Expand, and test results:

19. $(x+2y)^4$. 23. $(1-3x^2)^4$. 27. $(1-x)^7$.

20. $(2x-y)^3$. 24. $(5x^2-ab)^3$. 28. $(1-2x)^6$.

21. $(2x-5)^3$. 25. $(1+a^2b^2)^4$. 29. $(x-\frac{1}{2})^6$.

22. $(x^2-10)^4$. 26. $(2ax-b)^5$. 30. $(\frac{1}{2}x-\frac{1}{3}y)^4$.

Expand:

31. $\left(2a+\frac{1}{2}\right)^5$. 34. $\left(3a^2+\frac{b}{6}\right)^3$. 37. $\left(\frac{1}{2x}-2x\right)^5$.

32. $\left(\frac{x}{y}-\frac{y}{x}\right)^4$. 35. $\left(1+\frac{3x}{2}\right)^5$. 38. $\left(\frac{1}{a}-a\right)^6$.

33. $\left(\frac{x}{y}-\frac{y}{x}\right)^6$. 36. $\left(\frac{3}{5}+\frac{5x}{3}\right)^4$. 39. $\left(x+\frac{1}{x}\right)^7$.

40. Expand $(r-s-t)^3$.

SOLUTION

Since $(r-s-t)^3$ may be written in the binomial form, $(\overline{r-s}-t)^3$, we may substitute $(r-s)$ for a and t for x in

§ 265, $(a-x)^3 = a^3 - 3a^2x + 3ax^2 - x^3$.

Then, we have

$$\begin{aligned}(r-s-t)^3 &= (\overline{r-s}-t)^3\\ &= (r-s)^3 - 3(r-s)^2t + 3(r-s)t^2 - t^3\\ &= r^3 - 3r^2s + 3rs^2 - s^3 - 3t(r^2-2rs+s^2) + 3rt^2 - 3st^2 - t^3\\ &= r^3 - 3r^2s + 3rs^2 - s^3 - 3r^2t + 6rst - 3s^2t + 3rt^2 - 3st^2 - t^3\end{aligned}$$

41. Expand $(a+b-c-d)^3$.

SUGGESTION. $(a+b-c-d)^3 = (\overline{a+b}-\overline{c+d})^3$, a binomial form.

Expand:

42. $(a+x-y)^3$. 46. $(a+x+2)^3$.

43. $(a-m-n)^3$. 47. $(a-x-2)^3$.

44. $(a-x+y)^3$. 48. $(a+2b-3c)^3$.

45. $(a-x-y)^3$. 49. $(a+b+x+y)^3$.

EVOLUTION

267. Just as (§ 132) one of the *two* equal factors of a number is its **second**, or **square, root**; so one of the *three* equal factors of a number is its **third**, or **cube, root**; one of the *four* equal factors, the **fourth root**; etc.

The second root of a number, as a, is indicated by $\sqrt{a}$; the third root by $\sqrt[3]{a}$; the fourth root by $\sqrt[4]{a}$; the fifth root by $\sqrt[5]{a}$; etc.

The sign $\sqrt{\ }$ is called the **root sign**, or the **radical sign**; the small figure in its opening is called the **index** of the root.

When no index is written, the second, or square, root is meant.

268. The process $2^3 = 2 \cdot 2 \cdot 2 = 8$ illustrates *involution*.

The process $\sqrt[3]{8} = \sqrt[3]{2 \cdot 2 \cdot 2} = 2$ illustrates **evolution**, which will be defined here as the process of finding a root of a number, or as the *inverse of involution*.

269. You have learned (§ 132) that *every number has two square roots, one positive and the other negative.*

For example, $\sqrt{25} = +5$ or -5.

The roots may be written together, thus: ± 5, read '*plus or minus five*'; or ∓ 5, read '*minus or plus five*.'

270. The square root of -16 is not 4, for $4^2 = +16$; nor -4, for $(-4)^2 = +16$. No number so far included in our number system can be a square root of -16 or of any other negative number.

It would be inconvenient and confusing to regard $\sqrt{a}$ as a number only when a is positive. In order to preserve the generality of the discussion of number, it is necessary, therefore, to admit square roots of negative numbers into our number system. The square roots of -16 are written

$$\sqrt{-16} \text{ and } -\sqrt{-16}.$$

Such numbers are called **imaginary numbers** and, in contrast, numbers that do not involve a square root of a negative number are called **real numbers.**

271. Just as every number has two square roots, so every number has three cube roots, four fourth roots, etc.

For example, the cube roots of 8 are the roots of the equation $x^3 = 8$, which later will be found to be

$$2, \ -1+\sqrt{-3}, \text{ and } -1-\sqrt{-3}.$$

The present discussion is concerned only with *real* roots.

272. Since $2^3 = 8$, $\sqrt[3]{8} = 2$.

Since $(-2)^3 = -8$, $\sqrt[3]{-8} = -2$.

Since $2^4 = 16$ and $(-2)^4 = 16$, $\sqrt[4]{16} = \pm 2$.

Since $2^5 = 32$, $\sqrt[5]{32} = 2$.

Since $(-2)^5 = -32$, $\sqrt[5]{-32} = -2$.

A root is **odd** or **even** according as its index is odd or even.

273. It follows from the preceding illustrations and from the law of signs for involution (§ 258) that, for real roots:

Law of Signs. — *An odd root of a number has the same sign as the number.*

An even root of a number may have either sign.

274. A real root of a number, if it has the same sign as the number itself, is called a **principal root** of the number.

The principal square root of 25 is 5, but not -5. The principal cube root of 8 is 2; of -8 is -2.

275. Axiom 7. — *The same roots of equal numbers are equal.*

Thus, if $x = 16$, $\sqrt{x} = 4$; if $x = 8$, $\sqrt[3]{x} = 2$; etc.

276. Since $(2^2)^3 = 2^{2\times 3} = 2^6$, the principal cube root of 2^6 is

$$\sqrt[3]{2^6} = 2^{6 \div 3} = 2^2.$$

Hence, for evolution:

Law of Exponents. — *The exponent of any root of a number is equal to the exponent of the number divided by the index of the root.*

277. Since $(5a)^2 = 5^2a^2 = 25a^2$, the principal square root of $25a^2$ is

$$\sqrt{25a^2} = \sqrt{25} \cdot \sqrt{a^2} = 5a.$$

Hence, for principal roots:

Principle. — *Any root of a product may be obtained by taking that root of each of the factors and finding the product of the results.*

278. Since $\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4} = \frac{16}{81}$, the principal fourth root of $\frac{16}{81}$ is

$$\sqrt[4]{\frac{16}{81}} = \frac{\sqrt[4]{16}}{\sqrt[4]{81}} = \frac{\sqrt[4]{2^4}}{\sqrt[4]{3^4}} = \frac{2}{3}.$$

Hence, for principal roots:

Principle. — *Any root of the quotient of two numbers is equal to that root of the dividend divided by that root of the divisor.*

279. Evolution of monomials.

EXERCISES

1. Find the square root of $36a^6b^2$.

Solution

Since, in squaring a monomial, § 262, the coefficient is squared and the exponents of the letters are multiplied by 2, to find the square root, the *square root* of the coefficient must be found, and to it must be annexed the letters each with its exponent *divided* by 2.

The square root of 36 is 6, and the square root of the literal factors is a^3b. Therefore, the *principal* square root of $36a^6b^2$ is $6a^3b$.

The square root may also be $-6a^3b$, since $-6a^3b \times -6a^3b = 36a^6b^2$.

$$\therefore \sqrt{36a^6b^2} = \pm 6a^3b.$$

2. Find the cube root of $-125\,x^6y^{21}$.

Solution

$$\sqrt[3]{-125\,x^6y^{21}} = -5\,x^2y^7, \text{ the real root.}$$

To find the root of an integral term:

Rule. — *Find the required root of the numerical coefficient, annex to it the letters each with its exponent divided by the index of the root sought, and prefix the proper sign to the result.*

Find real roots:

3. $\sqrt[3]{a^3b^9c^{15}}$.
4. $\sqrt{a^6b^{16}c^{14}}$.
5. $\sqrt[5]{a^{10}x^5y^{30}}$.
6. $\sqrt[4]{a^{4n}b^8c^{12}}$.
7. $\sqrt{x^{4n}y^8z^{2m}}$.
8. $\sqrt[3]{-8\,a^6b^{15}}$.
9. $\sqrt[5]{-32\,x^{10}y^{10}}$.
10. $\sqrt{16\,x^4y^2}$.
11. $\sqrt[7]{-a^{21}b^{35}x^{14}}$.
12. $\sqrt[5]{-243\,y^{5n}}$.
13. $\sqrt{(-mb^5)^2}$.
14. $\sqrt[3]{(-a^2b)^9}$.
15. $-\sqrt[9]{a^{18}b^9c^{27}}$.
16. $-\sqrt[3]{-27\,p^9r^3}$.
17. $-\sqrt[7]{-128\,a^{14}n^{28}}$.

18. Find the cube root of $\dfrac{-8\,x^9y^6}{27\,m^3n^{12}}$.

Solution

$$\sqrt[3]{\frac{-8\,x^9y^6}{27\,m^3n^{12}}} = \frac{\sqrt[3]{-8\,x^9y^6}}{\sqrt[3]{27\,m^3n^{12}}} = \frac{-2\,x^3y^2}{3\,mn^4} = -\frac{2\,x^3y^2}{3\,mn^4}.$$

To find the root of a fractional term:

Rule. — *Find the required root of both numerator and denominator, and prefix the proper sign to the resulting fraction.*

Find real roots :

19. $\sqrt{\dfrac{64\,a^4b^6}{81\,m^2n^8}}$.
20. $\sqrt[7]{\dfrac{(x-y)^{14}}{128\,x^{14}}}$.
21. $\sqrt[3]{-125\dfrac{x^3}{y^9}}$.
22. $\sqrt[5]{\dfrac{-32\,a^5x^{10}}{243\,y^{15}}}$.
23. $\sqrt[3]{\left(-\dfrac{27\,a^3}{64\,b^6}\right)^2}$.
24. $\sqrt[3]{-\dfrac{125\,x^{12}y^{12}}{1728\,c^3}}$.

280. **To find the square root of a polynomial.**

EXERCISES

1. Derive the process for finding the square root of $a^2 + 2\,ab + b^2$.

PROCESS

$$\begin{array}{lll} & a^2 + 2\,ab + b^2 & \underline{|\,a + b} \\ & a^2 & \\ \text{Trial divisor, } 2\,a & & |\,2\,ab + b^2 \\ \text{Complete divisor, } 2\,a + b & & |\,2\,ab + b^2 \end{array}$$

Explanation. — Since $a^2 + 2\,ab + b^2$ is the square of $(a + b)$, we know that the square root of $a^2 + 2\,ab + b^2$ is $a + b$.

Since the first term of the root is a, it may be found by taking the square root of a^2, the first term of the power. On subtracting a^2, there is a remainder of $2\,ab + b^2$.

The second term of the root is known to be b, and that may be found by dividing the first term of the remainder by twice the part of the root already found. This divisor is called a *trial* divisor.

Since $2\,ab + b^2$ is equal to $b\,(2\,a + b)$, the complete divisor which multiplied by b produces the remainder $2\,ab + b^2$ is $2\,a + b$; that is, the complete divisor is found by adding the second term of the root to twice the root already found.

On multiplying the complete divisor by the second term of the root and subtracting, there is no remainder; then, $a + b$ is the required root.

2. Find the square root of $9\,x^2 - 30\,xy + 25\,y^2$.

PROCESS

$$\begin{array}{lll} & 9\,x^2 - 30\,xy + 25\,y^2 & \underline{|\,3\,x - 5\,y} \\ & 9\,x^2 & \\ \text{Trial divisor, } 6\,x & & |\,-30\,xy + 25\,y^2 \\ \text{Complete divisor, } 6\,x - 5\,y & & |\,-30\,xy + 25\,y^2 \end{array}$$

Find the square root of:

3. $4\,x^2 + 12\,x + 9$.
4. $x^2 + 2\,x + 1$.
5. $1 - 4\,m + 4\,m^2$.
6. $c^2 - 12\,c + 36$.
7. $4\,x^2 + 4\,x + 1$.
8. $16 + 24\,x + 9\,x^2$.

Since, in squaring $a + b + c$, $a + b$ may be represented by x, and the square of the number by $x^2 + 2\,xc + c^2$, the square root of a number whose root consists of *more than the two terms* may be obtained in the same way as in exercise 1, *by considering the terms already found as one term.*

9. Find the square root of $4x^4 + 12x^3 - 3x^2 - 18x + 9$.

PROCESS

$$\begin{array}{l|l}
 & 4x^4 + 12x^3 - 3x^2 - 18x + 9 \;\big|\; 2x^2 + 3x - 3 \\
 & 4x^4 \\
4x^2 & 12x^3 - 3x^2 \\
4x^2 + 3x & 12x^3 + 9x^2 \\
4x^2 + 6x & -12x^2 - 18x + 9 \\
4x^2 + 6x - 3 & -12x^2 - 18x + 9
\end{array}$$

EXPLANATION.—Proceeding as in exercise 2, we find that the first two terms of the root are $2x^2 + 3x$.

Considering $(2x^2 + 3x)$ as the first term of the root, we find the next term of the root as we found the second term, by dividing the remainder by twice the part of the root already found. Hence, the trial divisor is $4x^2 + 6x$, and the next term of the root is -3. Annexing this, as before, to the trial divisor already found, we find that the complete divisor is $4x^2 + 6x - 3$. Multiplying this by -3 and subtracting the product from $-12x^2 - 18x + 9$, we have no remainder.

Hence, the square root of the number is $2x^2 + 3x - 3$.

RULE. — *Arrange the terms of the polynomial with reference to the consecutive powers of some letter.*

Find the square root of the first term, write the result as the first term of the root, and subtract its square from the given polynomial.

Divide the first term of the remainder by twice the root already found, used as a trial divisor, and the quotient will be the next term of the root. Write this result in the root, and annex it to the trial divisor to form the complete divisor.

Multiply the complete divisor by this term of the root, and subtract the product from the first remainder.

Find the next term of the root by dividing the first term of the remainder by the first term of the trial divisor.

Form the complete divisor as before and continue in this manner until all the terms of the root have been found.

Find the square root of:

10. $25a^2 - 40a + 16$.

11. $900x^2 + 60x + 1$.

12. $x^2 + xy + \frac{1}{4}y^2$.

13. $4x^4 - 52x^2 + 169$.

14. $9x^4 - 12x^3 + 10x^2 - 4x + 1$.

15. $x^4 - 6x^3y + 13x^2y^2 - 12xy^3 + 4y^4$.

16. $x^8 + 2a^6x^2 - a^4x^4 - 2a^2x^6 + a^8$.

17. $25x^4 + 4 - 12x - 30x^3 + 29x^2$.

18. $1 - 2x + 3x^2 - 4x^3 + 3x^4 - 2x^5 + x^6$.

19. $\frac{a^4}{9} - \frac{4a^3}{3} + 4a^2 + \frac{a^2b}{3} - 2ab + \frac{b^2}{4}$.

20. Find *four* terms of the square root of $1 + x$.

SQUARE ROOT OF ARITHMETICAL NUMBERS

281. Compare the number of digits in the square root of each of the following numbers with the number of digits in the number itself:

NUMBER	ROOT	NUMBER	ROOT	NUMBER	ROOT
1	1	1'00	10	1'00'00	100
25	5	10'24	32	56'25'00	750
81	9	98'01	99	99'80'01	999

From the preceding comparison it may be observed that:

PRINCIPLE. — *If a number is separated into periods of two digits each, beginning at units, its square root will have as many digits as the number has periods.*

The left-hand period may be incomplete, consisting of only one digit.

282. If the number of units expressed by the tens' digit is represented by t and the number of units expressed by the units' digit by u, any number consisting of tens and units will be represented by $t + u$, and its square by $(t+u)^2$, or $t^2 + 2tu + u^2$.

Since $25 = 20 + 5$, $25^2 = (20 + 5)^2 = 20^2 + 2(20 \times 5) + 5^2 = 625$.

EXERCISES

283. 1. Find the square root of 3844.

FIRST PROCESS

	38′44	60 + 2
$t^2 =$	36 00	
$2t = 120$	2 44	
$u = 2$		
$2t + u = 122$	2 44	

EXPLANATION. — Separating the number into periods of two digits each (Prin., § 281), we find that the root is composed of two digits, tens and units. Since the largest square in 38 is 6, the tens of the root cannot be greater than 6 tens, or 60. Writing 6 tens in the root, squaring, and subtracting from 3844, we have a remainder of 244.

Since the square of a number composed of tens and units is equal to (*the square of the tens*) + (*twice the product of the tens and the units*) + (*the square of the units*), when the square of the tens has been subtracted, the remainder, 244, is twice the product of the tens and the units, plus the square of the units, or only a little more than twice the product of the tens and the units.

Therefore, 244 divided by twice the tens is approximately equal to the units. 2×6 tens, or 120, then, is a *trial*, or *partial divisor*. On dividing 244 by the trial divisor, the units' figure is found to be 2.

Since twice the tens are to be multiplied by the units, and the units also are to be multiplied by the units to obtain the square of the units, in order to abridge the process the tens and units are first added, forming the *complete divisor* 122, which is then multiplied by the units. Thus, $(120 + 2)$ multiplied by $2 = 244$.

Therefore, the square root of 3844 is 62.

SECOND PROCESS

	38′44	62
$t^2 =$	36	
$2t = 120$	2 44	
$u = 2$		
$2t + u = 122$	2 44	

EXPLANATION. — In practice it is usual to place the figures of the same order in the same column, and to disregard the ciphers on the right of the products.

Since any number may be regarded as composed of tens and units, the foregoing processes have a general application.

Thus, $346 = 34$ tens $+ 6$ units; $2377 = 237$ tens $+ 7$ units.

2. Find the square root of 104976.

SOLUTION

		10′49′76 | 324
		9
Trial divisor	$= 2 \times 30 = 60$	1 49
Complete divisor	$= 60 + 2 = 62$	1 24
Trial divisor	$= 2 \times 320 = 640$	25 76
Complete divisor	$= 640 + 4 = 644$	25 76

RULE. — *Separate the number into periods of two figures each, beginning at units.*

Find the greatest square in the left-hand period and write its root for the first figure of the required root.

Square this root, subtract the result from the left-hand period, and annex to the remainder the next period for a new dividend.

Double the root already found, with a cipher annexed, for a trial divisor, and by it divide the dividend. The quotient, or quotient diminished, will be the second figure of the root. Add to the trial divisor the figure last found, multiply this complete divisor by the figure of the root last found, subtract the product from the dividend, and to the remainder annex the next period for the next dividend.

Proceed in this manner until all the periods have been used. The result will be the square root sought.

1. When the number is not a perfect square, annex periods of decimal ciphers and continue the process.
2. Decimals are pointed off from the decimal point toward the right.
3. The square root of a common fraction may be obtained by finding the square root of both numerator and denominator separately or by reducing the fraction to a decimal and then finding the root.

Find the square root of:

3. 529.	**6.** 57121.	**9.** 2480.04.
4. 2209.	**7.** 42025.	**10.** 10.9561.
5. 4761.	**8.** 95481.	**11.** .001225.

Find the square root of:

12. $\frac{625}{729}$.	14. $\frac{169}{225}$.	16. $\frac{289}{324}$.	18. $\frac{576}{784}$.
13. $\frac{576}{841}$.	15. $\frac{196}{1156}$.	17. $\frac{361}{400}$.	19. $\frac{289}{961}$.

Find the square root to two decimal places:

20. $\frac{3}{4}$.	22. $\frac{5}{8}$.	24. $\frac{5}{6}$.	26. $\frac{7}{8}$.
21. $\frac{4}{5}$.	23. .6.	25. $\frac{2}{9}$.	27. $\frac{5}{16}$.

ROOTS BY FACTORING

284. The method of finding the cube root of polynomials and of arithmetical numbers, analogous to the one just given for square root, is beyond the scope of this text; but a method of finding the cube root, or any other root, of a number that is a perfect power of the same degree as the index of the required root is here mentioned because of its simplicity.

This method consists in factoring, grouping the factors, and taking the required root of each group.

Thus, $\sqrt[3]{42875} = \sqrt[3]{5 \cdot 5 \cdot 5 \times 7 \cdot 7 \cdot 7} = \sqrt[3]{5^3 \times 7^3} = 5 \times 7 = 35$;

also,
$$\sqrt{x^4 + 2x^3 - 3x^2 - 4x + 4} = \sqrt{(x-1)^2(x+2)^2}$$
$$= (x-1)(x+2)$$
$$= x^2 + x - 2.$$

EXERCISES

285. Find, by the method of factoring:

1. Square root of $a^6 - 12a^3 + 36$.
2. Cube root of $x^3 - 15x^2 + 75x - 125$.
3. Fourth root of $x^4 - 8x^3 + 24x^2 - 32x + 16$.
4. Fifth root of $x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$.

Find the indicated root:

5. $\sqrt[3]{3375}$.	7. $\sqrt[6]{262144}$.	9. $\sqrt[5]{4084101}$.
6. $\sqrt[4]{1296}$.	8. $\sqrt[5]{759375}$.	10. $\sqrt[8]{16777216}$.

RADICALS

286. Thus far the exponents used have been *positive integers* only, and the **laws of exponents** have been based on this idea; but since zero, fractional, and negative exponents may occur in algebraic processes, they must follow the same laws as are given for positive integral exponents; hence, it becomes necessary to discover *meanings* for these *new kinds* of exponents, because, for example, in a^0, a^{-2}, and $a^{\frac{2}{3}}$, the exponents $0, -2$, and $\frac{2}{3}$ cannot show how many times a is used as a factor (§ 9).

287. Meaning of zero and negative exponents.

By notation, §§ 9, 10, $\quad a^2 = 1 \cdot a \cdot a. \quad (1)$

Dividing both members of this equation by a, the first member by subtracting exponents (§ 32) and the second by taking out the factor a, we have

$$a^1 = 1 \cdot a. \quad (2)$$

Dividing (2) by a, $\quad a^0 = 1. \quad (3)$

Dividing (3) by a, $\quad a^{-1} = \frac{1}{a}. \quad (4)$

Dividing (4) by a, $\quad a^{-2} = \frac{1}{a^2}. \quad (5)$

The meaning of a zero exponent, illustrated in (3), and of a negative exponent, in (4) and (5), may be stated as follows:

Any number with a zero exponent is equal to 1.

Any number with a negative exponent is equal to the reciprocal of the same number with a numerically equal positive exponent.

288. The meaning of a negative exponent shows that:

PRINCIPLE. — *Any factor may be transferred from one term of a fraction to the other without changing the value of the fraction, provided the sign of the exponent is changed.*

289. Meaning of a fractional exponent.

Just as, § 276, $\sqrt{a^2} = a^{2 \div 2} = a$,

so $\sqrt[3]{a^2} = a^{2 \div 3} = a^{\frac{2}{3}}$. That is,

The numerator of a fractional exponent with positive integral terms indicates a power and the denominator a root.

Since the operations may be performed in either order:

The fractional exponent as a whole indicates a root of a power or a power of a root.

EXERCISES

290. Find a simple value for:

1. 5^0. 3. 2^{-5}. 5. $(-3)^0$. 7. $(a^n b^2 q)^0$.

2. 4^{-2}. 4. 3^{-3}. 6. $(-6)^{-2}$. 8. $(-\frac{1}{3})^{-2}$.

9. Which is the greater, $(\frac{1}{5})^2$ or $(\frac{1}{5})^3$? $(\frac{1}{5})^{-2}$ or $(\frac{1}{5})^{-3}$?

10. Write $5x^{-3}y^2$ with positive exponents.

SOLUTION. — By § 287, $5x^{-3}y^2 = 5y^2\dfrac{1}{x^3} = \dfrac{5y^2}{x^3}$.

Write with positive exponents:

11. $2x^{-1}$. 13. $a^{-1}b^{-1}$. 15. $4a^2c^{-2}$.

12. $5a^{-5}$. 14. $x^{-3}y^{-2}$. 16. $3ax^{-2}$

17. Write $\dfrac{3a^2y}{bx^2}$ without a denominator.

SOLUTION. — By § 288, $\dfrac{3a^2y}{bx^2} = 3a^2b^{-1}x^{-2}y$.

Write without a denominator:

18. $\dfrac{ax}{by}$. 19. $\dfrac{mn}{a^2}$. 20. $\dfrac{1}{a^{-2}b^2}$. 21. $\left(\dfrac{x}{y}\right)^2$.

22. Find the value of $16^{\frac{3}{4}}$.

FIRST SOLUTION. $16^{\frac{3}{4}} = \sqrt[4]{16^3} = \sqrt[4]{16 \cdot 16 \cdot 16}$

$= \sqrt[4]{(2 \cdot 2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2)}$

$= \sqrt[4]{(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2)}$

$= 2 \cdot 2 \cdot 2 = 8.$

SECOND SOLUTION. $16^{\frac{3}{4}} = (16^{\frac{1}{4}})^3 = 2^3 = 8.$

In numerical exercises it is usually best to find the root first.

Simplify, taking only principal roots:

23. $8^{\frac{1}{3}}$. 25. $64^{\frac{2}{3}}$. 27. $64^{-\frac{2}{3}}$.

24. $8^{\frac{2}{3}}$. 26. $32^{\frac{3}{5}}$. 28. $(-8)^{-\frac{4}{3}}$.

29. Which is the greater, $27^{\frac{2}{3}}$ or $(-27)^{-\frac{2}{3}}$? $(\frac{1}{4})^{\frac{3}{2}}$ or $(\frac{1}{4})^{-\frac{3}{2}}$?

30. Express $\sqrt[3]{a^2bc^{-4}}$ with positive fractional exponents.

SOLUTION. $\sqrt[3]{a^2bc^{-4}} = a^{\frac{2}{3}}b^{\frac{1}{3}}c^{-\frac{4}{3}} = \dfrac{a^{\frac{2}{3}}b^{\frac{1}{3}}}{c^{\frac{4}{3}}}$.

Express with positive fractional exponents:

31. $\sqrt{ab^3}$. 33. $(\sqrt{x})^3$. 35. $(\sqrt[3]{xy})^{-2}$.

32. $\sqrt{xy}$. 34. $(\sqrt[5]{y})^4$. 36. $5\sqrt{x^{-1}y^{-1}}$.

Express roots with radical signs and powers with positive exponents:

37. $a^{\frac{2}{5}}$. 39. $x^{\frac{5}{4}}$. 41. $x^{\frac{5}{6}}y^{\frac{1}{6}}$. 43. $a^{\frac{1}{2}} \div x^{\frac{1}{2}}$.

38. $x^{\frac{4}{5}}$. 40. $a^{\frac{1}{3}}b^{\frac{2}{3}}$. 42. $a^{\frac{1}{4}}b^{-\frac{3}{4}}$. 44. $x^{\frac{2}{3}} \div y^{\frac{4}{5}}$

Multiply:

45. a^3 by a^{-2}. 47. a^4 by a^{-4}. 49. a^2 by a^0.

46. a^2 by a^{-1}. 48. a by a^{-3}. 50. $x^{\frac{1}{2}}$ by $x^{\frac{1}{2}}$.

Divide:

51. a^5 by a^6. 53. a^2 by a^{-2}. 55. $x^{\frac{1}{2}}$ by $x^{\frac{1}{3}}$.

52. a^3 by a^0. 54. $x^{\frac{5}{2}}$ by $x^{-\frac{1}{2}}$. 56. $x^{n-\frac{3}{2}}$ by x^{n-2}.

Solve for values of x corresponding to principal roots by applying axioms 6 and 7 (§§ 261, 275), and test each result:

57. $x^{\frac{1}{2}}=7.$
58. $x^{\frac{3}{4}}=8.$
59. $x^{\frac{4}{3}}=81.$
60. $\frac{1}{3}x^{\frac{3}{2}}=72.$
61. $x^{-\frac{1}{2}}=6.$
62. $x^{-\frac{2}{3}}=144.$
63. $25x^{-\frac{2}{3}}=1.$
64. $x^{\frac{5}{3}}+32=0.$

291. An indicated root of a number is called a **radical**; the number whose root is required is called the **radicand**.

$\sqrt{5a}$, $(x^3)^{\frac{1}{3}}$, $\sqrt[3]{a^2+2}$, and $(x+y)^{\frac{1}{4}}$ are radicals whose radicands are, respectively, $5a$, x^3, a^2+2, and $x+y$.

292. The **order** of a radical is shown by the index of the root or by the denominator of the fractional exponent.

$\sqrt{a+x}$ and $(b-x)^{\frac{1}{2}}$ are radicals of the second order.

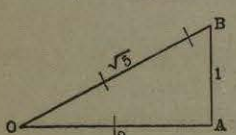
293. In the discussion and treatment of radicals only *principal roots* will be considered.

Thus, $\sqrt{16}$ will be taken to represent only the principal square root of 16, or 4. The other square root will be denoted by $-\sqrt{16}$.

294. **Graphical representation of a radical of the second order.**

In geometry it is shown that the hypotenuse of a right triangle is equal to the *square root of the sum of the squares of the other two sides;* consequently, a radical of the second order may be represented graphically by the *hypotenuse* of a right triangle whose other two sides are such that the sum of their squares is equal to the radicand.

Thus, to represent $\sqrt{5}$ graphically, since it may be observed that $5=2^2+1^2$, draw OA 2 units in length, then draw AB 1 unit in length in a direction perpendicular to OA. Draw OB, completing the right-angled triangle OAB. Then, the length of OB represents $\sqrt{5}$ in its relation to the unit length.



EXERCISES

295. Represent graphically:

1. $\sqrt{2}$.
2. $\sqrt{10}$.
3. $\sqrt{13}$.
4. $\sqrt{17}$.
5. $\sqrt{34}$.
6. $\sqrt{25}$.
7. $\sqrt{\frac{5}{4}}$.
8. $\sqrt{\frac{13}{36}}$.

296. A number that is, or may be, expressed as an integer or as a fraction with integral terms, is called a **rational number**.

3, $\frac{1}{2}$, $\sqrt{25}$, and .333 are rational numbers.

297. A number that cannot be expressed as an integer or as a fraction with integral terms is called an **irrational number**.

$\sqrt{2}$, $4^{\frac{1}{3}}$, $1+\sqrt{3}$, and $\sqrt{1+\sqrt{3}}$ are irrational numbers.

From § 294, it will be observed that the irrational number $\sqrt{5}$ can be represented graphically by a line of *exact* length, though it cannot be represented exactly by decimal figures, for $\sqrt{5}=2.236\ldots$, which is an endless decimal.

298. When the indicated root of a *rational* number cannot be obtained exactly, the expression is called a **surd**.

$\sqrt{2}$ is a surd, since 2 is rational but has no rational square root.

$\sqrt{1+\sqrt{3}}$ is not a surd, because $1+\sqrt{3}$ is not rational.

Radicals may be either rational or irrational, but surds are always irrational.

Both $\sqrt{4}$ and $\sqrt{3}$ are radicals, but only $\sqrt{3}$ is a surd.

299. A surd may contain a *rational factor*, that is, a factor whose radicand is a perfect power of a degree corresponding to the order of the surd.

The rational factor may be removed and written as the coefficient of the irrational factor.

In $\sqrt{8}=\sqrt{4\times 2}$ and $\sqrt[3]{54}=\sqrt[3]{27\times 2}$, the rational factors are $\sqrt{4}$ and $\sqrt[3]{27}$, respectively; that is, $\sqrt{8}=2\sqrt{2}$ and $\sqrt[3]{54}=3\sqrt[3]{2}$.

300. In the following pages it will be assumed that irrational numbers obey the same law as rational numbers. For proofs of the generality of these laws, the reader is referred to the author's *Advanced Algebra*.

REDUCTION OF RADICALS

301. To reduce a radical to its simplest form.

As the work progresses the student will discover the meaning of *simplest form*.

EXERCISES

302. 1. Reduce $\sqrt{20\,a^6}$ to its simplest form by writing the rational factor as the coefficient of the irrational factor.

PROCESS

$$\sqrt{20\,a^6} = \sqrt{4\,a^6 \times 5} = \sqrt{4\,a^6} \times \sqrt{5} = 2\,a^3\sqrt{5}$$

EXPLANATION. — Since the highest factor of $20\,a^6$ that is a perfect square is $4\,a^6$, $\sqrt{20\,a^6}$ is separated into two factors, a rational factor $\sqrt{4\,a^6}$, and an irrational factor $\sqrt{5}$; that is, § 277, $\sqrt{20\,a^6} = \sqrt{4\,a^6} \times \sqrt{5}$.

On finding the square root of $4\,a^6$ and prefixing the root to the irrational factor as a coefficient, the result is $2\,a^3\sqrt{5}$.

2. Reduce $\sqrt[3]{-864}$ to its simplest form.

PROCESS

$$\sqrt[3]{-864} = \sqrt[3]{-216 \times 4} = \sqrt[3]{-216} \times \sqrt[3]{4} = -6\sqrt[3]{4}$$

RULE. — *Separate the radical into two factors one of which is its highest rational factor.*

Find the required root of the rational factor, multiply the result by the coefficient, if any, of the given radical, and place the product as the coefficient of the irrational factor.

Reduce to simplest form:

3. $\sqrt{12}$.
4. $\sqrt{75}$.
5. $\sqrt[3]{16}$.
6. $\sqrt{128}$.
7. $\sqrt[3]{250}$.
8. $\sqrt[4]{32}$.
9. $\sqrt{18\,a^2}$.
10. $\sqrt{25\,b}$.
11. $\sqrt{98\,c^3}$.
12. $\sqrt{50\,a}$.
13. $\sqrt{243\,a^5x^{10}}$.
14. $\sqrt[3]{128\,a^6b^4}$.
15. $(a^3 + 5\,a^2)^{\frac{1}{2}}$.
16. $\sqrt{18\,x - 9}$.
17. $\sqrt[3]{x^6 - 2\,x^3}$.

18. Reduce $\sqrt{\frac{a^2}{2\,y^3}}$ to its simplest form; that is, to a radical having an integral radicand.

PROCESS

$$\sqrt{\frac{a^2}{2\,y^3}} = \sqrt{\frac{a^2 \times 2\,y}{2\,y^3 \times 2\,y}} = \sqrt{\frac{a^2}{4\,y^4}} \times \sqrt{2\,y} = \frac{a}{2\,y^2}\sqrt{2\,y}$$

EXPLANATION. — Since the denominator must be removed from the radical, and since the radical is of the second order, the denominator must be made a perfect square. The smallest factor that will do this is $2\,y$.

On multiplying both terms of the fraction by this factor, the largest rational factor of the resulting radical is found to be $\sqrt{\frac{a^2}{4\,y^4}}$, or $\frac{a}{2\,y^2}$.

Therefore, the irrational factor is $\sqrt{2\,y}$ and its coefficient is $\frac{a}{2\,y^2}$.

Reduce to simplest form:

19. $\sqrt{\frac{1}{2}}$.
20. $\sqrt{\frac{2}{3}}$.
21. $\sqrt{\frac{3}{8}}$.
22. $\sqrt[3]{\frac{4}{5}}$.
23. $\sqrt[3]{\frac{5}{12}}$.
24. $\sqrt{\frac{2\,a^3}{b}}$.
25. $\sqrt{\frac{5\,x^4y^2}{2\,a^2}}$.
26. $\sqrt[4]{\frac{x}{y}}$.
27. $\sqrt{\frac{2}{3\,y^5}}$.
28. $\sqrt{\frac{4\,a}{3\,x^2}}$.
29. $\sqrt{\frac{3\,x}{50\,a^3y}}$.

303. Although $\frac{3}{6} = \frac{1}{2}$, it does not follow that $64^{\frac{3}{6}} = 64^{\frac{1}{2}}$, for each fractional exponent denotes a power of a root of 64, and the roots and powers taken are not the same for $64^{\frac{1}{2}}$ as for $64^{\frac{3}{6}}$. By trial, however, it is found that each number is equal to 8; and in general it may be proved that

A number having a fractional exponent is not changed in value by reducing the fractional exponent to higher or lower terms.

EXERCISES

304. 1. Reduce $\sqrt[6]{9\,a^2}$ to its simplest form; that is, to a radical having the smallest index possible.

PROCESS

$$\sqrt[6]{9\,a^2} = \sqrt[6]{(3\,a)^2} = (3\,a)^{\frac{2}{6}} = (3\,a)^{\frac{1}{3}} = \sqrt[3]{3\,a}$$

2. Reduce $\sqrt[9]{64\,a^6b^{15}}$ to its simplest form.

PROCESS

$$\sqrt[9]{64\,a^6b^{15}} = \sqrt[9]{2^6a^6b^6b^9} = b\,(2\,ab)^{\frac{6}{9}} = b\,(2\,ab)^{\frac{2}{3}} = b\sqrt[3]{4\,a^2b^2}$$

Simplify:

3. $\sqrt[4]{36}$. 5. $\sqrt[4]{1600}$. 7. $\sqrt[4]{9\,a^2b^2c^6}$.

4. $\sqrt[4]{25}$. 6. $\sqrt[6]{27\,a^3}$. 8. $\sqrt[4]{121\,a^6x^4}$.

305. The student has doubtless discovered that:

A radical is in its **simplest form** when the index of the root is as small as possible, and when the radicand is integral and contains no factor that is a perfect power whose exponent corresponds with the index of the root.

$\sqrt{7}$ is in its simplest form; but $\sqrt{\frac{2}{4}}$ is not in its simplest form, because $\frac{2}{4}$ is not integral in form; $\sqrt{8}$ is not in its simplest form, because 4, a factor of 8, has an exact square root; $\sqrt[6]{25}$, or $25^{\frac{1}{6}}$, is not in its simplest form, because $25^{\frac{1}{6}} = (5^2)^{\frac{1}{6}} = 5^{\frac{2}{6}} = 5^{\frac{1}{3}}$, or $\sqrt[3]{5}$.

MISCELLANEOUS EXERCISES

306. Reduce to simplest form:

1. $\sqrt{600}$. 5. $\sqrt[3]{189}$. 9. $\sqrt[4]{144}$. 13. $\sqrt{\frac{1}{3}}$.

2. $\sqrt{500}$. 6. $\sqrt{84}$. 10. $\sqrt[6]{81}$. 14. $\sqrt{\frac{1}{x^3}}$.

3. $\sqrt[5]{160}$. 7. $\sqrt[3]{72}$. 11. $\sqrt[6]{343}$.

4. $\sqrt[3]{3000}$. 8. $\sqrt[3]{192}$. 12. $\sqrt[4]{289}$. 15. $\sqrt[3]{\frac{a}{3\,b^2}}$.

16. $\sqrt{405\,a^5y^2}$. 18. $\sqrt{8-20\,b^2}$. 20. $\sqrt[6]{a^4b^2c^4d^8}$.

17. $(135\,x^4y^5)^{\frac{1}{3}}$. 19. $5\sqrt{4\,a^2+4}$. 21. $(16\,x-16)^{\frac{1}{2}}$.

307. A surd that has a rational coefficient is called a **mixed surd.**

$2\sqrt{2}$, $a\sqrt[3]{x^2}$, and $(a-b)\sqrt{a+b}$ are mixed surds.

308. A surd that has no rational coefficient except unity is called an **entire surd.**

$\sqrt{5}$, $\sqrt[3]{11}$, and $\sqrt{a^2+x^2}$ are entire surds.

309. To reduce a mixed surd to an entire surd.

EXERCISES

1. Express $2\,a\sqrt{5\,b}$ as an entire surd.

PROCESS

$$2\,a\sqrt{5\,b} = \sqrt{4\,a^2}\sqrt{5\,b} = \sqrt{4\,a^2 \times 5\,b} = \sqrt{20\,a^2b}$$

RULE. — *Raise the coefficient to a power corresponding to the index of the given radical, and introduce the result under the radical sign as a factor.*

Express as entire surds:

2. $2\sqrt{2}$. 5. $3\sqrt[3]{3}$. 8. $\frac{1}{2}\sqrt{2}$. 11. $\frac{4}{5}\sqrt{4\frac{3}{8}}$.

3. $3\sqrt{5}$. 6. $4\sqrt{5}$. 9. $\frac{3}{4}\sqrt{x^5}$. 12. $\frac{3}{2}\sqrt{\frac{23}{27}a^2}$.

4. $5\sqrt{2}$. 7. $\frac{1}{2}\sqrt{8}$. 10. $\frac{1}{2}\sqrt{bc}$. 13. $\frac{2}{3}\sqrt[3]{1\frac{1}{8}}$.

310. To reduce radicals to the same order.

EXERCISES

1. Reduce $\sqrt[4]{3}$, $\sqrt{2}$, and $\sqrt[3]{4}$ to radicals of the same order.

PROCESS

$$\sqrt[4]{3} = 3^{\frac{1}{4}} = 3^{\frac{3}{12}} = \sqrt[12]{3^3} = \sqrt[12]{27}$$
$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{6}{12}} = \sqrt[12]{2^6} = \sqrt[12]{64}$$
$$\sqrt[3]{4} = 4^{\frac{1}{3}} = 4^{\frac{4}{12}} = \sqrt[12]{4^4} = \sqrt[12]{256}$$

RULE. — *Express the given radicals with fractional exponents having a common denominator.*

Raise each number to the power indicated by the numerator of its fractional exponent, and indicate the root expressed by the common denominator.

Reduce to radicals of the same order:

2. $\sqrt{2}$ and $\sqrt[4]{3}$.
3. $\sqrt{5}$ and $\sqrt[3]{6}$.
4. $\sqrt[4]{7}$ and $\sqrt{10}$.
5. $\sqrt[6]{10}$, $\sqrt{2}$, and $\sqrt[3]{5}$.
6. $\sqrt[6]{4}$, $\sqrt[4]{2}$, and $\sqrt{3}$.
7. $\sqrt[10]{13}$, $\sqrt{5}$, and $\sqrt[5]{4}$.
8. $\sqrt{3}$, $\sqrt[3]{5}$, and $\sqrt[4]{\frac{1}{27}}$.
9. $\sqrt{ab}$, $\sqrt[3]{ab^2}$, and $\sqrt[4]{2}$.
10. $\sqrt{a}$, $\sqrt[3]{b}$, $\sqrt[4]{x}$, and $\sqrt[6]{y}$.
11. $\sqrt[3]{a+b}$ and $\sqrt{x+y}$.
12. Which is greater, $\sqrt[5]{5}$ or $\sqrt{2}$? $\sqrt[3]{4}$ or $\sqrt{3}$?
13. Which is greater, $\sqrt[3]{3}$ or $\sqrt[4]{4}$? $3\sqrt{2}$ or $3\sqrt[3]{4}$?

Arrange in order of ascending value:

14. $\sqrt[3]{3}$, $\sqrt{2}$, and $\sqrt[6]{7}$.
15. $\sqrt{2}$, $\sqrt[3]{4}$, and $\sqrt[4]{5}$.
16. $\sqrt[3]{2}$, $\sqrt[5]{3}$, and $\sqrt[15]{30}$.
17. $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt{2\frac{1}{2}}$, and $\sqrt[3]{4}$.
18. $\sqrt{7}$, $\sqrt[4]{48}$, $\sqrt[3]{4}$, and $\sqrt[6]{63}$.
19. $\sqrt[3]{4}$, $\sqrt[4]{5}$, $\sqrt[6]{13}$, $\sqrt[12]{150}$.

ADDITION AND SUBTRACTION OF RADICALS

311. Radicals that in their simplest form are of the same order and have the same radicand are called **similar radicals.**

Thus, $2\sqrt{3}$ and $7\sqrt{3}$ are similar radicals.

312. PRINCIPLE.— *Only similar radicals can be united into one term by addition or subtraction.*

EXERCISES

313. 1. Find the sum of $\sqrt{50}$, $2\sqrt[6]{8}$, and $6\sqrt{\frac{1}{2}}$.

PROCESS

$$\begin{aligned}\sqrt{50} &= 5\sqrt{2}\\ 2\sqrt[6]{8} &= 2\sqrt{2}\\ 6\sqrt{\tfrac{1}{2}} &= 3\sqrt{2}\\ \hline \text{Sum} &= 10\sqrt{2}\end{aligned}$$

EXPLANATION. — To ascertain whether the given expressions are similar radicals, each may be reduced to its simplest form. Since, in their simplest form, they are of the same order and have the same radicand, they are similar, and their sum is obtained by prefixing the sum of the coefficients to the common radical factor.

Find the sum of:

2. $\sqrt{50}$, $\sqrt{18}$, and $\sqrt{98}$.
3. $\sqrt{27}$, $\sqrt{12}$, and $\sqrt{75}$.
4. $\sqrt{20}$, $\sqrt{80}$, and $\sqrt{45}$.
5. $\sqrt{28}$, $\sqrt{63}$, and $\sqrt{700}$.
6. $\sqrt[3]{250}$, $\sqrt[3]{16}$, and $\sqrt[3]{54}$.
7. $\sqrt[3]{128}$, $\sqrt[3]{686}$, and $\sqrt[3]{\frac{1}{4}}$.
8. $\sqrt[3]{135}$, $\sqrt[3]{320}$, and $\sqrt[3]{625}$.
9. $\sqrt[3]{500}$, $\sqrt[3]{108}$, and $\sqrt[3]{-32}$.
10. $\sqrt{\frac{1}{2}}$, $\sqrt{12\frac{1}{2}}$, $\sqrt{\frac{1}{8}}$, and $\sqrt{1\frac{1}{8}}$.
11. $\sqrt{\frac{1}{3}}$, $\sqrt{75}$, $\frac{2}{3}\sqrt{3}$, and $\sqrt{12}$.
12. $\sqrt{\frac{3}{4}}$, $\frac{1}{3}\sqrt{3}$, $\frac{7}{6}\sqrt[4]{9}$, and $\sqrt{147}$.
13. $\sqrt[3]{40}$, $\sqrt{28}$, $\sqrt[6]{25}$, and $\sqrt{175}$.
14. $\sqrt{147}$, $4\sqrt{20}$, $\sqrt{75}$, and $\sqrt{605}$.
15. $\sqrt[3]{192}$, $\sqrt{80}$, $4\sqrt{45}$, and $5\sqrt[3]{24}$.

Simplify:

16. $\sqrt{245}-\sqrt{405}+\sqrt{45}$.
17. $\sqrt{12}+3\sqrt{75}-2\sqrt{27}$.
18. $5\sqrt{72}+3\sqrt{18}-\sqrt{50}$.
19. $\sqrt[3]{128}+\sqrt[3]{686}-\sqrt[3]{54}$.
20. $\sqrt{112}-\sqrt{343}+\sqrt{448}$.
21. $\sqrt{\frac{a}{x^2}}+\sqrt{\frac{a}{y^2}}-\sqrt{\frac{a}{z^2}}$.
22. $\sqrt{\frac{ax^4}{by^2}}-\sqrt{\frac{16\,ax^2}{by^2}}$.
23. $\sqrt{\frac{a}{bc}}+\sqrt{\frac{b}{ac}}+\sqrt{\frac{c}{ab}}$.
24. $\sqrt{(a+b)^2c}-\sqrt{(a-b)^2c}$.
25. $\sqrt[3]{abx}-\sqrt[6]{a^2b^2x^2}+\sqrt[9]{8\,a^3b^3x^3}$.
26. $\sqrt{3\,x^3+30\,x^2+75\,x}-\sqrt{3\,x^3-6\,x^2+3\,x}$.
27. $\sqrt{5\,a^5+30\,a^4+45\,a^3}-\sqrt{5\,a^5-40\,a^4+80\,a^3}$.
28. $\sqrt{50}+\sqrt[6]{9}-4\sqrt{\frac{1}{2}}+\sqrt[3]{-24}+\sqrt[9]{27}-\sqrt[4]{64}$.
29. $\sqrt{\frac{2}{3}}+6\sqrt{\frac{5}{4}}-\frac{1}{5}\sqrt{18}+\sqrt[4]{36}-\sqrt[8]{\frac{16}{81}}+\sqrt[6]{125}-2\sqrt{\frac{2}{25}}$.

MULTIPLICATION OF RADICALS

314. $$a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{1}{2}+\frac{1}{3}} = a^{\frac{3}{6}+\frac{2}{6}} = a^{\frac{5}{6}}.$$

That is, $$\sqrt{a} \times \sqrt[3]{a} = \sqrt[6]{a^3} \times \sqrt[6]{a^2} = \sqrt[6]{a^5}.$$

Since fractional exponents to be united by addition must be expressed with a common denominator, radicals to be united by multiplication must be expressed with a common root index.

EXERCISES

315. 1. Multiply $\sqrt{7}$ by $\sqrt{5}$; $5\sqrt{3}$ by $2\sqrt{15}$.

PROCESSES

$$\sqrt{7} \times \sqrt{5} = \sqrt{7 \times 5} = \sqrt{35}$$

$$5\sqrt{3} \times 2\sqrt{15} = 5 \times 2\sqrt{3 \times 15} = 10\sqrt{45} = 10\sqrt{9}\sqrt{5} = 30\sqrt{5}$$

2. Multiply $2\sqrt{3}$ by $3\sqrt[3]{2}$.

PROCESS

$$2\sqrt{3} = 2 \cdot 3^{\frac{1}{2}} = 2 \cdot 3^{\frac{3}{6}} = 2\sqrt[6]{27}$$

$$3\sqrt[3]{2} = 3 \cdot 2^{\frac{1}{3}} = 3 \cdot 2^{\frac{2}{6}} = 3\sqrt[6]{4}$$

$$2\sqrt{3} \times 3\sqrt[3]{2} = 2\sqrt[6]{27} \times 3\sqrt[6]{4} = 2 \times 3\sqrt[6]{27 \times 4} = 6\sqrt[6]{108}$$

RULE. — *If the radicals are not of the same order, reduce them to the same order.*

Multiply the coefficients for the coefficient of the product and the radicands for the radical factor of the product; simplify the result, if necessary.

Multiply:

3. $\sqrt{2}$ by $\sqrt{8}$.

4. $\sqrt{2}$ by $\sqrt{6}$.

5. $\sqrt{3}$ by $\sqrt{15}$.

6. $2\sqrt{5}$ by $3\sqrt{10}$.

7. $3\sqrt{20}$ by $2\sqrt{2}$.

8. $2\sqrt[3]{3}$ by $3\sqrt[3]{45}$.

9. $2\sqrt[4]{6}$ by $3\sqrt{6}$.

10. $3\sqrt{3}$ by $2\sqrt[3]{5}$.

11. $2\sqrt[3]{24}$ by $\sqrt[3]{18}$.

12. $\sqrt{2xy}$ by $3\sqrt[3]{x^2y^3}$.

Find the value of:

13. $\sqrt{mn} \times \sqrt[4]{m^2n} \times \sqrt[8]{mn^4}$.

14. $\sqrt{2axy} \times \sqrt[3]{xy} \times \sqrt[4]{a^2xy}$.

15. $\sqrt{a-b} \times \sqrt[4]{a^2b^2} \times \sqrt[4]{(a-b)^{-2}}$.

16. $\sqrt{\frac{2}{3}} \times \sqrt{\frac{4}{5}} \times \sqrt{\frac{3}{4}}$.

17. $\sqrt[3]{\frac{2}{3}} \times \sqrt[3]{\frac{3}{4}} \times \sqrt{\frac{1}{2}}$.

18. $16^{\frac{1}{3}} \times 2^{\frac{1}{2}} \times 32^{\frac{5}{6}}$.

19. $27^{\frac{1}{4}} \times 9^{\frac{1}{3}} \times 81^{\frac{1}{3}}$.

20. Multiply $2\sqrt{2}+3\sqrt{3}$ by $5\sqrt{2}-2\sqrt{3}$.

SOLUTION

$$\begin{array}{l} 2\sqrt{2} + 3\sqrt{3} \\ \underline{5\sqrt{2} - 2\sqrt{3}} \\ 20 \quad + 15\sqrt{6} \\ \underline{\qquad - \ 4\sqrt{6} - 18} \\ 20 \quad + 11\sqrt{6} - 18 = 2 + 11\sqrt{6}. \end{array}$$

Multiply:

21. $\sqrt{5}+\sqrt{3}$ by $\sqrt{5}-\sqrt{3}$.

22. $\sqrt{7}+\sqrt{2}$ by $\sqrt{7}-\sqrt{2}$.

23. $\sqrt{6}-\sqrt{5}$ by $\sqrt{6}-\sqrt{5}$.

24. $5-\sqrt{5}$ by $1+\sqrt{5}$.

25. $4\sqrt{7}+1$ by $4\sqrt{7}-1$.

26. $2\sqrt{2}+\sqrt{3}$ by $4\sqrt{2}+\sqrt{3}$.

27. $a^2-ab\sqrt{2}+b^2$ by $a^2+ab\sqrt{2}+b^2$.

28. $x\sqrt{x}-x\sqrt{y}+y\sqrt{x}-y\sqrt{y}$ by $\sqrt{x}+\sqrt{y}$.

Expand:

29. $(\sqrt{3+\sqrt{5}})(\sqrt{3-\sqrt{5}})$.

30. $(\sqrt{9+\sqrt{6}})(\sqrt{9-\sqrt{6}})$.

31. $(\sqrt{6+\sqrt{11}})(\sqrt{6-\sqrt{11}})$.

32. $(\sqrt{5a+a\sqrt{5}})(\sqrt{5a-a\sqrt{5}})$.

DIVISION OF RADICALS

316. $a^{\frac{1}{2}} \div a^{\frac{1}{3}} = a^{\frac{1}{2}-\frac{1}{3}} = a^{\frac{3}{6}-\frac{2}{6}} = a^{\frac{1}{6}}.$

That is, $\sqrt{a} \div \sqrt[3]{a} = \sqrt[6]{a^3} \div \sqrt[6]{a^2} = \sqrt[6]{a^3 \div a^2} = \sqrt[6]{a}.$

In division, when one fractional exponent is subtracted from another, the exponents must be expressed with a common denominator. When one radical is divided by another, the radicals must be expressed with a common root index.

EXERCISES

317. 1. Divide $\sqrt{60}$ by $\sqrt{12}$.

PROCESS

$$\sqrt{60} \div \sqrt{12} = \sqrt{60 \div 12} = \sqrt{5}$$

2. Divide $\sqrt[3]{2}$ by $\sqrt{2}$.

PROCESS

$$\sqrt[3]{2} \div \sqrt{2} = \sqrt[6]{4} \div \sqrt[6]{8} = \sqrt[6]{\tfrac{4}{8}} = \sqrt[6]{\tfrac{32}{64}} = \tfrac{1}{2}\sqrt[6]{32}$$

RULE. — *If necessary, reduce the radicals to the same order.*

To the quotient of the coefficients annex the quotient of the radicands written under the common radical sign, and reduce the result to its simplest form.

Find quotients:

3. $\sqrt{50} \div \sqrt{8}$.
4. $\sqrt{72} \div 2\sqrt{6}$.
5. $4\sqrt{5} \div \sqrt{40}$.
6. $6\sqrt{7} \div \sqrt{126}$.
7. $\sqrt[3]{4} \div \sqrt{2}$.
8. $7\sqrt{75} \div 5\sqrt{28}$.
9. $\sqrt[3]{16} \div \sqrt[6]{32}$.
10. $\sqrt{2ab^3} \div \sqrt[4]{a^4b^4}$.
11. $\sqrt[3]{a^2x^2} \div \sqrt{2ax}$.
12. $\sqrt[3]{9a^2b^2} \div \sqrt{3ab}$.
13. $\sqrt[4]{4x^2y^2} \div \sqrt[3]{2xy}$.
14. $\sqrt{a-b} \div \sqrt{a+b}$.

15. Divide $\sqrt{15} - \sqrt{3}$ by $\sqrt{3}$.
16. Divide $\sqrt{6} - 2\sqrt{3} + 4$ by $\sqrt{2}$.
17. Divide $\sqrt{2} + 2 + \frac{1}{3}\sqrt{42}$ by $\frac{1}{3}\sqrt{6}$.
18. Divide $5\sqrt{2} + 5\sqrt{3}$ by $\sqrt{10} + \sqrt{15}$.
19. Divide $5 + 5\sqrt{30} + 36$ by $\sqrt{5} + 2\sqrt{6}$.

INVOLUTION AND EVOLUTION OF RADICALS

318. In finding powers and roots of radicals, it is frequently convenient to use fractional exponents.

EXERCISES

319. 1. Find the cube of $2\sqrt{ax^5}$.

SOLUTION. $(2\sqrt{ax^5})^3 = 2^3(a^{\frac{1}{2}}x^{\frac{5}{2}})^3 = 8a^{\frac{3}{2}}x^{\frac{15}{2}} = 8\sqrt{a^3x^{15}} = 8ax^7\sqrt{ax}.$

2. Find the square of $3\sqrt[6]{x^5}$.

SOLUTION. $(3\sqrt[6]{x^5})^2 = 9(x^{\frac{5}{6}})^2 = 9x^{\frac{5}{3}} = 9\sqrt[3]{x^5} = 9x\sqrt[3]{x^2}.$

Square:

3. $3\sqrt{cb}$.
4. $2\sqrt[3]{5x}$.
5. $x\sqrt[3]{2x^3}$.
6. $n^2\sqrt{4b}$.

Cube:

7. $2\sqrt{5}$.
8. $3\sqrt{2}$.
9. $2\sqrt[3]{a^2}$.
10. $\sqrt[4]{a^2b^3}$.

Involve as indicated:

11. $(-2\sqrt{2ab})^4$.
12. $(-\sqrt{2}\sqrt[6]{x})^3$.
13. $(-\sqrt{2}\sqrt[3]{cx^2})^4$.
14. $(-2\sqrt{x}\sqrt[3]{y})^5$.

15. Obtain by the binomial formula the cube of $\sqrt{2} + 1$.

SOLUTION.
$$\begin{aligned}(\sqrt{2}+1)^3 &= (\sqrt{2})^3 + 3(\sqrt{2})^2 \cdot 1 + 3\sqrt{2} \cdot 1^2 + 1^3 \\ &= 2\sqrt{2} + 6 + 3\sqrt{2} + 1 \\ &= 7 + 5\sqrt{2}.\end{aligned}$$

Expand:

16. $(2+\sqrt{6})^2$.
17. $(2+\sqrt{2})^2$.
18. $(2+\sqrt{5})^3$.
19. $(2-\sqrt{3})^3$.
20. $(\sqrt{7}-\sqrt{6})^2$.
21. $(2\sqrt{2}-\sqrt{3})^2$.

22. What is the fourth root of $\sqrt{2x}$?

SOLUTION. $\sqrt[4]{\sqrt{2x}} = [(2x)^{\frac{1}{2}}]^{\frac{1}{4}} = (2x)^{\frac{1}{8}} = \sqrt[8]{2x}.$

Find the square root of:

23. $\sqrt{2}$.
24. $\sqrt[3]{5}$.
25. $\sqrt[6]{x^2}$.
26. $\sqrt[5]{x^{12}}$.

Find the cube root of:

27. $\sqrt{2x}$.
28. $\sqrt[4]{8m^2x^3}$.
29. $-27\sqrt{x^6}$.
30. $-64\sqrt[5]{a^2y^3}$.

Simplify the following indicated roots:

31. $\sqrt{\sqrt[3]{4a^2x^4}}$.
32. $\sqrt[3]{\sqrt{a^{12}x^4}}$.
33. $(\sqrt{8a^3x^3})^{\frac{1}{3}}$.

320. A binomial, one or both of whose terms are surds, is called a **binomial surd**.

$\sqrt{2} + \sqrt{5}$, $2 + \sqrt{5}$, $\sqrt[3]{2} + 1$, and $\sqrt{3} - \sqrt[3]{2}$ are binomial surds.

321. A binomial surd whose surd or surds are of the second order is called a **binomial quadratic surd.**

$\sqrt{2} + \sqrt{5}$ and $2 + \sqrt{5}$ are binomial quadratic surds.

322. Two binomial quadratic surds that differ only in the sign of one of the terms are called **conjugate surds.**

$3 + \sqrt{5}$ and $3 - \sqrt{5}$ are conjugate surds; also $\sqrt{3} + \sqrt{2}$ and $\sqrt{3} - \sqrt{2}$.

323. The square root of a binomial quadratic surd by inspection.

The square of a binomial may be written in the form

$$(a+b)^2 = (a^2+b^2) + 2ab.$$

Thus, $\quad (\sqrt{2}+\sqrt{6})^2 = (2+6) + 2\sqrt{12} = 8 + 2\sqrt{12}.$

Therefore, the terms of the square root of $8 + 2\sqrt{12}$ may be obtained by separating $\sqrt{12}$ into two factors such that the sum of their squares is 8. They are $\sqrt{2}$ and $\sqrt{6}$.

That is, $\quad \sqrt{8 + 2\sqrt{12}} = \sqrt{2} + \sqrt{6}.$

PRINCIPLE. — *The terms of the square root of a binomial quadratic surd that is a perfect square may be obtained by dividing the irrational term by 2 and then separating the quotient into two factors, the sum of whose squares is the rational term.*

EXERCISES

324. 1. Find the square root of $14 + 8\sqrt{3}$.

SOLUTION

$$14 + 8\sqrt{3} = 14 + 2(4\sqrt{3}) = 14 + 2\sqrt{48}.$$

Since $\quad \sqrt{48} = \sqrt{6} \times \sqrt{8}$ and $14 = 6 + 8,$

$$\sqrt{14 + 8\sqrt{3}} = \sqrt{6} + \sqrt{8} = \sqrt{6} + 2\sqrt{2}.$$

2. Find the square root of $11 - 6\sqrt{2}$.

SOLUTION

$$\sqrt{11 - 6\sqrt{2}} = \sqrt{11 - 2\sqrt{18}} = \sqrt{9} - \sqrt{2} = 3 - \sqrt{2}.$$

Find the square root of:

3. $12 + 2\sqrt{35}$.
4. $16 - 2\sqrt{60}$.
5. $15 + 2\sqrt{26}$.
6. $16 - 2\sqrt{55}$.
7. $11 + 2\sqrt{30}$.
8. $7 - 2\sqrt{10}$.
9. $12 - 6\sqrt{3}$.
10. $17 + 12\sqrt{2}$.
11. $12 + 4\sqrt{5}$.
12. $11 + 4\sqrt{7}$.
13. $15 - 6\sqrt{6}$.
14. $18 + 6\sqrt{5}$.

RATIONALIZATION

325. Suppose that it is required to find the approximate value of $\frac{1}{\sqrt{3}}$, having given $\sqrt{3} = 1.732 \cdots$.

```
1.732 ···)1.000000(.577 ···        3)1.732 ···
          8660                        .577 ···
          ···
```

We may obtain a decimal approximately equal to $\frac{1}{\sqrt{3}}$, as in the first process (incomplete), by dividing 1 by $1.732 \cdots$; but a great saving of labor may be effected by first changing the fraction to an equal fraction having a *rational* denominator, thus:

$$\frac{1}{\sqrt{3}} = \frac{1 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{3}}{3},$$

and employing the second process.

326. The process of multiplying an expression containing a surd by any number that will make the product rational is called **rationalization.**

327. The factor by which a surd expression is multiplied to render the product rational is called the **rationalizing factor.**

328. The process of reducing a fraction having an irrational denominator to an equal fraction having a rational denominator is called **rationalizing the denominator.**

EXERCISES

329. Find the value of each of the following to the nearest third decimal place, taking $\sqrt{2}=1.41421$, $\sqrt{3}=1.73205$, and $\sqrt{5}=2.23607$:

1. $\frac{5}{\sqrt{2}}$. 4. $\frac{3}{\sqrt{6}}$. 7. $\frac{10}{\sqrt{45}}$.

2. $\sqrt{\frac{2}{3}}$. 5. $\frac{6}{\sqrt{8}}$. 8. $\frac{15}{\sqrt{50}}$.

3. $\frac{2}{\sqrt{5}}$. 6. $\frac{4}{\sqrt{12}}$. 9. $\frac{1}{\sqrt{125}}$.

Rationalize the denominator of each of the following, using the smallest, or lowest, rationalizing factor possible:

10. $\frac{\sqrt{a}}{\sqrt{b}}$. 14. $\frac{2\sqrt{a}}{\sqrt{by}}$. 18. $\frac{\sqrt{r-1}}{\sqrt{r+1}}$.

11. $\frac{1}{\sqrt{x^7}}$. 15. $\frac{\sqrt{6}}{\sqrt[3]{12}}$. 19. $\frac{\sqrt{a+b}}{\sqrt{a-b}}$.

12. $\frac{ax}{\sqrt{2a^3x}}$. 16. $\frac{\sqrt{a}}{\sqrt[3]{ax^2}}$. 20. $\sqrt{\frac{x-2}{x+2}}$.

13. $\frac{\sqrt{63x}}{\sqrt{8b^2}}$. 17. $\sqrt{\frac{x^2y}{3xy^2}}$. 21. $\sqrt[3]{\frac{a+b}{(a-b)^2}}$.

330. *The product of any two conjugate surds is rational.*

For, by § 94, $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=(\sqrt{a})^2-(\sqrt{b})^2=a-b.$

Hence, *a binomial quadratic surd may be rationalized by multiplying it by its conjugate.*

EXERCISES

331. 1. Rationalize the denominator of $\frac{2}{3-\sqrt{5}}$.

SOLUTION

$$\frac{2}{3-\sqrt{5}}=\frac{2(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})}=\frac{2(3+\sqrt{5})}{9-5}=\frac{3+\sqrt{5}}{2}.$$

2. Rationalize the denominator of $\frac{6}{\sqrt{7}+\sqrt{3}}$.

SOLUTION

$$\frac{6}{\sqrt{7}+\sqrt{3}}=\frac{6(\sqrt{7}-\sqrt{3})}{(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})}=\frac{6(\sqrt{7}-\sqrt{3})}{7-3}=\frac{3(\sqrt{7}-\sqrt{3})}{2}.$$

Rationalize the denominator of:

3. $\frac{3}{2+\sqrt{3}}$. 6. $\frac{1}{\sqrt{3}-\sqrt{2}}$. 9. $\frac{b}{a+2\sqrt{b}}$.

4. $\frac{5}{\sqrt{5}-\sqrt{3}}$. 7. $\frac{3}{2-\sqrt{2}}$. 10. $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$.

5. $\frac{8}{\sqrt{3}+\sqrt{5}}$. 8. $\frac{6}{\sqrt{6}-2\sqrt{3}}$. 11. $\frac{3-\sqrt{ab}}{3+\sqrt{ab}}$.

12. $\frac{19}{3\sqrt{3}-2\sqrt{2}}$. 14. $\frac{x}{x+\sqrt{x^2-1}}$.

13. $\frac{x-3}{\sqrt{x+1}+2}$. 15. $\frac{x+y}{\sqrt{x+y}+\sqrt{x-y}}$.

Reduce to a decimal, to the nearest hundredth:

16. $\frac{3}{2-\sqrt{3}}$. 17. $\frac{4}{3+\sqrt{5}}$. 18. $\frac{5}{\sqrt{3}-\sqrt{2}}$.

RADICAL EQUATIONS

332. An equation involving an irrational root of an unknown number is called an **irrational,** or **radical, equation.**

$x^{\frac{1}{2}} = 3$, $\sqrt{x+1} = \sqrt{x-4} + 1$, and $\sqrt[3]{x-1} = 2$ are radical equations.

333. A radical equation may be freed of radicals, wholly or in part, by raising both members, suitably prepared, to the same power. If the given equation contains more than one radical, involution may have to be repeated.

When the following equations have been freed of radicals, the resulting equations will be found to be simple equations. Other varieties of radical equations are treated subsequently.

EXERCISES

334. 1. Given $\sqrt{2x} + 4 = 10$, to find the value of x.

SOLUTION

$$\sqrt{2x} + 4 = 10.$$

Transposing, $\sqrt{2x} = 6.$

Squaring, $2x = 36.$

$\therefore x = 18.$

VERIFICATION. — Substituting 18 for x in the given equation and (§ 293) considering only the *positive* value of $\sqrt{2x}$, we have $\sqrt{36} + 4 = 10$; that is, $10 = 10$; hence, the equation is satisfied for $x = 18$.

2. Given $\sqrt{x-7} + \sqrt{x} = 7$, to find the value of x.

SOLUTION

$$\sqrt{x-7} + \sqrt{x} = 7.$$

Transposing, $\sqrt{x-7} = 7 - \sqrt{x}.$

Squaring, $x - 7 = 49 - 14\sqrt{x} + x.$

Transposing and combining, $14\sqrt{x} = 56.$

Dividing by 14, $\sqrt{x} = 4.$

Squaring, $x = 16.$

VERIFICATION. $\sqrt{16-7} + \sqrt{16} = \sqrt{9} + \sqrt{16} = 3 + 4 = 7$; that is, $7 = 7$.

3. Solve, if possible, the equation $\sqrt{x-7} - \sqrt{x} = 7.$

SOLUTION

Transposing, squaring, simplifying, etc.,

$$\sqrt{x} = -4.$$

Squaring, $x = 16.$

VERIFICATION. — Substituting 16 for x in the given equation and (§ 293) considering only the *positive* value of $\sqrt{x-7}$ and of $\sqrt{x}$, the first member becomes

$$\sqrt{16-7} - \sqrt{16} = \sqrt{9} - \sqrt{16} = 3 - 4 = -1;$$

but the second member of the given equation is 7; hence, $x = 16$ does not satisfy the equation.

That is, the equation has no root, or is *impossible*.

General Directions. — *Transpose so that the radical term, if there is but one, or the most complex radical term, if there is more than one, may constitute one member of the equation.*

Then raise each member to a power corresponding to the order of that radical and simplify.

If the equation is not freed of radicals by the first involution, proceed again as at first.

Solve, and verify results, denoting impossible equations:

4. $\sqrt{x+1} = 3.$
5. $\sqrt{x+5} = 4.$
6. $\sqrt{x-7} = 1.$
7. $\sqrt{x-a^2} = b.$
8. $\sqrt[3]{x-1} = 2.$
9. $\sqrt[3]{x-a^3} = a.$
10. $\sqrt{x} + b = a.$
11. $1 + \sqrt{x} = 5.$
12. $2\sqrt{x} = 6 - \sqrt{x}.$
13. $\sqrt{x+16} - \sqrt{x} = 2.$
14. $\sqrt{2x} - \sqrt{2x-3} = 1.$
15. $\sqrt{2x} + \sqrt{2x-3} = 1.$
16. $\sqrt{x^2+x+1} = 2 - x.$
17. $3\sqrt{x^2-9} = 3x - 3.$
18. $\sqrt{3x+7} + \sqrt{3x} = 7.$
19. $2\sqrt{x} + \sqrt{4x-11} = 1.$
20. $5 - \sqrt{x+5} = \sqrt{x}.$
21. $\sqrt{x^2-5x+7} + 2 = x.$

Solve, and verify results, denoting impossible equations:

22. $4-\sqrt{4-8x+9x^2}=3x.$

23. $\sqrt{3x-5}+\sqrt{3x+7}=6.$

24. $\sqrt{4x+5}-2\sqrt{x-1}=9.$

25. $\sqrt{2x-1}+\sqrt{2x+4}=5.$

26. $\sqrt{5x-1}-1=\sqrt{5x+16}.$

27. $\sqrt{7+3\sqrt{5x-16}}-4=0.$

28. $2x-\sqrt{4x^2-\sqrt{16x^2-7}}=1.$

29. $\sqrt{4x}-\sqrt{x}=\sqrt{9x-32}.$

30. $\sqrt{2(x^2+3x-5)}=(x+2)\sqrt{2}.$

31. $\sqrt{2(x+1)}+\sqrt{2x-1}=\sqrt{8x+1}.$

32. Solve the equation $\sqrt{2x}-\sqrt{2x-7}=\dfrac{3}{\sqrt{2x-7}}.$

SUGGESTION.—Clear the equation of fractions.

33. Solve the equation $\dfrac{\sqrt{3x}+15}{\sqrt{3x}+5}=\dfrac{\sqrt{3x}+6}{\sqrt{3x}+1}.$

SUGGESTION.—Some labor may be saved by reducing each fraction to a mixed number and *simplifying before clearing of fractions.*

Thus,
$$1+\frac{10}{\sqrt{3x}+5}=1+\frac{5}{\sqrt{3x}+1}.$$

Canceling, and dividing both members by 5,
$$\frac{2}{\sqrt{3x}+5}=\frac{1}{\sqrt{3x}+1}.$$

Solve and verify:

34. $\dfrac{\sqrt{s-1}}{\sqrt{s+5}}=\dfrac{\sqrt{s-3}}{\sqrt{s-1}}.$

35. $\dfrac{\sqrt{t-3}}{\sqrt{t+1}}=\dfrac{\sqrt{t-4}}{\sqrt{t-2}}.$

36. $\dfrac{\sqrt{2x}+9}{\sqrt{2x}-7}=\dfrac{\sqrt{2x}+20}{\sqrt{2x}-12}.$

37. $\dfrac{\sqrt{x}+18}{\sqrt{x}+2}=\dfrac{32}{\sqrt{x}+6}+1.$

REVIEW

335. 1. Distinguish between involution and evolution.

2. Give the law of exponents for involution; for evolution.

3. For what values of n between 1 and 12 is $(-2)^n$ positive? negative? What is the sign for any power of a positive number?

4. How is a fraction raised to a power? How is the root of a fraction found? Raise $\frac{9}{16}$ to the second power. $\sqrt{\frac{9}{16}}=?$

5. How is the power of a product found? the root of a product?

6. What operation is indicated by a radical sign? In what other way may this operation be indicated? Illustrate.

7. How many values has $\sqrt{25}$? What is the principal square root of 25? What is the principal cube root of -8?

8. What is the index of a root? What index is meant when none is expressed?

9. Distinguish between real and imaginary numbers and illustrate each.

10. What is the sign of an odd root of a number? of an even root of a number?

11. What is the meaning of x^0?

12. When a number has a fractional exponent, what does the numerator of the exponent show? the denominator?

13. What is the meaning of x^{-2}? How may any factor be transferred from one term of a fraction to the other?

Illustrate by writing without a denominator: $\dfrac{x^2}{a^{-3}b}.$

14. Expand $(x-y)^6$ by the binomial formula. How does the number of terms correspond with the exponent of the power? What is the coefficient of the first and last terms? of the second term? How are the coefficients of the other terms obtained?

15. Define and illustrate radical, radicand, entire surd, and mixed surd.

16. What is meant by the order of a radical? Illustrate by giving radicals of different orders.

17. How may a radical of the second order be represented graphically?

Illustrate by representing graphically $\sqrt{26}$.

18. What is a rational number? an irrational number?

From the following select the rational numbers:

$$8;\ \tfrac{2}{9};\ \sqrt{3};\ \sqrt[3]{8};\ -7\tfrac{2}{3};\ \sqrt{25};\ 5^{\frac{1}{2}}.$$

19. Are $\sqrt[3]{15}$ and $\sqrt{1+\sqrt{2}}$ radicals? surds? Are all radicals surds? Are all surds radicals?

20. How may the coefficient of a radical be placed under the radical sign?

Express as entire surds: $\frac{1}{3}\sqrt{2}$; $9\sqrt{bc}$; $\frac{2}{7}\sqrt[3]{x^2y}$.

21. When is a radical in its simplest form?

Illustrate by reducing $\sqrt{40\,b^2c}$, $\sqrt{\frac{2}{3}}$, and $\sqrt[6]{4}$, each to its simplest form.

22. What are similar radicals? When numbers have fractional exponents with different denominators, what must be done to the fractional exponents before the numbers can be multiplied? Find the value of $5^{\frac{1}{3}} \times 10^{\frac{1}{2}} \times 6^{\frac{1}{4}}$.

23. What is a binomial surd? a binomial quadratic surd? What are conjugate surds?

24. Define rationalization; rationalizing factor.

How may a binomial quadratic surd be rationalized?

Rationalize the denominator of $\dfrac{7\,x}{\sqrt{a}+\sqrt{b}}$.

25. What is a radical equation? Give general directions for solving a radical equation.

Solve and verify: $\sqrt{x}+\sqrt{3+x}=3$.

QUADRATIC EQUATIONS

336. The equation $x-2=0$ is of the *first* degree and has *one* root, $x=2$. Similarly, $x-3=0$ is of the first degree and has one root, $x=3$. Consequently, the product of these two simple equations, which is

$$(x-2)(x-3)=0, \text{ or } x^2-5\,x+6=0,$$

is of the *second degree* and has *two* roots, 2 and 3.

337. An equation that, when simplified, contains the *square* of the unknown number, but no higher power, is called an equation of the *second degree*, or a **quadratic equation.**

It is evident, therefore, that quadratic equations may be of two kinds — those which contain only the second power of the unknown number, and those which contain both the second and first powers.

$x^2=15$ and $3\,x^2+2x=4$ are quadratic equations.

PURE QUADRATIC EQUATIONS

338. An equation that contains only the second power of the unknown number is called a **pure quadratic.**

$2\,x^2=8$ and $4\,x^2-2\,x^2=16$ are pure quadratic equations.

339. The equation $x^2=16$ has two roots, for it may be reduced to the form $(x-4)(x+4)=0$, which is equivalent to the two simple equations,

$$x-4=0 \text{ and } x+4=0,$$

each of which has one root, namely, $+4$ and -4. That is,

Principle. — *Every pure quadratic equation has two roots, numerically equal but opposite in sign.*