GRAPHIC SOLUTIONS

SIMPLE EQUATIONS

237. When related quantities in a series are to be compared, as, for instance, the population of a town in successive years, recourse is often had to a method of representing quantities by *lines*. This is called the **graphic** method.

By this method, quantity is photographed in the process of change. The whole range of the variation of a quantity, presented in this vivid pictorial way, is easily comprehended at a glance; it stamps itself on the memory.

238. In Fig. 1 is shown the population of a town throughout its variations during the first 13 years of the town's existence.

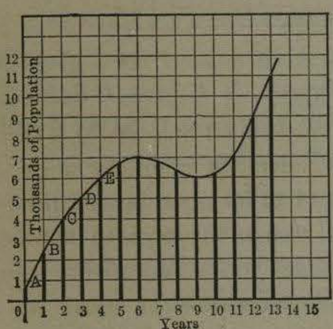

Fig. 1.

The population at the end of 2 years, for example, is represented by the length of the heavy black line drawn upward from 2, and is 4000; the population at the end of 6 years is 7000; at the end of 10 years, 6300 approximately; and so on.

239. Every point of the curved line shown in Fig. 1 exhibits a pair of corresponding values of two related quantities, years and population. For instance, the position of E shows that the population at the end of 4 years was 6000.

Such a line is called a **graph**.

Graphs are useful in numberless ways. The statistician uses them to present information in a telling way. The broker or merchant uses them to compare the rise and fall of prices. The physician uses them to record the progress of diseases. The engineer uses them in testing materials and in computing. The scientist uses them in his investigations of the laws of nature. In short, graphs may be used whenever two related quantities are to be compared throughout a series of values.

240. The graph in Fig. 2 represents the rate in gallons per day per person at which water was used in New York City during a certain day of 24 hours.

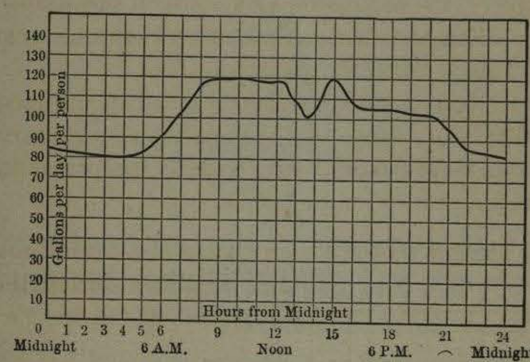

Fig. 2.

Thus, if each horizontal space represents 1 hour (from midnight) and each vertical space 10 gallons, at midnight water was being used at the rate of about 84 gallons per day per person; at 6 A.M., about 91 gallons; at 1 P.M., the 13th hour, about 108 gallons; etc.

1. What was the approximate consumption of water at 2 A.M.? at noon? at 1:30 P.M.? at 3 P.M.? at 6 P.M.?

2. What was the maximum rate during the day? the minimum rate? at what time did each occur?

3. During what hours was the rate most uniform? What was the rate at the middle of each hour?

4. What was the average increase per hour between 6 A.M. and 8 A.M.? the average decrease between 4 P.M. and 8 P.M.?

241. The graphs in Fig. 3 present to the eye in a forceful way the remarkable contrasts in the months of July of the years 1901 and 1904 by showing the *average daily maximum temperature* for ten cities that cover the part of the United States east of the Rocky Mountains.

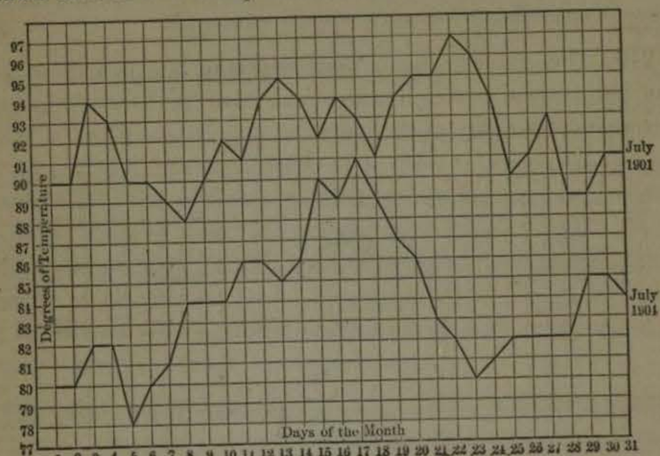

Fig. 3.

The vertical spaces for 0° to 76° inclusive are omitted. In the following 'temperature' means 'average maximum temperature.'

1. In which year was the month of July the hotter?

2. On how many days of July, 1901, was the temperature *below* 90°? How many days of July, 1904, had a temperature *above* 90°?

3. What was the *highest* temperature for July, 1901, and on what day did it occur? for July, 1904? Give the *lowest* temperature for July of each of these years and the date of its occurrence.

4. Which year had the smaller range of temperature for July? How many degrees hotter was the Fourth of July, 1901, than the Fourth of July, 1904?

5. Find the difference between the highest temperature of July, 1901, and the lowest temperature of July, 1904.

242. Fig. 4 gives two graphs,—one showing the height of water above zero of the gage in the Cumberland River at Nashville, the other showing the same thing for the Arkansas River at Little Rock, from daily observations taken in September of the year 1908.

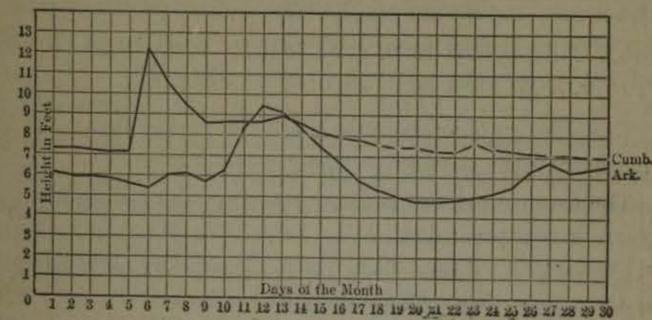

Fig. 4.

In giving heights read the graphs to the nearest tenth of a foot.

1. When was the water in the Cumberland highest? lowest? What was the maximum height? the minimum? the range between them?

2. How many days *later* than the Cumberland was the Arkansas at its maximum? How many days *earlier* than the Cumberland was the Arkansas at its minimum?

3. What was the range for the Arkansas during the month?

4. State the difference between the maximum readings for the two rivers; between their minimum readings.

5. On what days were the readings higher for the Arkansas than for the Cumberland?

6. What part of the month shows the greatest and most rapid changes in the height of the Cumberland? the Arkansas?

7. Which river had the least variation in height during the last half of the month?

8. Give the time of the greatest change in a single day for either river. How much was this change?

EXERCISES

243. 1. Letting each horizontal space represent 10 years and each vertical space 1 million of population, locate points from the pairs of corresponding values (years and millions of population given below) and connect these points with a line, thus constructing a population graph of the United States:

1810, 7.2; 1820, 9.6; 1830, 12.8; 1840, 17.1; 1850, 23.2; 1860, 31.4; 1870, 38.6; 1880, 50.2; 1890, 62.6; 1900, 76.3.

2. From the graph of exercise 1, tell the period during which the increase in the population was greatest; least.

3. The average price of tin in cents per pound for the months of a certain year was: Jan., 23.4; Feb., 24.7; Mar., 26.2; Apr., 27.3; May, 29.3; June, 29.3; July, 28.3; Aug., 28.1; Sept., 26.6; Oct., 25.8; Nov., 25.4; Dec., 25.3.

Draw a graph to show the variation in the price of tin during the year with each horizontal space representing 1 month and each vertical space 1 cent.

4. Construct the graph of exercise 3, letting each horizontal space represent 1 cent and each vertical space 1 month.

244. Let x and y be two *algebraic* quantities so related that $y = 2x - 3$. It is evident that we may give x a series of values, and obtain a corresponding series of values of y; and that *the number of such pairs of values of x and y is unlimited.* All of these values are represented in the graph of $y = 2x - 3$. Just as in the preceding illustrations, so in the graph of $y = 2x - 3$, Fig. 5, values of x are represented by lines laid off on or parallel to an **x-axis**, X′X, and values of y by lines laid off on or parallel to a **y-axis**, Y′Y, usually drawn perpendicular to the x-axis.

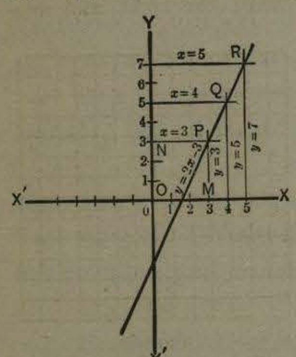

FIG. 5.

For example, the position of P shows that $y = 3$ when $x = 3$; the position of Q shows that $y = 5$ when $x = 4$; the position of R shows that $y = 7$ when $x = 5$; etc.

Evidently every point of the graph gives a pair of corresponding values of x and y.

245. Conversely, to locate any point with reference to two axes for the purpose of representing a pair of corresponding values of x and y, the value of x may be laid off on the x-axis as an *x-distance*, or **abscissa**, and that of y on the y-axis as a *y-distance*, or **ordinate**. If from each of the points on the axes obtained by these measurements, a line parallel to the other axis is drawn, the intersection of these two lines locates the point.

Thus, in Fig. 5, to represent the corresponding values $x = 3$, $y = 3$, a point P may be located by measuring 3 units from O to M on the x-axis and 3 units from O to N on the y-axis, and then drawing a line from M parallel to OY, and one from N parallel to OX, producing these lines until they intersect.

246. The abscissa and ordinate of a point referred to two perpendicular axes are called the **rectangular coördinates**, or simply the **coördinates**, of the point.

Thus, in Fig. 5, the coördinates of P are $OM(= NP)$ and $MP(= ON)$.

247. By universal custom *positive* values of x are laid off from O as a zero-point, or **origin**, toward the *right*, and *negative* values toward the *left*. Also *positive* values of y are laid off *upward* and *negative* values *downward*.

The point A in Fig. 6 may be designated as 'the point (2, 3),' or by the equation $A = (2, 3)$.

Similarly,

$B = (-2, 4)$, $C = (-3, -1)$, and $D = (1, -2)$.

The abscissa is always written first.

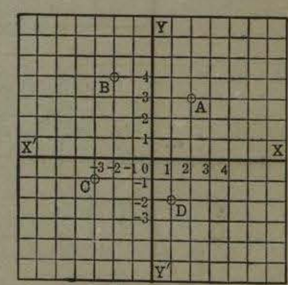

FIG. 6.

248. Plotting points and constructing graphs.

EXERCISES

NOTE. — The use of paper ruled in small squares, called coördinate paper, is advised in plotting graphs.

Draw two axes at right angles to each other and locate:

1. $A = (3, 2)$. 5. $E = (5, 5)$. 9. $L = (0, 4)$.
2. $B = (3, -2)$. 6. $F = (-5, 5)$. 10. $M = (0, -5)$.
3. $C = (4, 3)$. 7. $G = (-2, 5)$. 11. $N = (3, 0)$.
4. $D = (4, -3)$. 8. $H = (-3, -4)$. 12. $P = (-6, 0)$.

13. Where do all points having the abscissa 0 lie? the ordinate 0?

14. What are the coördinates of the origin?

15. Construct the graph of the equation $2y - x = 2$.

SOLUTION

Solving for y, $y = \frac{1}{2}(x + 2)$.

Values are now given to x and corresponding values are computed for y by means of this equation. The numbers substituted for x need not be large. Convenient numbers to be substituted for x in this instance are the even integers from -6 to $+6$.

When $x = -6$, $y = -2$. These values locate the point $A = (-6, -2)$.
When $x = -4$, $y = -1$. These values serve to locate $B = (-4, -1)$.
Other points may be located in the same way.

A record of the work should be kept as follows:

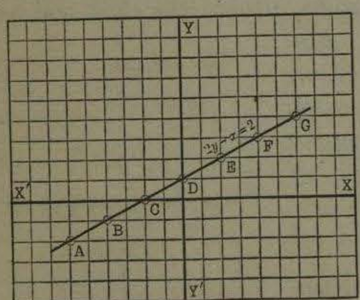

FIG. 7.

$y = \frac{1}{2}(x + 2)$

x	y	POINT
-6	-2	A
-4	-1	B
-2	0	C
0	1	D
2	2	E
4	3	F
6	4	G

A line drawn through A, B, C, D, etc., is the graph of $2y - x = 2$.

Construct the graph of each of the following:

16. $y = 3x - 7$. 19. $3x - y = 4$. 22. $3x = 2y$.
17. $y = 2x + 1$. 20. $4x - y = 10$. 23. $2x + y = 1$.
18. $y = 2x - 1$. 21. $x - 2y = 2$. 24. $2x + 3y = 6$.

249. It can be proved by the principle of the similarity of triangles that:

PRINCIPLE. — *The graph of a simple equation is a straight line.*

For this reason simple equations are sometimes called **linear equations.**

250. Since a straight line is determined by two points, to plot the graph of a linear equation, *plot two points and draw a straight line through them.*

It is often convenient to plot the points where the graph intersects the axes. To find where it intersects the x-axis, let $y = 0$; to find where it intersects the y-axis, let $x = 0$.

Thus, in $y = \frac{1}{2}(x + 2)$, when $y = 0$, $x = -2$, locating C, Fig. 7; when $x = 0$, $y = 1$, locating D.

Draw a straight line through C and D.

If the equation has no absolute term, $x = 0$ when $y = 0$, and this method gives only one point. In any case it is desirable, *for the sake of accuracy*, to plot points some distance apart, as A and G, in Fig. 7.

EXERCISES

251. Construct the graph of each of the following:

1. $y = x - 2$. 8. $2x - 3y = 6$. 15. $8x - 3y = -6$.
2. $y = 2 - x$. 9. $2x + 3y = 0$. 16. $-2x + y = -3$.
3. $y = 9 - 4x$. 10. $x - 4y = 3$. 17. $-3x + 4y = 8$.
4. $y = 4x - 9$. 11. $7x - y = 14$. 18. $5x + 3y = 7\frac{1}{2}$.
5. $y = 10 - 2x$. 12. $4 - x = 2y$. 19. $x - \frac{1}{2}y = 3$.
6. $y = 2x - 10$. 13. $3x + 4y = 12$. 20. $\frac{1}{2}x + \frac{1}{3}y = 2$.
7. $y = 2x - 4$. 14. $5x - 2y = 10$. 21. $.7x - .3y = .4$.

252. Graphic solution of simultaneous linear equations.

I. Let it be required to solve graphically the equations

$$\begin{cases} y = 2 + x, & (1) \\ y = 6 - x. & (2) \end{cases}$$

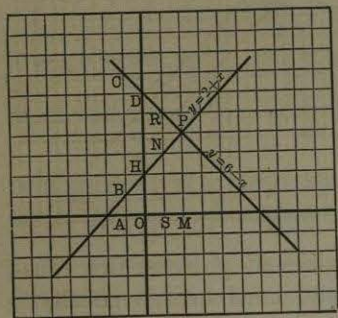

FIG. 8.

As in § 248, construct the graph of each equation, as shown in Fig. 8.

1. When $x = -1$, the value of y in (1) is represented by AB, and in (2) by AC.

Therefore, when $x = -1$, the equations are not satisfied by the same values of y.

2. Compare the values of y when $x = 0$; when $x = 1$; 2.

3. For what value of x are the values of y in the two equations equal, or coincident?

4. What values of x and y will satisfy *both* equations?

The required values of x and y, then, are represented graphically by the coördinates of P, *the intersection of the graphs.*

II. Let the given equations

be $$\begin{cases} x + y = 7, \\ 2x + 2y = 14. \end{cases}$$

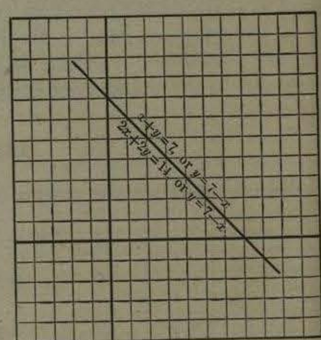

FIG. 9.

5. What happens if we try to eliminate either x or y?

6. Since $y = 7 - x$ in both equations, what will be the relative positions of any two points plotted for the same value of x? the relative positions of the two graphs?

7. The algebraic analysis shows that the equations are **indeterminate.**

The graphic analysis also shows that the equations are indeterminate, for *their graphs coincide.*

III. Let the given equations

be $$\begin{cases} y = 6 - x, & (1) \\ y = 4 - x. & (2) \end{cases}$$

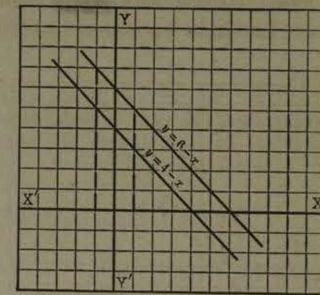

FIG. 10.

8. When $x = -1$, how much greater is the value of y in (1) than in (2), as shown both by the equations and their graphs?

9. Compare the y's for other values of x.

10. For every value of x the values of y in the two equations differ by 2, and the graphs are 2 units apart, vertically.

In algebraic language, the equations cannot be simultaneous; that is, they are **inconsistent.**

In graphical language, their graphs *cannot intersect*, being parallel straight lines.

253. PRINCIPLES. — 1. *A single linear equation involving two unknown numbers is indeterminate.*

2. *Two linear equations involving two unknown numbers are determinate, provided the equations are independent and simultaneous.*

They are satisfied by one, and only one, pair of common values.

3. *The pair of common values is represented graphically by the coördinates of the intersection of their graphs.*

EXERCISES

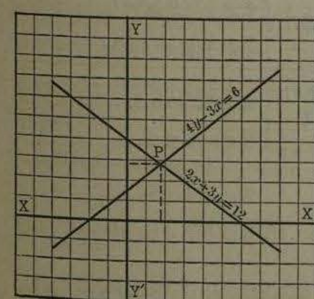

FIG. 11.

254. 1. Solve graphically the equations $$\begin{cases} 4y - 3x = 6, \\ 2x + 3y = 12. \end{cases}$$

SOLUTION. — On plotting the graphs of both equations, as in § 248, it is found that they intersect at a point P, whose coördinates are 1.8 and 2.8, approximately.

Hence, $x = 1.8$ and $y = 2.8$.

The coördinates of P are estimated to the nearest tenth.

NOTE. — In solving simultaneous equations by the graphic method the same axes must be used for the graphs of both equations.

Construct the graphs of each of the following systems of equations. Solve, if possible. If there is no solution, tell why.

2. $\begin{cases} x - y = 1, \\ x + y = 9. \end{cases}$

3. $\begin{cases} x + y = 3, \\ x + 2y = 4. \end{cases}$

4. $\begin{cases} x = 4 + y, \\ y = 3 + x. \end{cases}$

5. $\begin{cases} 2x - y = 5, \\ 4x + y = 16. \end{cases}$

6. $\begin{cases} 3x = y + 9, \\ 2y = 6x - 18. \end{cases}$

7. $\begin{cases} y = 4x, \\ x - y = 3. \end{cases}$

8. $\begin{cases} x = \frac{1}{2}(y + 4), \\ y = 2(x - 2). \end{cases}$

9. $\begin{cases} x + y = -3, \\ x - 2y = -12. \end{cases}$

10. $\begin{cases} x + y = 4, \\ y = 2 - x. \end{cases}$

11. $\begin{cases} x = 2(y + 1), \\ 21 = 2(2x + y). \end{cases}$

12. $\begin{cases} x + y = 8, \\ 2x - 6y = -9. \end{cases}$

13. $\begin{cases} 2x - 5y = 5, \\ 10y = 2x + 1. \end{cases}$

14. $\begin{cases} 3y = 2x - 7, \\ 2x = 6 + 3y. \end{cases}$

15. $\begin{cases} 3(x - 4) = 2y, \\ 6(y + 6) = 9x. \end{cases}$

16. $\begin{cases} 10x + y = 14, \\ 8x - 5y = -2. \end{cases}$

17. $\begin{cases} 2x + 3y = 8, \\ 3x + 2y = 8. \end{cases}$

18. $\begin{cases} 4y + 3x = 5, \\ 4x - 3y = 3. \end{cases}$

19. $\begin{cases} x + 3y = -6, \\ 2x - 4y = -12. \end{cases}$

20. $\begin{cases} 4x - 10y = 0, \\ 2x + y = 12. \end{cases}$

21. $\begin{cases} x - 2y = 2, \\ 2y - 6x = 3. \end{cases}$

22. $\begin{cases} 3x + 4y = 10, \\ 6x + 8y = 20. \end{cases}$

23. $\begin{cases} \frac{3}{2}x + \frac{5}{2}y = 3\frac{1}{2}, \\ 10x - 2y = 14. \end{cases}$

REVIEW

255. 1. Distinguish between integral and fractional equations; between dependent and independent equations.

2. What is meant by the root of an equation? by solving an equation?

3. Define and illustrate equivalent equations.

4. What is a formula? Give a simple formula that has been used in solving some problem.

5. Find three values for x and y in $x + y = 15$. What kind of an equation is this, and why?

6. Define simultaneous equations; elimination. State the axiom upon which elimination by addition is based; elimination by comparison.

7. Outline the method of elimination by addition or subtraction; by substitution.

8. State what is meant by a graph. Of what practical use are graphs?

9. Define abscissa; ordinate; coördinates; origin.

10. In making graphs, where are positive values of x and y laid off? negative values? Interpret the equation $A = (-4, 3)$.

11. What is the abscissa of any point of the y-axis? the ordinate of any point of the x-axis? What are the coördinates of the origin?

12. Why are simple equations sometimes called linear equations?

13. Construct the graph of $2y = 3x - 4$.

14. How many points is it necessary to plot in drawing the graph of a simple equation? Why?

15. Tell how to determine where a graph crosses the x-axis; the y-axis.

16. In drawing the graph of the equation $3y = 2x$, what is the result, if the only points plotted are those where the graph intersects the axes? What must be done in a case like this?

17. Of what does the graphical solution of two simultaneous simple equations consist?

18. Solve graphically and algebraically:

$$\begin{cases} 2x - 3y = 10, \\ 5x + 2y = \ \ 6. \end{cases}$$

Compare the results obtained.

19. If a system of two linear equations is indeterminate, how will this fact be shown by the graphs of the equations referred to the same axes?

20. Draw the graphs of the two equations

$$\begin{cases} x + y = 6, \\ x = 13 - y, \end{cases}$$

and tell the algebraic meaning of the fact that the two graphs do not intersect.

21. From the following select the integral equations; the fractional equations; the numerical equations; the literal equations; the indeterminate equations:

(1)	(3)	(5)
$3x + 5y = 19.$	$ax + bx = c.$	$5x + 2 = 3x - 10.$
(2)	**(4)**	**(6)**
$\frac{7}{2x} + \frac{15}{3x} = 51.$	$\frac{2x}{3} + \frac{25x}{9} = 31.$	$\frac{a}{bx} + \frac{c}{dy} = a^2b^2.$

22. Classify the following sets of equations as equivalent equations, dependent equations, independent equations, simultaneous equations, or inconsistent equations:

(1) $\begin{cases} 3x = 18, \\ x + 5 = 11. \end{cases}$ (3) $\begin{cases} x + \ \ y = 6, \\ 3x - \ \ y = 2. \end{cases}$

(2) $\begin{cases} x + y = \ \ 7, \\ x + y = 11. \end{cases}$ (4) $\begin{cases} x - \ \ y = 2, \\ 3x - 3y = 6. \end{cases}$

INVOLUTION

256. The process of finding any required power of an expression is called **involution.**

257. The following illustrate powers of positive numbers, of negative numbers, of powers, of products, and of quotients, and show that every case of involution is an example of multiplication of *equal* factors.

POWERS OF A POSITIVE NUMBER	POWERS OF A NEGATIVE NUMBER	POWERS OF A POWER
$2 = 2^1$	$-2 = (-2)^1$	$4 = 2^2$
2	-2	4
$4 = 2^2$	$4 = (-2)^2$	$16 = (2^2)^2 = 2^4$
2	-2	4
$8 = 2^3$	$-8 = (-2)^3$	$64 = (2^2)^3 = 2^6$
2	-2	4
$16 = 2^4$	$16 = (-2)^4$	$256 = (2^2)^4 = 2^8$

POWER OF A PRODUCT

$$(2 \cdot 3)^2 = (2 \cdot 3) \times (2 \cdot 3) = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2.$$

POWER OF A QUOTIENT

$$\left(\frac{2}{3}\right)^2 = \frac{2}{3} \cdot \frac{2}{3} = \frac{2^2}{3^2}.$$

258. From these examples and § 78, it is seen that, for involution:

Law of Signs. — *All powers of a positive number are positive; even powers of a negative number are positive, and odd powers are negative.*

259. From the examples in § 257 observe that, for involution:

Law of Exponents. — *The exponent of a power of a number is equal to the exponent of the number multiplied by the exponent of the power to which the number is to be raised.*

260. The last two examples in § 257 illustrate the following:

Principles. — 1. *Any power of a product is equal to the product of its factors each raised to that power.*

2. *Any power of the quotient of two numbers is equal to the quotient of the numbers each raised to that power.*

261. Axiom 6. — *The same powers of equal numbers are equal.*

Thus, if $x = 3$, $x^2 = 3^2$, or 9; also $x^4 = 3^4$, or 81; etc.

262. Involution of monomials.

EXERCISES

1. What is the third power of $4a^3b$?

Solution

$$(4a^3b)^3 = 4a^3b \times 4a^3b \times 4a^3b = 64a^9b^3.$$

2. What is the fifth power of $-2ab^2$?

Solution

$$(-2ab^2)^5 = -2ab^2 \times -2ab^2 \times -2ab^2 \times -2ab^2 \times -2ab^2 = -32a^5b^{10}.$$

To raise an integral term to any power:

Rule. — *Raise the numerical coefficient to the required power and annex to it each letter with an exponent equal to the product of its exponent by the exponent of the required power.*

Make the power positive or negative according to the law of signs.

Raise to the power indicated:

3. $(ab^2c^3)^2$.
4. $(a^3b^2c)^4$.
5. $(2a^2c)^3$.
6. $(7a^2m^5)^2$.
7. $(-4c^2y^5)^3$.
8. $(-2a^3n^5)^4$.
9. $(abcx)^m$.
10. $(2e^2x^c)^6$.
11. $(-1)^{99}$.
12. $(-1)^{200}$.
13. $(3bc)^n$.
14. $(2a^2x^3)^n$.

15. What is the square of $-\frac{5a^3x^2}{7b^2c}$?

Solution

$$\left(-\frac{5a^3x^2}{7b^2c}\right)^2 = -\frac{5a^3x^2}{7b^2c} \times -\frac{5a^3x^2}{7b^2c} = \frac{25a^6x^4}{49b^4c^2}.$$

To raise a fraction to any power:

Rule. — *Raise both numerator and denominator to the required power and prefix the proper sign to the result.*

Raise to the power indicated:

16. $\left(\frac{1}{4b}\right)^2$.
17. $\left(\frac{2x}{7y}\right)^2$.
18. $\left(\frac{3x^2}{10y^3}\right)^2$.
19. $\left(-\frac{5}{ab}\right)^2$.
20. $\left(-\frac{2}{3x}\right)^4$.
21. $\left(-\frac{3x}{2y}\right)^3$.
22. $\left(-\frac{2a}{b}\right)^6$.
23. $\left(-\frac{b^3c^2}{a^2x}\right)^2$.
24. $\left(\frac{a^2b^3}{xy^4}\right)^n$.

263. Involution of polynomials.

The following are type forms of *squares* of polynomials:

§ 85, $(a+x)^2 = a^2 + 2ax + x^2$.

§ 88, $(a-x)^2 = a^2 - 2ax + x^2$.

§ 91, $(a-x+y)^2 = a^2 + x^2 + y^2 - 2ax + 2ay - 2xy$.

EXERCISES

264. Raise to the second power:

1. $2a+b$.
2. $2a-b$.
3. $x+3y$.
4. $3x-4y^3$.
5. $5m^4-11$.
6. $4rs^2+t^3$.
7. $2a+3b-4c$.
8. $5a^2-1+4n^3$.
9. $3r^2+2s+t^4$.

Raise to the required power by multiplication:

10. $(x+y)^3$.
11. $(x-y)^3$.
12. $(x+y)^4$.
13. $(x-y)^4$.
14. $(x+y)^5$.
15. $(x-y)^5$.

THE BINOMIAL FORMULA

265. By actual multiplication,

$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3.$$
$$(a-x)^3 = a^3 - 3a^2x + 3ax^2 - x^3.$$
$$(a+x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4.$$
$$(a-x)^4 = a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4.$$

From the *expansions* just given, and as many others as the student may wish to obtain by multiplication, the following *observations* may be made in regard to any *positive integral power* of any *binomial*, the letter a standing for the first term and x for the second:

1. *The number of terms is one greater than the index of the required power.*

2. *The first term contains a only; the last term x only; all other terms contain both a and x.*

3. *The exponent of a in the first term is the same as the index of the required power and it decreases 1 in each succeeding term; the exponent of x in the second term is 1, and it increases 1 in each succeeding term.*

4. *In each term the sum of the exponents of a and x is equal to the index of the required power.*

5. *The coefficient of the first term is 1; the coefficient of the second term is the same as the index of the required power.*

6. *The coefficient of any term may be found by multiplying the coefficient of the preceding term by the exponent of a in that term, and dividing this product by the number of the term.*

7. *All the terms are positive, if both terms of the binomial are positive.*

8. *The terms are alternately positive and negative, if the second term of the binomial is negative.*

EXERCISES

266. 1. Write by inspection the fifth power of $(b-y)$.

SOLUTION

Substituting b for a and y for x and applying the observations of § 265, (2 and 3 for the letters and exponents, 5 and 6 for the coefficients, and 8 for the signs) we have

$$(b-y)^5 = b^5 - 5b^4y + 10b^3y^2 - 10b^2y^3 + 5by^4 - y^5.$$

NOTE. — Observe that *the coefficients of the latter half of the expansion are the same as those of the first half, written in the reverse order.*

Expand:

2. $(m+n)^5$.	7. $(x-y)^4$.	12. $(x+4)^3$.
3. $(m-n)^5$.	8. $(c-n)^6$.	13. $(x+5)^3$.
4. $(a-c)^3$.	9. $(x-a)^7$.	14. $(x-2)^3$.
5. $(a+b)^3$.	10. $(d-y)^8$.	15. $(x+2)^4$.
6. $(b+d)^4$.	11. $(b+y)^6$.	16. $(a-3)^4$.

17. Write the expansion of $(2c^2-5)^4$.

SOLUTION

§ 265, $(a-x)^4 = a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4.$

Substituting $2c^2$ for a and 5 for x, we have

$$(2c^2-5)^4 = (2c^2)^4 - 4(2c^2)^3 5 + 6(2c^2)^2 5^2 - 4(2c^2)5^3 + 5^4$$
$$= 16c^8 - 160c^6 + 600c^4 - 1000c^2 + 625.$$

18. Expand $(1+x^2)^3$, and test the result.

SOLUTION

§ 265, $(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3.$

Substituting 1 for a and x^2 for x, we have

$$(1+x^2)^3 = 1^3 + 3(1)^2(x^2) + 3(1)(x^2)^2 + (x^2)^3$$
$$= 1 + 3x^2 + 3x^4 + x^6.$$

TEST. — When $x=1$, $(1+x^2)^3 = 8$, and $1 + 3x^2 + 3x^4 + x^6 = 8$; hence, $(1+x^2)^3 = 1 + 3x^2 + 3x^4 + x^6$, and the expansion is correct.

Expand, and test results:

19. $(x+2y)^4$.
20. $(2x-y)^3$.
21. $(2x-5)^3$.
22. $(x^2-10)^4$.
23. $(1-3x^2)^4$.
24. $(5x^2-ab)^3$.
25. $(1+a^2b^2)^4$.
26. $(2ax-b)^5$.
27. $(1-x)^7$.
28. $(1-2x)^6$.
29. $(x-\frac{1}{2})^6$.
30. $(\frac{1}{2}x-\frac{1}{3}y)^4$.

Expand:

31. $\left(2a+\frac{1}{2}\right)^5$.
32. $\left(\frac{x}{y}-\frac{y}{x}\right)^4$.
33. $\left(\frac{x}{y}-\frac{y}{x}\right)^6$.
34. $\left(3a^2+\frac{b}{6}\right)^3$.
35. $\left(1+\frac{3x}{2}\right)^5$.
36. $\left(\frac{3}{5}+\frac{5x}{3}\right)^4$.
37. $\left(\frac{1}{2x}-2x\right)^5$.
38. $\left(\frac{1}{a}-a\right)^6$.
39. $\left(x+\frac{1}{x}\right)^7$.

40. Expand $(r-s-t)^3$.

SOLUTION

Since $(r-s-t)^3$ may be written in the binomial form, $(\overline{r-s}-t)^3$, we may substitute $(r-s)$ for a and t for x in

§ 265, $(a-x)^3 = a^3 - 3a^2x + 3ax^2 - x^3$.

Then, we have

$$\begin{aligned}(r-s-t)^3 &= (\overline{r-s}-t)^3\\ &= (r-s)^3 - 3(r-s)^2t + 3(r-s)t^2 - t^3\\ &= r^3 - 3r^2s + 3rs^2 - s^3 - 3t(r^2 - 2rs + s^2) + 3rt^2 - 3st^2 - t^3\\ &= r^3 - 3r^2s + 3rs^2 - s^3 - 3r^2t + 6rst - 3s^2t + 3rt^2 - 3st^2 - t^3\end{aligned}$$

41. Expand $(a+b-c-d)^3$.

SUGGESTION. $(a+b-c-d)^3 = (\overline{a+b} - \overline{c+d})^3$, a binomial form.

Expand:

42. $(a+x-y)^3$.
43. $(a-m-n)^3$.
44. $(a-x+y)^3$.
45. $(a-x-y)^3$.
46. $(a+x+2)^3$.
47. $(a-x-2)^3$.
48. $(a+2b-3c)^3$.
49. $(a+b+x+y)^3$.

EVOLUTION

267. Just as (§ 132) one of the *two* equal factors of a number is its **second, or square, root**; so one of the *three* equal factors of a number is its **third, or cube, root**; one of the *four* equal factors, the **fourth root**; etc.

The second root of a number, as a, is indicated by $\sqrt{a}$; the third root by $\sqrt[3]{a}$; the fourth root by $\sqrt[4]{a}$; the fifth root by $\sqrt[5]{a}$; etc.

The sign $\sqrt{\ }$ is called the **root sign**, or the **radical sign**; the small figure in its opening is called the **index** of the root.

When no index is written, the second, or square, root is meant.

268. The process $2^3 = 2 \cdot 2 \cdot 2 = 8$ illustrates *involution.*

The process $\sqrt[3]{8} = \sqrt[3]{2 \cdot 2 \cdot 2} = 2$ illustrates **evolution**, which will be defined here as the process of finding a root of a number, or as the *inverse of involution.*

269. You have learned (§ 132) that *every number has two square roots, one positive and the other negative.*

For example, $\sqrt{25} = +5$ or -5.

The roots may be written together, thus: ± 5, read '*plus or minus five*'; or ∓ 5, read '*minus or plus five.*'

270. The square root of -16 is not 4, for $4^2 = +16$; nor -4, for $(-4)^2 = +16$. No number so far included in our number system can be a square root of -16 or of any other negative number.

It would be inconvenient and confusing to regard $\sqrt{a}$ as a number only when a is positive. In order to preserve the generality of the discussion of number, it is necessary, there-