cease to have any points in common, showing that the given equations have *no common real values* of x and y.

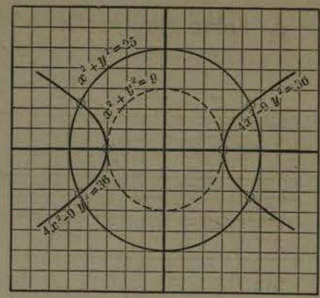
It is shown by the algebraic solution of the equations that there are two roots and that both are *imaginary*.

A system of two independent simultaneous equations in x and y, one simple and the other quadratic, has two roots.

The roots are real and unequal if the graphs intersect, real and equal if the graphs are tangent to each other, and imaginary if the graphs have no points in common.

5. Solve graphically

$$\begin{cases} 4x^2 - 9y^2 = 36, \\ x^2 + y^2 = 25. \end{cases}$$



SOLUTION.—The graphs (see exercises 4 and 1, § 375) show that both of the given equations are satisfied by *four* different pairs of real values of x and y:

$$\begin{cases} x = 4.5; & 4.5; & -4.5; & -4.5; \\ y = 2.2; & -2.2; & -2.2; & 2.2. \end{cases}$$

6. What would be the nature of the roots in exercise 5, if the second equation were $x^2 + y^2 = 9$?

SOLUTION.—Imagine the circle $x^2 + y^2 = 25$ in exercise 5 to become smaller and smaller until it coincides with the circle $x^2 + y^2 = 9$ (see dotted circle in the cut). The four real unequal roots represented by the coördinates of the points of intersection of the graphs come together in pairs at the points $(3, 0)$ and $(-3, 0)$ where the circle $x^2 + y^2 = 9$ is tangent to the hyperbola $4x^2 - 9y^2 = 36$; consequently, the equations $4x^2 - 9y^2 = 36$ and $x^2 + y^2 = 9$ have two pairs of *equal real roots*, namely:

$$\begin{cases} x = 3, & 3, & -3, & -3; \\ y = 0, & 0, & 0, & 0. \end{cases}$$

A system of two independent simultaneous quadratic equations in x and y has four roots.

An intersection of the graphs represents a real root, and a point of tangency, a pair of equal real roots. If there are less than four real roots, the other roots are imaginary.

It is not possible to solve *any* two simultaneous equations in x and y, that involve quadratics, *by quadratic methods*, but approximate values of the real roots may *always* be found *by the graphic method.*

Find by graphic methods, to the nearest tenth, the real roots of the following, and the number of imaginary roots, if there are any. Discuss the graphs and the roots.

7. $\begin{cases} 4x^2 - 9y^2 = 36, \\ x - 3y = 1. \end{cases}$

8. $\begin{cases} 4x^2 - 9y^2 = 36, \\ 4x^2 + 9y^2 = 36. \end{cases}$

9. $\begin{cases} 4x^2 - 9y^2 = 36, \\ 4y = x^2 - 16. \end{cases}$

10. $\begin{cases} 9x^2 + 16y^2 = 144, \\ 3x + 4y = 12. \end{cases}$

11. $\begin{cases} 9x^2 + 16y^2 = 144, \\ x^2 + y^2 = \frac{49}{4}. \end{cases}$

12. $\begin{cases} x^2 + y^2 = 4, \\ x = y - 5. \end{cases}$

13. $\begin{cases} x^2 - 4y^2 = 4, \\ x^2 + y^2 = 4. \end{cases}$

14. $\begin{cases} x - y = 2, \\ xy = -1. \end{cases}$

15. $\begin{cases} x^2 + y^2 = 9, \\ y = x^2 - 5x + 6. \end{cases}$

16. $\begin{cases} x^2 + y^2 = 9, \\ x = y^2 + 5y + 6. \end{cases}$

17. $\begin{cases} y = x^2 - 4, \\ x = (y+1)(y+4). \end{cases}$

18. $\begin{cases} y = x^2 - 5x + 4, \\ x = y^2 - 4y + 3. \end{cases}$

379. Another graphic method of solving quadratic equations in one unknown number (§ 356).

It has been seen that the real roots of *simultaneous equations* are the coördinates of the points where their graphs intersect or are tangent to each other, and that when there is no point in common, the roots are imaginary.

In §§ 356, 357, it was found that the real roots of a *quadratic equation* were the abscissas of the points where the graph of the quadratic expression crossed or touched the x-axis, and that when it had no point in common with the x-axis, the roots were imaginary.

In other words, the solution of a quadratic equation in x was made to depend upon the solution of the simultaneous system,

(I). $\begin{cases} y = ax^2 + bx + c, & \text{(a parabola)} \\ y = 0, & \text{(a straight line)} \end{cases}$

the second being the equation of the x-axis.

In the following method, by substituting y for x^2 in the given equation,

$$ax^2 + bx + c = 0,$$

the equation is divided into the simultaneous system,

(II). $\begin{cases} ay + bx + c = 0, & \text{(a straight line)} \\ y = x^2. & \text{(a parabola)} \end{cases}$

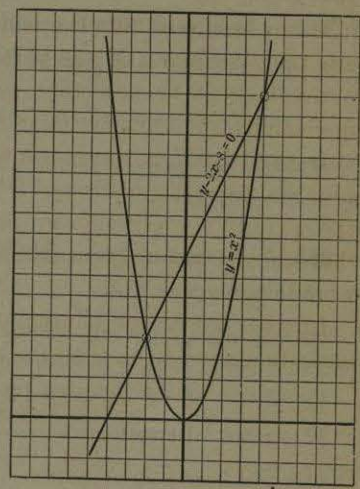
The solution of this system for x gives the required roots of $ax^2 + bx + c = 0$.

It will be observed that whether system (I) or (II) is used, the solution requires the construction of a parabola and a straight line, but the advantage of using (II) instead of (I) lies in the fact that the parabola $y = x^2$ is the same for all quadratic equations in x and when once constructed can be used for solving any number of equations, while with (I) a different parabola must be constructed for each equation solved.

EXERCISES

380. 1. Solve graphically the equation $x^2 - 2x - 8 = 0$.

SOLUTION



Substituting y for x^2, we have

$$y - 2x - 8 = 0.$$

Consequently, the values of x that satisfy the system,

$$\begin{cases} y - 2x - 8 = 0, \\ y = x^2, \end{cases}$$

are the same as those that satisfy the given equation.

Constructing the graph of $y = x^2$, we have the parabola shown in the figure.

Constructing the graph of $y - 2x - 8 = 0$, a straight line, we find that it intersects the parabola at $x = -2$ and $x = 4$.

Hence, the roots of the equation $x^2 - 2x - 8 = 0$ are -2 and 4.

Solve graphically, giving roots to the nearest tenth:

2. $x^2 + x - 2 = 0$.
3. $x^2 - x - 6 = 0$.
4. $x^2 - 3x - 4 = 0$.
5. $x^2 - 2x - 15 = 0$.
6. $x^2 + 5x + 14 = 0$.
7. $x^2 - 7x + 18 = 0$.
8. $2x^2 - x = 6$.
9. $2x^2 - x - 15 = 0$.
10. $3x^2 + 5x - 28 = 0$.
11. $6x^2 - 7x - 20 = 0$.
12. $8x^2 + 14x - 15 = 0$.
13. $15x^2 + 2x - 20 = 0$.