15. Define and illustrate radical, radicand, entire surd, and mixed surd.

16. What is meant by the order of a radical? Illustrate by giving radicals of different orders.

17. How may a radical of the second order be represented graphically?

Illustrate by representing graphically $\sqrt{26}$.

18. What is a rational number? an irrational number?

From the following select the rational numbers:

$$8;\ \tfrac{2}{9};\ \sqrt{3};\ \sqrt[3]{8};\ -7\tfrac{2}{3};\ \sqrt{25};\ 5^{\frac{1}{2}}.$$

19. Are $\sqrt[3]{15}$ and $\sqrt{1+\sqrt{2}}$ radicals? surds? Are all radicals surds? Are all surds radicals?

20. How may the coefficient of a radical be placed under the radical sign?

Express as entire surds: $\frac{1}{3}\sqrt{2}$; $9\sqrt{bc}$; $\frac{2}{7}\sqrt[3]{x^2y}$.

21. When is a radical in its simplest form?

Illustrate by reducing $\sqrt{40\,b^2c}$, $\sqrt{\frac{2}{3}}$, and $\sqrt[6]{4}$, each to its simplest form.

22. What are similar radicals? When numbers have fractional exponents with different denominators, what must be done to the fractional exponents before the numbers can be multiplied? Find the value of $5^{\frac{1}{3}} \times 10^{\frac{1}{2}} \times 6^{\frac{1}{4}}$.

23. What is a binomial surd? a binomial quadratic surd? What are conjugate surds?

24. Define rationalization; rationalizing factor.

How may a binomial quadratic surd be rationalized?

Rationalize the denominator of $\dfrac{7x}{\sqrt{a}+\sqrt{b}}$.

25. What is a radical equation? Give general directions for solving a radical equation.

Solve and verify: $\sqrt{x}+\sqrt{3+x}=3$.

QUADRATIC EQUATIONS

336. The equation $x-2=0$ is of the *first* degree and has *one* root, $x=2$. Similarly, $x-3=0$ is of the first degree and has one root, $x=3$. Consequently, the product of these two simple equations, which is

$$(x-2)(x-3)=0, \text{ or } x^2-5x+6=0,$$

is of the *second degree* and has *two* roots, 2 and 3.

337. An equation that, when simplified, contains the *square* of the unknown number, but no higher power, is called an equation of the *second degree*, or a **quadratic equation.**

It is evident, therefore, that quadratic equations may be of two kinds — those which contain only the second power of the unknown number, and those which contain both the second and first powers.

$x^2=15$ and $3x^2+2x=4$ are quadratic equations.

PURE QUADRATIC EQUATIONS

338. An equation that contains only the second power of the unknown number is called a **pure quadratic.**

$2x^2=8$ and $4x^2-2x^2=16$ are pure quadratic equations.

339. The equation $x^2=16$ has two roots, for it may be reduced to the form $(x-4)(x+4)=0$, which is equivalent to the two simple equations,

$$x-4=0 \text{ and } x+4=0,$$

each of which has one root, namely, $+4$ and -4. That is,

Principle. — *Every pure quadratic equation has two roots, numerically equal but opposite in sign.*

EXERCISES

340. 1. Given $10x^2 = 99 - x^2$, to find the value of x.

SOLUTION

$$10x^2 = 99 - x^2.$$

Transposing, etc., $\quad 11x^2 = 99.$

Dividing by 11, $\quad x^2 = 9.$

Taking the square root of each member, § 275,

$$x = \pm 3.$$

NOTE. — Strictly speaking, the last equation should be $\pm x = \pm 3$, which stands for the equations, $+x = +3$, $+x = -3$, and $-x = -3$, and $-x = +3$. But since the last two equations may be derived from the first two, by changing signs, the first two express *all* the values of x.

For convenience, then, the two expressions, $x = +3$ and $x = -3$, are written $x = \pm 3$.

Consequently, in finding the square roots of the members of an equation, it will be sufficient to write the double sign before the root of *one* member.

2. Find the roots of the equation $3x^2 = 24$.

SOLUTION

$$3x^2 = 24.$$

Dividing by 3, $\quad x^2 = 8.$

Taking the square root, $\quad x = \pm 2\sqrt{2}.$

VERIFICATION. — The given equation becomes $24 = 24$ and is therefore satisfied when either $+2\sqrt{2}$ or $-2\sqrt{2}$ is substituted for x.

Solve, and verify each result:

3. $3x^2 - 5 = 22.$

4. $2x^2 + 3x^2 = 80.$

5. $4x^2 = \frac{1}{9}.$

6. $\frac{3}{4}x^2 - 5 = 22.$

7. $5x^2 - 75 = 2x^2.$

8. $2x^2 - 25 = 73.$

9. $7x^2 = 4x^2 + 24.$

10. $4n^2 + 9 = 5n^2 - 7.$

11. $(x+2)^2 - 4(x+2) = 4.$

12. $(3u - 8)(3u + 8) = 17.$

13. $(x+1)^3 - (x-1)^3 = 38.$

14. $(x+1)^2 = x(3x+2) - 3.$

15. $5(s+2) = 3s^2 + s(5 - s).$

16. $(2r+1)(2r+3) = 8(r+3).$

Problems

341. 1. The length of a 10-acre field is 4 times its width. What are its dimensions?

2. How many rods of fence will inclose a square garden whose area is $2\frac{1}{2}$ acres?

3. A 4-foot length of stone curbing contains 3840 cubic inches. If its width is 5 times its thickness, find these dimensions.

4. The width of the largest American flag is $\frac{3}{5}$ of its length. The area of the flag is 1500 square feet. What are the dimensions of the flag?

5. A farmer sold his pumpkins for \$50. The number of tons was 8 times the number of dollars he received per ton. Find the number of tons sold and the price per ton.

6. A lace worker in Switzerland received \$12 for a piece of work. The number of days he worked on it was $\frac{1}{12}$ of the number of cents he earned per day. What were his daily earnings?

7. The area of the face of a square drawing board is 5 square feet. Find the dimensions of a rectangular board 3 inches longer and 3 inches narrower, if its face has the same area.

8. A shipment of railroad ties measuring 400,000 board feet contained as many car loads as there were board feet in a tie. If each car held 250 ties, find the total number of ties and the number of board feet in one tie.

Formulæ

342. Solve the following formulæ from *physics*:

1. $s = \frac{1}{2}gt^2$, for t.

2. $E = \frac{1}{2}Mv^2$, for v.

3. $P = I^2R$, for I.

4. $F = \frac{mv^2}{R}$, for v.

5. $G = \frac{mm'}{d^2}$, for d.

6. When $g = 32.16$, formula 1 gives the number of feet (s) through which a body will fall in t seconds, starting from rest. How long will it take a brick to fall to the sidewalk from the top of a building 100.5 feet high?

7. To lighten a balloon at the height of 2500 feet, a bag of sand was let fall. Find the time, to the nearest tenth of a second, required for it to reach the earth.

Solve the following *geometrical* formulæ:

8. $c^2 = a^2 + b^2$, for b.

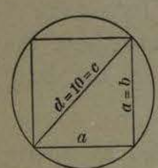
9. $4\,m^2 = 2(a^2 + b^2) - c^2$, for m.

10. $A = .7854\,d^2$, for d.

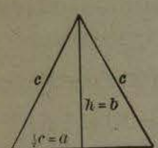
11. $V = \frac{1}{3}\pi r^2 h$, for r.

12. Using formula 8, find the hypotenuse (c) of a right triangle whose other two sides are $a = 8$ and $b = 6$.

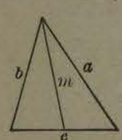
13. By means of formula 8, find the side (a) of a right triangle whose hypotenuse (c) is 5 and whose side (b) is 3.



14. From formula 8 and the accompanying figure find, to the nearest tenth, the side (a) of a square inscribed in a circle whose diameter (d) is 10.



15. Using formula 8 and the accompanying figure, deduce a formula for the altitude (h) of an equilateral triangle in terms of its side (c).



16. From formula 9, find the length of the median (m) to the side (c) of the triangle in the accompanying figure, if $a = 11$, $b = 8$, and $c = 9$.

17. Substituting in formula 10, find, to the nearest tenth of a foot, the diameter (d) of a circle whose area (A) is 1000 square feet.

18. Using formula 11, find, to the nearest centimeter, the radius (r) of the base of a conical vessel 20 centimeters high ($h = 20$) that will hold a liter of water ($V = 1$ liter $= 1000$ cu. cm.; $\pi = 3.1416$).

AFFECTED QUADRATIC EQUATIONS

343. A quadratic equation that contains both the second and the first powers of one unknown number is called an **affected quadratic.**

$x^2 + 3\,x = 10$ and $4\,x^2 - x = 3$ are affected quadratics.

344. To solve affected quadratics by factoring.

Reduce the equation to the form $ax^2 + bx + c = 0$, factor the first member, and equate each factor to zero, as in § 163, thus obtaining two simple equations together equivalent to the given quadratic.

Thus, $3\,x^2 = 10\,x - 3.$

Transposing, $3\,x^2 - 10\,x + 3 = 0.$

Factoring, $(x - 3)(3\,x - 1) = 0.$

$\therefore x - 3 = 0$ or $3\,x - 1 = 0$;

whence, $x = 3$ or $\frac{1}{3}$.

EXERCISES

345. Solve by factoring, and verify results:

1. $x^2 - 5\,x + 6 = 0.$
2. $x^2 + 10\,x + 21 = 0.$
3. $x^2 + 12\,x - 28 = 0.$
4. $x^2 - 20\,x + 51 = 0.$
5. $x^2 - 5\,x = 24.$
6. $x^2 - 1 = 3(x + 1).$
7. $x^2 + 10\,x = 39.$
8. $60 + x^2 = 17\,x.$
9. $x(x - 1) = 42.$
10. $x^2 - 3 = 2(x + 6).$
11. $x^2 - 11(x + 3) = 9.$
12. $55 + x(x + 16) = 0.$
13. $2\,x^2 - 7\,x + 3 = 0.$
14. $2\,z^2 - z - 3 = 0.$
15. $3\,v^2 - 2\,v - 8 = 0.$
16. $10\,r^2 - 27\,r + 5 = 0.$
17. $6(s^2 + 1) = 13\,s.$
18. $2\,x^2 + 7\,x = 4.$
19. $4\,x^2 + 6 = 11\,x.$
20. $3\,w^2 + 3\,w = 6.$
21. $2\,t(t + 3) + 4 = 0.$
22. $3(x^2 - 2) - 7\,x = 0.$
23. $4\,y^2 + 8\,y + 3 = 0.$
24. $10(2 - 3\,x + x^2) = 0.$

346. First method of completing the square.

Since $(x+a)^2 = x^2 + 2ax + a^2,$

the general form of the perfect square of a binomial is

$$x^2 + 2ax + a^2.$$

Consequently, *an expression like* $x^2 + 2ax$ *may be made a perfect square by adding the term* a^2, which it will be observed *is the square of half the coefficient of* x.

Thus, to solve $x^2 + 6x = -5$

by the method of taking the square root of both members (the method used in solving pure quadratics), we must **complete the square** in the first member.

The number to be added is the square of half the coefficient of x; that is, $(\frac{6}{2})^2$, or 9. *The same number must be added to the second member to preserve the equality.*

Therefore, Ax. 1, $x^2 + 6x + 9 = -5 + 9;$

that is, $x^2 + 6x + 9 = 4.$

Taking the square root, § 275, $x + 3 = \pm 2;$

whence, $x = -3 + 2 \text{ or } -3 - 2.$

$$\therefore x = -1 \text{ or } -5.$$

EXERCISES

347. 1. Solve the equation $x^2 - 5x - 14 = 0.$

SOLUTION

$$x^2 - 5x - 14 = 0.$$

Transposing, $x^2 - 5x = 14.$

Completing the square, $x^2 - 5x + \frac{25}{4} = 14 + \frac{25}{4} = \frac{81}{4}.$

Taking the square root, $x - \frac{5}{2} = \pm \frac{9}{2};$

whence, $x = \frac{5}{2} + \frac{9}{2} \text{ or } \frac{5}{2} - \frac{9}{2}.$

$$\therefore x = 7 \text{ or } -2.$$

VERIFICATION. — Either 7 or -2 substituted for x in the given equation reduces it to $0 = 0$; that is, the given equation is satisfied by these values of x.

2. Solve the equation $4x^2 + 4x - 2 = 0.$

SOLUTION

$$4x^2 + 4x - 2 = 0.$$

Transposing, $4x^2 + 4x = 2.$

Dividing by 4, $x^2 + x = \frac{1}{2}.$

Completing the square, $x^2 + x + \frac{1}{4} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$

Taking the square root, $x + \frac{1}{2} = \pm \frac{1}{2}\sqrt{3}.$

$$\therefore x = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3} \text{ or } -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{3},$$

which would usually be written, $x = \frac{1}{2}(-1 \pm \sqrt{3}).$

Steps in the solution of an affected quadratic equation by the first method of completing the square are:

1. *Transpose so that the terms containing* x^2 *and* x *are in one member and the known terms in the other.*
2. *Make the coefficient of* x^2 *positive unity by dividing both members by the coefficient of* x^2.
3. *Complete the square by adding to each member the square of half the coefficient of* x.
4. *Find the square root of both members.*
5. *Solve the two simple equations thus obtained.*

Solve, and verify all results:

3. $x^2 - 4x = 5.$
4. $x^2 - 6x = 7.$
5. $8x = x^2 - 9.$
6. $x^2 + 2x = 15.$
7. $x^2 - 2x = 24.$
8. $x^2 + 8x = -15.$
9. $63 = x^2 + 2x.$
10. $x^2 - 12x = -11.$
11. $x^2 - 40 = 6x.$
12. $y^2 = 10 - 3y.$
13. $v^2 + 5v = 14.$
14. $n(n-1) = 2.$
15. $v^2 + 3v = 1.$
16. $2x(x-2) = 8.$
17. $r^2 + 4r - 7 = 0.$
18. $l^2 - 11l + 28 = 0.$
19. $5x^2 - 3x - 2 = 0.$
20. $3x^2 - 6x = -2.$

348. Hindoo method of completing the square.

EXERCISES

1. Solve the general quadratic equation $ax^2 + bx + c = 0$.

SOLUTION

	$ax^2 + bx + c = 0.$	(1)
Transposing c,	$ax^2 + bx = -c.$	(2)
Multiplying by a,	$a^2x^2 + abx = -ac.$	(3)
Completing the square,	$a^2x^2 + abx + \frac{b^2}{4} = \frac{b^2}{4} - ac.$	(4)
Multiplying by 4,	$4a^2x^2 + 4abx + b^2 = b^2 - 4ac.$	(5)
Taking the square root,	$2ax + b = \pm\sqrt{b^2 - 4ac}.$	(6)
	$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$	(7)

It is evident that (5) can be obtained by multiplying (2) by $4a$ and adding b^2 to both members. Hence, when a quadratic has the general form of (1), if the absolute term is transposed to the second member, as in (2), the square may be completed and **fractions avoided** by

Multiplying by 4 times the coefficient of x^2 and adding to each member the square of the coefficient of x in the given equation.

This is called the **Hindoo method** of completing the square.

NOTE. — The formula for the values of x, given in (7) and known as the **quadratic formula**, may be used in obtaining the roots of any quadratic by substituting the numerical values of a, b, and c found in the given equation after it is reduced to the form $ax^2 + bx + c = 0$.

Solve by the Hindoo method, then by the quadratic formula:

2. $3x^2 + 4x = 4.$
3. $2x^2 - 11x + 12 = 0.$
4. $5x^2 - 14x = -8.$
5. $2x^2 + 5x = 7.$
6. $2x^2 + 7x = -6.$
7. $3x^2 - 7x = -2.$
8. $4x^2 - x - 3 = 0.$
9. $5x^2 - 7x = -2.$
10. $6x^2 + 5x = -1.$
11. $2 + 5x + 2x^2 = 0.$
12. $6x^2 + 2 = 7x.$
13. $4x^2 + 4x = 3.$
14. $3x + 2x^2 = 9.$
15. $15x^2 - 7x - 2 = 0.$

349. Miscellaneous equations to be solved by any method.

EXERCISES

General Directions. — 1. *Reduce the equation to the general form $ax^2 + bx + c = 0$.*

2. *If the factors are readily seen, solve by factoring.*

3. *If the factors are not readily seen, solve by completing the square or by formula.*

NOTE. — In reducing fractional equations to the general form, observe the cautions given on page 146.

Solve, and verify each result:

1. $x^2 + 5 = 6x.$
2. $x^2 = 3x + 10.$
3. $5x + 3x^2 = 2.$
4. $2x^2 - 7x = 2.$
5. $x^2 - 12x = 28.$
6. $x^4 - 16 = x^2(x^2 - 1).$
7. $x^2 - 13x - 30 = 0.$
8. $x^2 + 4(x - 3) = 0.$
9. $6 + 11x + 3x^2 = 0.$
10. $(x + 5)^2 = 10x + 74.$
11. $4x^2 - 3x - 2 = 0.$
12. $(x + 7)(x - 9) = 1 - 2x.$
13. $x + \frac{1}{x} - \frac{5}{2} = 0.$
14. $\frac{x}{9(x-1)} = \frac{x-2}{6}.$
15. $\frac{x}{12} + \frac{x^2 - 15}{5x} = \frac{x}{5}.$
16. $\frac{x}{x-5} - \frac{x-5}{x} = \frac{3}{2}.$
17. $\frac{x+4}{x-2} + 3 = \frac{(x+3)^2}{x^2-9}.$
18. $\frac{x^2}{x-2} = \frac{4}{x-2} + 5.$
19. $\frac{x}{x+2} + \frac{1}{2} = \frac{x+2}{2x}.$
20. $\frac{5x}{x+7} + \frac{x+6}{x+3} = 3.$
21. $\frac{x+2}{x-7} - \frac{x+5}{x-5} = 1.$
22. $\frac{2x-3}{x^2-3x} = 2 - \frac{3}{x^2-3x}.$
23. $\frac{x-2}{x+2} - \frac{x+2}{2-x} = \frac{40}{x^2-4}.$

Find roots to two decimal places:

24. $x^2 - 4x - 1 = 0.$
25. $v^2 + 6v + 7 = 0.$
26. $u^2 + 5u + 5.5 = 0.$
27. $t^2 - 12t + 16.5 = 0.$

Literal Equations

350. The methods of solution for literal quadratic equations are the same as for numerical quadratics. The method by factoring (§ 344) is recommended when the factors can be seen readily. If it is necessary to complete the square, the first method (§ 346) is usually more advantageous, provided the coefficient of x^2 is $+1$, otherwise the Hindoo method (§ 348) is better, because by its use fractions are avoided. Results may be tested by substituting simple numerical values for the literal known numbers.

EXERCISES

351. Solve for x by the method best adapted:

1. $x^2 - ax = ab - bx.$
2. $x^2 + ax = ac + cx.$
3. $x^2 = (m - n)x + mn.$
4. $5x - 2ax = x^2 - 10a.$
5. $x^2 + 3bx = 5cx + 15bc.$
6. $6x^2 + 3ax = 2bx + ab.$
7. $acx^2 - bcx - bd + adx = 0.$
8. $x^2 + 4mx + 3nx + 12mn = 0.$
9. $x^2 = 4ax - 2a^2.$
10. $x^2 - ax - a^2 = 0.$
11. $4ax - x^2 = 3a^2.$
12. $5ax + 6a^2 = 6x^2.$
13. $21b^2 - 4bx = x^2.$
14. $\frac{7m^2}{12} - mx = \frac{x^2}{3}.$
15. $\frac{x^2}{3b} = \frac{5x}{4} + \frac{b}{3}.$
16. $\frac{x}{x-1} - \frac{x}{x+1} = m.$
17. $x + \frac{a^2}{x} = \frac{a^2}{b} + b.$
18. $2x - \frac{3x^2}{a} = a - 2x.$
19. $\frac{1}{ax+4} = 1 - \frac{ax-4}{16}.$
20. $x^2 + \frac{a}{b}x = \frac{a+b}{b}.$
21. $x^2 + 2 = \left(\frac{2a^2+1}{a}\right)x.$
22. $x^2 - \frac{2x}{ab} = \frac{4(ab-1)}{ab}.$

23. $x^2 - 2(a - b)x = 4ab.$
24. $x^2 - 2x(m - n) = 2mn.$
25. $x^2 + 2(a + 8)x = -32a.$
26. $x^2 + x + bx + b = a(x + 1).$
27. $a(2x - 1) + 2bx - b = x(2x - 1).$
28. $x^2 + 4(a - 1)x = 8a - 4a^2.$
29. $\frac{1}{a+b+x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{x}.$
30. $\frac{2a+x}{2a-x} + \frac{a-2x}{a+2x} = \frac{8}{3}.$
31. $a(x - 2a + b) + a(x + a - b) = x^2 - (a - b)^2.$

RADICAL EQUATIONS

352. In §§ 333, 334, the student learned how to free radical equations of radicals, the cases treated there being such as lead to simple equations. The radical equations in this chapter lead to quadratic equations, but the methods of freeing them of radicals are the same as in the cases already discussed.

EXERCISES

353. 1. Solve the equation $2\sqrt{x} - x = x - 8\sqrt{x}.$

Solution

$$2\sqrt{x} - x = x - 8\sqrt{x}.$$

Dividing by $\sqrt{x}$, $\quad 2 - \sqrt{x} = \sqrt{x} - 8.$

Transposing, etc., $\quad \sqrt{x} = 5.$

Squaring, $\quad x = 25.$

Verification. — When $x = 25$,

$$\text{1st member} = 2\sqrt{25} - 25 = 10 - 25 = -15;$$

$$\text{2d member} = 25 - 8\sqrt{25} = 25 - 40 = -15.$$

Hence, $x = 25$ is a root of the equation; $x = 0$, the root of the equation $\sqrt{x} = 0$, also is a root of the given equation, removed by dividing both members by $\sqrt{x}$.

2. Solve and verify $\sqrt{x+1}+\sqrt{x-2}-\sqrt{2x-5}=0.$

SOLUTION

$$\sqrt{x+1}+\sqrt{x-2}-\sqrt{2x-5}=0.$$

Transposing, $\sqrt{x+1}+\sqrt{x-2}=\sqrt{2x-5}.$

Squaring, $x+1+2\sqrt{x^2-x-2}+x-2=2x-5.$

Simplifying, $\sqrt{x^2-x-2}=-2.$

Squaring, $x^2-x-2=4.$

Solving, $x=-2$ or $3.$

VERIFICATION. — Substituting -2 for x in the given equation,

$$\sqrt{-1}+\sqrt{-4}-\sqrt{-9}=0;$$

that is, $\sqrt{-1}+2\sqrt{-1}-3\sqrt{-1}=0.$

Therefore, -2 is a root of the given equation.

Substituting 3 for x in the given equation,

$$\sqrt{4}+\sqrt{1}-\sqrt{1}=0,$$

which is not true according to the convention adopted in § 293.

Hence, 3 is not to be regarded as a root of the given equation.

NOTE. — The equation could be verified for $x=3$, if the negative square root of 1 were taken in the second term and the positive square root in the third, thus:

$$\sqrt{4}+\sqrt{1}-\sqrt{1}=2+(-1)-(+1)=0.$$

This is an improper method of verification, however, for it has been agreed previously that the square root sign shall denote only the *positive* square root.

Solve, and verify each result:

3. $8\sqrt{x}-8x=\frac{3}{2}.$

4. $3x+\sqrt{x}=5\sqrt{4x}.$

5. $x-1+\sqrt{x+5}=0.$

6. $x-5-\sqrt{x-3}=0.$

7. $\sqrt{4x+17}+\sqrt{x+1}-4=0.$

8. $1+\sqrt{(3-5x)^2+16}=2(3-x).$

9. $\sqrt{1+x\sqrt{x^2+12}}=1+x.$

10. $\sqrt{x-1}+\sqrt{2x-1}-\sqrt{5x}=0.$

11. $\sqrt{2x-7}-\sqrt{2x}+\sqrt{x-7}=0.$

12. $\sqrt{a+x}-\sqrt{a-x}=\sqrt{2x}.$

13. $\sqrt{x-a}+\sqrt{b-x}=\sqrt{b-a}.$

14. $\sqrt{2x+\sqrt{10x+1}}=\sqrt{2x}+1.$

15. $\sqrt{6+x}+\sqrt{x}-\sqrt{10-4x}=0.$

16. $\sqrt{4x-3}-\sqrt{2x+2}=\sqrt{x-6}.$

17. $\sqrt{2x+3}-\sqrt{x+1}=\sqrt{5x-14}.$

18. $\sqrt{x^2+8}-\dfrac{6}{\sqrt{x^2+8}}=x.$

19. $\sqrt{a-x}+\sqrt{b-x}=\sqrt{a+b-2x}.$

Find roots to two decimal places:

20. $x-\sqrt{6x}=6.$

21. $3\sqrt{x}=2(x-2).$

22. $2+2\sqrt{x+3}+x=0.$

23. $\sqrt{x-2}+\sqrt{2x}=3.$

Problems

354. 1. The sum of two numbers is 8, and their product is 15. Find the numbers.

SOLUTION. — Let $x=$ one number.

Then, $8-x=$ the other.

Since their product is 15, $(8-x)x=15.$

Solving, $x=3$ or $5,$

and $8-x=5$ or $3.$

Therefore, the numbers are 3 and 5.

2. Divide 20 into two parts whose product is 96.

3. Divide 14 into two parts whose product is 45.

4. Find two consecutive positive integers the sum of whose squares is 61.

Solve the following problems and verify each solution:

5. A plumber received \$24 for some work. The number of hours that he worked was 20 less than the number of cents per hour that he earned. Find his hourly wage.

6. A man's life insurance for a certain time cost \$20.80. If the number of weeks was 12 more than the number of cents he paid per week, what was the weekly premium?

7. The area of a revolving floor in a hall in Paris is 2650 square feet. The width is 3 feet less than the length. What are the dimensions of the floor?

8. The base of the tower of the Metropolitan Life Building in New York City is 6375 square feet in area. The length of the base is 10 feet greater than the width. What are the dimensions of the base?

9. The 1860 bunches of asparagus from an acre of land were sold in boxes each holding 1 less than $\frac{1}{2}$ as many bunches as there were boxes. Find the number of bunches in a box.

10. One of the largest electric signs in the world is 14,560 square feet in area. The length of the sign lacks 22 feet of being twice the width. Find the dimensions of the sign.

11. The area of the plate glass floor of the highest bridge in the world, built across Royal Gorge in Colorado, is 5060 square feet. The length is 10 feet more than 10 times its width. What is its length?

12. If the average amount deposited in the postal savings banks of Canada by each depositor one year had been \$70 less, and if the number of depositors had been equal to the average number of dollars each deposited, the total deposit would have been \$40,000. Find the average amount each deposited.

13. The length of a steel barge used for coal on the Ohio River is 5 feet more than 5 times its width. If the depth is 8 feet and the capacity of the barge is 28,080 cubic feet, what is the width? the length?

14. The area of the plate used in a giant camera is $37\frac{1}{3}$ square feet. The width of the plate is 8 inches more than $\frac{1}{2}$ the length. Find the dimensions of the plate.

15. The *Chester*, a United States scouting cruiser, steamed 106.12 knots on its trial voyage. The number of knots that it went in one hour was 1.47 less than 7 times the number of hours spent on its trial voyage. Find its speed per hour.

16. A piece of silk made from spiders' web and exhibited at the Paris Exposition was 36 times as long as it was wide. If its width had been increased 9 inches, it would have contained $13\frac{1}{2}$ square yards. Find its length and width.

17. The area of one side wall of a square reservoir, cut in the solid rock at Bowling Green, Ohio, is 2200 square feet, and the depth of the reservoir is 2 feet more than $\frac{1}{5}$ of its length. Find its three dimensions.

18. Some boys laid out basket-ball grounds 30 feet greater in length than in width, but to change the area to the prescribed limit of 3500 square feet, they reduced the length 10 feet. How much too large had they laid out the grounds?

19. A party hired a coach for \$12. In consequence of the failure of 3 of them to pay, each of the others had to pay 20 cents more. How many persons were in the party?

SOLUTION

Let $x =$ the number of persons.

Then, $x - 3 =$ the number that paid.

$\frac{12}{x} =$ the number of dollars each should have paid,

and $\frac{12}{x-3} =$ the number of dollars each paid.

Therefore, $\frac{12}{x-3} - \frac{1}{5} = \frac{12}{x}.$

Solving, $x = 15 \text{ or } -12.$

The second value of x is evidently inadmissible, since there could not be a negative number of persons.

Hence, the number of persons in the party was 15.

20. A club had a dinner that cost \$ 60. If there had been 5 persons more, the share of each would have been \$ 1 less. How many persons were there in the club?

21. A party of young people agreed to pay \$ 8 for a sleigh ride. As 4 were obliged to be absent, the cost for each of the rest was 10 cents greater. How many went on the ride?

22. Find two consecutive integers the sum of whose reciprocals is $\frac{9}{20}$.

23. The dry gum used per day in gumming United States postage stamps costs \$ 48. If 400 pounds more were used at the same total cost, the price per pound would be 2 cents less. How much gum is used daily?

24. A man earned \$ 48 by shearing sheep. The number of cents he earned per fleece was 2 more than the number of days he worked. How many sheep did he shear, if he averaged 100 a day?

25. The weight of 80 four-inch spikes was 3 pounds less than the weight of 80 five-inch spikes. If 1 pound of the former contained 6 spikes more than 1 pound of the latter, how many of each kind weighed 1 pound?

26. A tub of dairy butter weighed 20 pounds less than a tub of creamery butter, and 360 pounds of dairy butter required 3 tubs more than the same amount of creamery butter. What weight of butter was there in a tub of each kind?

27. Mr. Field paid \$ 8 for one mile of No. 9 steel wire and \$ 2.88 for one mile of No. 14 wire. The No. 9 wire weighed 224 pounds more, and cost $\frac{1}{2}$ cent per pound less, than the No. 14 wire. Find the cost of each per pound.

28. A boy earned \$ 420 by delivering bills. If he had received 50 cents more per thousand, he would have earned as much by delivering 70 thousand less than he did. How much was he paid per thousand?

29. A merchant sold a hunting coat for \$ 11, and gained a per cent equal to the number of dollars the coat cost him. What was his per cent of gain?

30. A moving picture film 150 feet long is made up of a certain number of individual pictures. If these pictures were $\frac{1}{4}$ of an inch longer, there would be 600 less for the same length of film. How long is each separate picture?

31. In Detroit, a machine for making pills turned out 250,000 pills per hour. If, in boxing these, 5 pills more were put into each box, the number of boxes would be 2500 less. How many pills does each box contain?

32. Two coats of paint applied to the sides of a barn having an area of 195 square yards required 69 pounds of paint. One pound covered $1\frac{1}{2}$ square yards more for the second coat than for the first. What area did 1 pound of paint cover for each coat?

33. A train started 16 minutes late, but finished its run of 120 miles on time by going 5 miles per hour faster than usual. What was the usual rate per hour?

34. Two automobiles went a distance of 60 miles, one making 6 miles per hour faster time than the other and completing the journey $\frac{5}{6}$ of an hour sooner. How long was each on the way?

35. The distance covered by an aeroplane on one occasion was 45 miles. If its rate per minute had been $\frac{1}{4}$ of a mile more, the distance would have been covered in 9 minutes less time. Find the speed of the aeroplane per minute.

36. If each cable of the Manhattan Bridge contained 9 strands more and each strand 20 wires more, the number of wires in a strand would be 6 times as many as the number of strands in a cable. How many strands are there in a cable, if there are 9472 wires in a cable?

37. To run around a track 1320 feet in circumference took one man 5 seconds less time than it took another who ran 2 feet per second slower. How long did it take each man?

38. Some rugs made in India have 400 knots to the square inch. A fast weaver ties 40 knots more per minute than a boy. If the former weaves a square inch in $13\frac{1}{3}$ minutes less time than the latter, how many knots does each tie per minute?

39. A cistern can be filled by two pipes in 24 minutes. If it takes the smaller pipe 20 minutes longer to fill the cistern than the larger pipe, in what time can the cistern be filled by each pipe?

Solution

Let x = the number of minutes required by the larger pipe.

Then, $x + 20$ = the number of minutes required by the smaller pipe.

Since $\frac{1}{x}$ = the part that the larger pipe fills in one minute,

$\frac{1}{x+20}$ = the part that the smaller pipe fills in one minute,

and $\frac{1}{24}$ = the part that both pipes fill in one minute.

Then, $$\frac{1}{x} + \frac{1}{x+20} = \frac{1}{24}.$$

Solving, $x = 40$ or -12.

The negative value is inadmissible. Hence, the larger pipe can fill the cistern in 40 minutes, and the smaller pipe in 60 minutes.

40. A city reservoir can be filled by two of its pumps in 3 days. The larger pump alone would take $1\frac{3}{4}$ days less time than the smaller. In what time can each fill the reservoir?

41. A company owned two plants that together made 25,200 concrete building blocks in 12 days. Working alone, one plant would have required 7 days more time than the other. What was the daily capacity of each plant?

42. The number of strawberry baskets made by a machine was 12 more per minute than the number of peach baskets made by another machine. One day the former machine started 45 minutes after the latter, but each finished 2400 baskets at the same instant. Find the rate of each per minute.

Formulæ

355. 1. In any right-angled triangle (Fig. 1), $c^2 = a^2 + b^2$. Find all sides when $a = c - 2$ and $b = c - 4$.

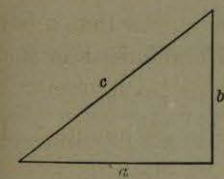


Fig. 1.

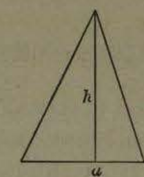


Fig. 2.

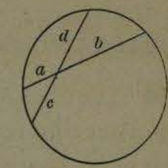


Fig. 3.

2. The area (A) of a triangle (Fig. 2) is expressed by the formula $A = \frac{1}{2}ah$. If the altitude (h) of a triangle is 2 inches greater than the base (a) and the area is 60 square inches, what is the length of the base?

3. If two chords intersect in a circle, as shown in Fig. 3, $a \times b$ is always equal to $c \times d$. Compute a and b when $c = 4$, $d = 6$, and $b = a + 5$.

4. The formula $h = a + vt - 16t^2$ gives, approximately, the height (h) of a body at the end of t seconds, if it is thrown vertically *upward*, starting with a velocity of v feet per second from a position a feet high.

Solve for t, and find how long it will take a skyrocket to reach a height of 796 feet, if it starts from a platform 12 feet high with an initial velocity of 224 feet per second.

5. How long will it take a bullet to reach a height of 25,600 feet, if it is fired vertically upward from the level of the ground with an initial velocity of 1280 feet per second?

6. When a body is thrown vertically *downward*, an approximate formula for its height is $h = a - vt - 16t^2$, in which h, a, v, and t stand for the same elements as in exercise 4.

Solve for t, and find when a ball thrown vertically downward from the Eiffel tower, height 984 feet, with an initial velocity of 24 feet per second, will be 368 feet above ground.

Find, to the nearest second, when it will reach the ground.

GRAPHIC SOLUTIONS

QUADRATIC EQUATIONS — ONE UNKNOWN NUMBER

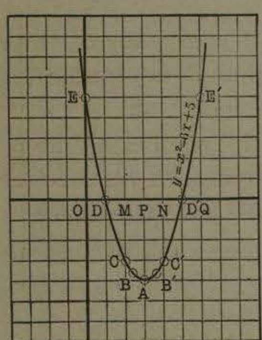
356. Let it be required to solve graphically, $x^2 - 6x + 5 = 0$.

To solve the equation graphically, we must first draw the graph of $x^2 - 6x + 5$. To do this, let $y = x^2 - 6x + 5$.

The graph of $y = x^2 - 6x + 5$ will represent *all* the corresponding real values of x and of $x^2 - 6x + 5$, and among them will be the values of x that make $x^2 - 6x + 5$ equal to zero, that is, the *roots* of the equation $x^2 - 6x + 5 = 0$.

When the coefficient of x^2 is $+1$, as in this instance, it is convenient to take for the first value of x a number equal to half the coefficient of x with its sign changed. Next, values of x differing from this value *by equal amounts* may be taken.

Thus, first substituting $x = 3$, it is found that $y = -4$, locating the point $A = (3, -4)$. Next give values to x differing from 3 by equal amounts, as $2\frac{1}{2}$ and $3\frac{1}{2}$, 2 and 4, 1 and 5, 0 and 6. It will be found that y has the same value for $x = 3\frac{1}{2}$ as for $x = 2\frac{1}{2}$, for $x = 4$ as for $x = 2$, etc. The table below gives a record of the points and their coördinates:



$y = x^2 - 6x + 5$

x	y	POINTS
3	-4	A
$2\frac{1}{2}, 3\frac{1}{2}$	$-3\frac{3}{4}$	B, B'
2, 4	-3	C, C'
1, 5	0	D, D'
0, 6	5	E, E'

Plotting the points A; B, B'; C, C'; etc., whose coördinates are given in the preceding table, and drawing a smooth curve through them, we obtain the graph of $y = x^2 - 6x + 5$ as shown in the figure.

It will be observed from the work of the preceding page that:

When $x = 3$, $x^2 - 6x + 5 = -4$, which is represented by the *negative* ordinate PA.

When $x = 2$ and also when $x = 4$, $x^2 - 6x + 5 = -3$, which is represented by the equal *negative* ordinates MC and NC'.

When $x = 0$ and also when $x = 6$, $x^2 - 6x + 5 = 5$, represented by the equal *positive* ordinates OE and QE'.

Thus, it is seen that the ordinates change sign as the curve crosses the x-axis.

At D and at D', where the ordinates are equal to 0, the value of $x^2 - 6x + 5$ is 0, and the abscissas are $x = 1$ and $x = 5$.

Hence, the roots of the given equation are 1 and 5.

The curve obtained by plotting the graph of $x^2 - 6x + 5$, or of any quadratic expression of the form $ax^2 + bx + c$, is a **parabola**.

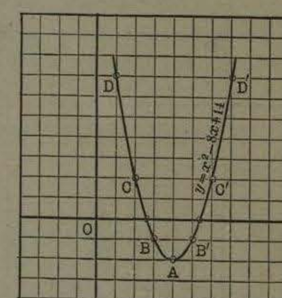
EXERCISES

357. 1. Solve graphically the equation $x^2 - 8x + 14 = 0$.

SOLUTION. — Since the coefficient of x is -8, § 356, first substitute 4 for x. Points and their coördinates are given in the table:

$y = x^2 - 8x + 14$

x	y	POINTS
4	-2	A
3, 5	-1	B, B'
2, 6	2	C, C'
1, 7	7	D, D'



Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 8x + 14$, which *crosses* the x-axis approximately at $x = 2.6$ and $x = 5.4$.

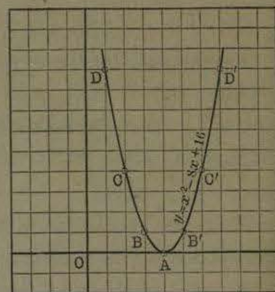
Hence, to the nearest tenth, the roots of $x^2 - 8x + 14 = 0$ are 2.6 and 5.4.

2. Solve graphically the equation $x^2 - 8x + 16 = 0$.

SOLUTION. — Since the coefficient of x is -8, § 356, first substitute 4 for x. Points and their coördinates are given in the table:

$y = x^2 - 8x + 16$

x	y	POINTS
4	0	A
3, 5	1	B, B'
2, 6	4	C, C'
1, 7	9	D, D'



Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 8x + 16$, which *touches* the x-axis at $x = 4$.

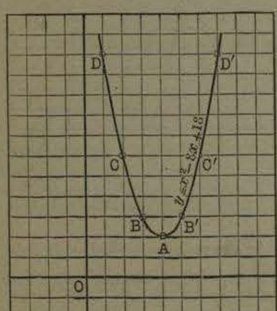
This fact is interpreted graphically to mean that the roots of the equation $x^2 - 8x + 16 = 0$ are *equal*, both being represented by the abscissa of the point of contact.

Hence, the roots are 4 and 4.

NOTE. — The student may show that 4 and 4 are the roots by solving the equation algebraically.

3. Solve graphically the equation $x^2 - 8x + 18 = 0$.

SOLUTION. — Since the coefficient of x is -8, § 356, first substitute 4 for x. Points and their coördinates are given in the table:



$y = x^2 - 8x + 18$

x	y	POINTS
4	2	A
3, 5	3	B, B'
2, 6	6	C, C'
1, 7	11	D, D'

Plotting these points and drawing a smooth curve through them, we

have the graph of $y = x^2 - 8x + 18$, which neither crosses nor touches the x-axis. This fact is interpreted graphically to mean that the roots of the equation $x^2 - 8x + 18 = 0$ are *imaginary*.

NOTE. — The student may show that the roots are imaginary by solving the equation algebraically.

In each of the preceding graphs, the point A, whose ordinate is the least algebraically that any point in the graph has, is called the **minimum point**.

When the coefficient of x^2 is $+1$, it is evident that:

PRINCIPLES. — 1. *If the minimum point lies below the x-axis, the roots are real and unequal.*

2. *If the minimum point lies on the x-axis, the roots are real and equal.*

3. *If the minimum point lies above the x-axis, the roots are imaginary.*

Solve graphically, giving real roots to the nearest tenth:

4. $x^2 - 4x + 3 = 0$.
5. $x^2 - 6x + 7 = 0$.
6. $x^2 - 4x = -2$.
7. $x^2 + 2(x + 1) = 0$.
8. $x^2 - 4x + 6 = 0$.
9. $x^2 - 2x - 2 = 0$.
10. $x^2 = 6x - 9$.
11. $x^2 + 4x + 2 = 0$.
12. $x^2 - 2x + 6 = 0$.
13. $x^2 - 4x - 1 = 0$.

14. Solve graphically $4x - 2x^2 + 1 = 0$.

SUGGESTION. — On dividing both members of the given equation by -2, the coefficient of x^2, the equation becomes

$$x^2 - 2x - \tfrac{1}{2} = 0.$$

The roots may be found by plotting the graph of $y = x^2 - 2x - \frac{1}{2}$.

Solve graphically, giving real roots to the nearest tenth:

15. $2x^2 + 8x + 7 = 0$.
16. $2x^2 - 12x + 15 = 0$.
17. $12x - 4x^2 - 1 = 0$.
18. $11 + 8x + 2x^2 = 0$.

NOTE. — Another method of solving quadratic equations graphically is given in § 379.