REVIEW

126. 1. What are positive numbers? negative numbers? In the following expression point out the positive numbers; the negative numbers. Perform the indicated operations:

$3\,ax + 7\,by - 9\,bx + 10\,by - 4\,ax - 3\,bx + 4\,ax - 2\,ax - 12\,by.$

2. What two meanings has the minus sign in algebra? If distance north is positive, what is the meaning of -150 miles? $+75$ miles?

3. Distinguish between arithmetical numbers and algebraic numbers.

4. Instead of subtracting a number (positive or negative), what may be done to secure the same result? Illustrate by subtracting -7 from $+12$. What is the absolute value of each of these numbers?

5. What is transposition? Give the principle relating to transposition.

6. State the law of signs for multiplication; for division.

7. What is the sign of the product of an even number of negative factors? of an odd number of negative factors?

8. In what respect do $(a-b)$ and $(b-a)$ differ? Expand $(a-b)^2$ and $(b-a)^2$ and compare the results.

9. For what values of n is $x^n + y^n$ divisible by $x+y$? by $x-y$? When is $x^n - y^n$ divisible by $x+y$? by $x-y$?

10 State the law of signs for the quotient when $x^n + y^n$ or $x^n - y^n$ is divided by $x+y$ or $x-y$; the law of exponents.

11. What must be added to $x^2 - 10\,x$ to make it the square of $x-5$? to a^2+b^2 to make it the square of $a+b$? to $x^4 + x^2y^2 + y^4$ to make it the square of x^2+y^2?

12. How may a parenthesis preceded by a minus sign be removed from an algebraic expression without changing the value of the expression?

13. Add $3\,a+5\,b-11\,c$, $b-2\,a+c$, $2\,c+3\,a-b$, $7\,c-b+6\,a$, $5\,b-4\,a-2\,c$, $b-a$, $c+b-a$, and $c-4\,a$.

14. Subtract the sum of $x-2\,y+3\,z-5\,w$ and $7\,x+w-2\,z$ from $10\,x-y+z-8\,w$.

15. If $x=r^2+rs-s^2$, $y=2\,r^2+4\,rs+2\,s^2$, and $z=r^2-3\,rs-s^2$, find the value of $x+y-z$.

Expand, and test each result:

16. $(r^3+7\,r^2s-3\,rs^2+2\,s^3)(r^2+2\,rs+s^2)$.

17. $(3\,l^{3r}+6\,l^{2r}m-12\,l^rm^2+3\,m^3)(4\,l^{2r}+3\,l^rm+2\,m^2)$.

18. $(x^4+y^4-4\,xy^3+5\,x^2y^2+3\,x^3y)(x^3+3\,x^2y+3\,xy^2+y^3)$.

Expand by inspection, and test each result:

19. $(3\,a+7\,b)^2$.

20. $(9\,w-2\,v)^2$.

21. $(x+2\,y)(x-2\,y)$.

22. $(a-3)(a+10)$.

23. $(7\,r+4\,s)(7\,r-4\,s)$.

24. $(3\,x-5\,y+z)^2$.

25. $(2\,c+d)(3\,c+2\,d)$.

26. $(5\,a-3\,b)(2\,a+2\,b)$.

Divide, and test each result:

27. $2\,l^6+5\,l^4r^2-3\,l^5r-6\,l^3r^3+3\,lr^5-r^6$ by $2\,l^2-3\,lr+r^2$.

28. $3\,x^4+8\,y^2-10\,yz-8\,x^2z-3\,z^2+10\,x^2y$ by $x^2+2\,y-3\,z$.

29. $4\,x^{4n}-25\,x^{2n}y^{2n}-10\,x^ny^{3n}-y^{4n}$ by $2\,x^{2n}-5\,x^ny^n-y^{2n}$.

Find quotients by inspection:

30. $\dfrac{y^4-1}{y+1}$.

31. $\dfrac{x^3-64}{x-4}$.

32. $\dfrac{a^5+32}{a+2}$.

33. $\dfrac{9\,x^2-16\,y^2}{3\,x+4\,y}$.

34. $\dfrac{c^3+125\,d^3}{c+5\,d}$.

35. $\dfrac{243-x^5}{3-x}$.

36. Simplify $17\,x-\{3\,y+4\,z-[z+5\,a+x-3\,a-2\,y]\}$.

37. Simplify $a+2\,b-[4\,c+2\,(a+2\,b)-\overline{b+4\,c}-a]+b$.

When $x = 2$, $y = -3$, and $z = 5$, find the value of:

38. $xz - (x + y + z)$.

39. $3(x - y) + 2(y - x) - zy$.

40. $x^2 - 3x(y + z) + y^2 - z$.

41. $(x - y)(y + z) - z^2(y - z)$.

42. When $a = 2$ and $b = 3$, prove that

$$b(ab + b - 2a) = ab^2 + b^2 - 2ab.$$

43. Collect similar terms within parentheses:

$ax^2 - cy + ax - 2ax^2 + 2cy^2 - ax - cy^2 + ax^2 + cy$.

44. Collect the coefficients of x and of y in

$7ax - 3by - 22a^2x + 5ay - 17bx + cy - 4x + 13y$.

Solve for x, and test results:

45. $3x + 7(x - 2) - 13 = 12 - 3x$.

46. $20 = 7 - 5(3 - x) + 9(x + 2)$.

47. $x^3 - 1 = x(x^2 - x) + x + 3 + x^2$.

48. $(x - 4)(x + 3) = (x + 6)^2 - 3 + 2x$.

49. $4a - 3(b + x) - 5a = 7b + 4a - 5(x + a)$.

50. $(a - x)(b - x) - b\{x - (a - x) - x\} = x^2 - 2bx + ab$.

Solve, and test:

51. $\begin{cases} 3x + 2y = 13, \\ 2x + y = 8. \end{cases}$

52. $\begin{cases} 7x + 4y = 39, \\ x - y = 4. \end{cases}$

53. $\begin{cases} x - y = 3, \\ 3x - 2y = 13. \end{cases}$

54. $\begin{cases} 5x + 7y = 24, \\ 3x - 2y = 2. \end{cases}$

55. $\begin{cases} 3x + 5y = 13, \\ 2x - 3y = -4. \end{cases}$

56. $\begin{cases} 9x - 2y = 15, \\ 5x + 7y = 57. \end{cases}$

Supply the missing coefficients in the following equations:

57. $3a - {*}b + 6a + 5b - {*}xy = {*}a + b - 2xy$.

58. $x^2 + 2xy + 3y^2 - [2x^2 + {*}y^2] = {*}x^2 + {*}xy$.

59. $6m^2 + 9mn - 3n^2 - [3m^2 + {*}mn] + n^2 = {*}m^2 - mn - {*}n^2$.

FACTORING

127. In *multiplication*, we find the product of two or more given numbers; in **factoring**, we have given the product to find the numbers that were multiplied to produce it.

These numbers are called the **factors** of the product.

128. A number that has no factors except itself and 1 is called a **prime number.**

MONOMIALS

129. To factor a monomial.

While in factoring it is usually the *prime* factors that are sought, this is not generally true in the case of monomials, because the factors of a monomial, except those of the coefficient, are evident.

Thus, $2a^3b^2$ shows its prime factors as well as though written $2 \cdot a \cdot a \cdot a \cdot b \cdot b$, but $81a^3b^2$ should be written $3^4a^3b^2$ to be considered in factored form.

However, it is often desirable to separate a monomial into two factors, one being given or both being specified in some way.

EXERCISES

130. 1. In each of the following, if xy is one factor, find the other: $6x^5y$, $15x^4y^2$, $2x^3y^3$, $a^2x^2b^2y^2$, $-mnxy$, $-xy$.

2. In each of the following, if abc is one factor, find the other: a^2bc, ab^2c, abc^2, $-a^2b^2c^2$, $-a^2bc$, $-\frac{1}{3}abc$.

3. Find two equal positive factors of x^2; of $9a^2x^2$; of $64m^4$.

4. Find two equal negative factors of $25x^2$; of $16a^2$; of $9a^6$.

131. A factor of two or more numbers is called a **common factor** of them.

132. One of the two equal factors of a number is called its **square root**.

Every number has two square roots, one *positive* and the other *negative*.

The square root of 25 is 5 or -5, for $5 \cdot 5 = 25$ and $(-5)(-5) = 25$.

In factoring, usually only the *positive* square root is taken.

133. To factor an expression whose terms have a common monomial factor.

EXERCISES

1. Find the factors of $3xy - 6x^2y + 9xy^2$.

PROCESS

$$3xy - 6x^2y + 9xy^2$$
$$= 3xy(1 - 2x + 3y)$$

EXPLANATION. — By examining the terms of the expression, it is seen that the monomial $3xy$ is a factor of every term. Dividing by this common factor gives the other factor.

Hence, the factors of $3xy - 6x^2y + 9xy^2$ are $3xy$ and $1 - 2x + 3y$.

TEST. — The product of the factors should equal the given expression; thus, $3xy(1 - 2x + 3y) = 3xy - 6x^2y + 9xy^2$.

Factor, and test each result:

2. $5x^3 - 5x^2$.	12. $x^{12} + x^{11} + x^{10} - x^9$.
3. $8x^2 + 2x^4$.	13. $ac - bc - cy - abc$.
4. $3x^3 - 6x^2y$.	14. $3x^3y^3 - 3x^2y^2 + 12xy$.
5. $4a^2 - 6ab$.	15. $3m^5 - 12m^3n^2 + 6mn^4$.
6. $5m^2 - 3mn$.	16. $9a^2by - 18ab^2y^2 + 24a^3b^2y^2$.
7. $3x^3y^2 - 3x^2y^3$.	17. $12x^2y^2z^3 - 16x^2y^2z^2 - 20x^3y^3z^3$.
8. $4a^3b - 6a^2b^2$.	18. $25c^2dx^3 + 35c^3d^2x^4 - 55c^2d^2x^5$.
9. $5m^4n - 10m^3n^2$.	19. $16a^2b^3c^4 - 24a^3b^2c^3 + 32a^3b^4c^3$.
10. $3x^4 - 9x^3 - 6x^2$.	20. $14a^2mn^2 - 21a^3m^2n^3 - 49a^4mn^2$.
11. $3a^4 - 2a^3b + a^2$.	21. $60m^2n^3r^2 - 45m^3n^2r^3 + 90m^4n^3r^2$.

BINOMIALS

134. To factor the difference of two squares.

By multiplication, § 94, $(a + b)(a - b) = a^2 - b^2$.

Therefore, $\quad a^2 - b^2 = (a + b)(a - b)$.

Hence, to factor the difference of two squares,

RULE. — *Find the square roots of the two terms, and make their sum one factor and their difference the other.*

EXERCISES

135. Factor, and test each result:

1. $x^2 - m^2$.	3. $x^2 - 1$.	5. $n^2 - 4^2$.
2. $a^2 - y^2$.	4. $a^2 - 3^2$.	6. $1 - \frac{1}{c^2}$.

7. Factor $a^2x^2 - 4c^2$.

SOLUTION

$$a^2x^2 - 4c^2 = (ax)^2 - (2c)^2$$
$$= (ax + 2c)(ax - 2c).$$

Factor, and test each result:

8. $x^2 - 81$.	12. $a^8 - b^2$.	16. $1 - 144m^2$.
9. $b^4 - 49$.	13. $m^2 - n^{12}$.	17. $64x^2 - a^2c^2$.
10. $25y^2 - 1$.	14. $81 - x^6y^6$.	18. $81a^4 - 100$.
11. $m^4 - 16n^2$.	15. $9a^2 - 49b^2$.	19. $121n^2 - 36r^2$.

136. To factor the sum or the difference of two cubes.

By applying the principles of §§ 111–113,

$$\frac{a^3 + b^3}{a + b} = a^2 - ab + b^2 \text{ and } \frac{a^3 - b^3}{a - b} = a^2 + ab + b^2.$$

Then, § 30, $\quad a^3 + b^3 = (a + b)(a^2 - ab + b^2)$,

and $\quad a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

By use of these forms any expression that can be written as the sum or the difference of two cubes may be factored.

EXERCISES

137. Factor, and test each result:

1. $x^3 + y^3$.
2. $x^3 - y^3$.
3. $m^3 - 1$.
4. $1 + m^3$.
5. $r^3 + 2^3$.
6. $\frac{a^3}{8} - 1$.

7. Factor $x^6 + y^6$.

SOLUTION

$$x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4).$$

8. Factor $a^9 - 8b^3$.

SOLUTION

$$a^9 - 8b^3 = (a^3)^3 - (2b)^3 = (a^3 - 2b)(a^6 + 2a^3b + 4b^2).$$

Factor, and test each result:

9. $x^3 - y^6$.
10. $8r^9 + s^9$.
11. $a^3b^3 - 27$.
12. $v^6 + 64t^3$.
13. $x^3y^6z^9 + 1$.
14. $n^3 - 1000$.

By applying §§ 111–113, as in § 136, any expression that can be written as the sum or the difference of the same odd powers of two numbers may be resolved into two factors.

Thus, $a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$,
and $a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.

Factor:

15. $m^5 + n^5$.
16. $m^5 - n^5$.
17. $a^5 + 32$.
18. $32 - a^5$.
19. $1 + s^5t^5$.
20. $x^5 - y^{10}$.

TRINOMIALS

138. To factor a trinomial that is a perfect square.

By multiplication, § 85,

$$(a + b)(a + b) = a^2 + 2ab + b^2.$$

Then, $\quad \boldsymbol{a^2 + 2ab + b^2 = (a + b)(a + b)}$.

Also, § 88, $\quad (a - b)(a - b) = a^2 - 2ab + b^2$.

Then, $\quad \boldsymbol{a^2 - 2ab + b^2 = (a - b)(a - b)}$.

These trinomials, $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$, are **perfect squares**, for each may be separated into two *equal* factors.

They are types, showing the form of all **trinomial squares**, for a and b may represent any two numbers.

Hence, to factor a trinomial square,

RULE. — *Connect the square roots of the terms that are squares with the sign of the other term, and indicate that the result is to be taken twice as a factor.*

EXERCISES

139. Factor, and test each result:

1. $x^2 + 2xy + y^2$.
2. $p^2 - 2pq + q^2$.
3. $c^2 + 2cd + d^2$.
4. $t^2 - 2tu + u^2$.
5. $x^2 - 2x + 1$.
6. $x^2 + 2x + 1$.

140. From the forms in the preceding discussion and exercises it is seen that a trinomial is a perfect square, if these two conditions are fulfilled:

1. Two terms, as $+a^2$ and $+b^2$, must be perfect squares.
2. The other term must be numerically equal to twice the product of the square roots of the terms that are squares.

Thus, $25x^2 - 20xy + 4y^2$ is a perfect square, for $25x^2 = (5x)^2$, $4y^2 = (2y)^2$, and $-20xy = -2(5x)(2y)$.

EXERCISES

141. Discover which of the following are perfect squares, factor such as are, and test each result:

1. $x^2 + 6x + 9$.
2. $4 - 4a + a^2$.
3. $r^2 - 8r + 16$.
4. $m^2 - mn + n^2$.
5. $1 + 4b + 4b^2$.
6. $1 - 6a^3 + 9a^6$.
7. $m^2 + m + \frac{1}{4}$.
8. $3x^2 + 8xy + 2y^2$.
9. $16p^2 - 24p + 9$.
10. $x^4 + 2x^2y^2 + y^4$.
11. $9 + 42b^3 + 49b^6$.
12. $1 - 6m^4 + 9m^8$.
13. $4x^2y^2 - 20xy + 25$.
14. $4x^2 + 12xyz + 9y^2z^2$.

142. **To factor a trinomial of the form $x^2 + px + q$.**

By multiplication, § 97,

$$(x+a)(x+b) = x^2 + (a+b)x + ab.$$

Then, $$x^2 + (a+b)x + ab = (x+a)(x+b).$$

This trinomial consists of an x^2-term, an x-term, and a term without x, that is, an **absolute term**. Therefore, it has the type form $x^2 + px + q$.

Hence, if a trinomial of this form is factorable, it may be factored by *finding two factors of q (the absolute term) such that their sum is p (the coefficient of x), and adding each factor of q to x.*

Thus,
$$x^2 + 8x + 15 = (x+3)(x+5),$$
$$x^2 - 8x + 15 = (x-3)(x-5),$$
$$x^2 + 2x - 15 = (x-3)(x+5),$$
$$x^2 - 2x - 15 = (x+3)(x-5).$$

EXERCISES

143. 1. Resolve $x^2 - 13x - 48$ into two binomial factors.

Solution. — The first term of each factor is evidently x.

Since the product of the second terms of the two binomial factors is -48, the second terms must have opposite signs; and since their algebraic sum, -13, is negative, the negative term must be numerically larger than the positive term.

The two factors of -48 whose sum is negative may be 1 and -48, 2 and -24, 3 and -16, 4 and -12, or 6 and -8. Since the algebraic sum of 3 and -16 is -13, 3 and -16 are the factors of -48 sought.

$$\therefore x^2 - 13x - 48 = (x+3)(x-16).$$

2. Factor $72 - m^2 - m$.

Solution. — Arranging the trinomial according to the descending powers of m,

$$72 - m^2 - m = -m^2 - m + 72$$

Making m^2 positive,
$$= -(m^2 + m - 72)$$
$$= -(m-8)(m+9)$$
$$= (-m+8)(m+9)$$
$$= (8-m)(9+m).$$

Separate into factors, and test each result by assigning a numerical value to each letter:

3. $x^2 + 7x + 12.$
4. $y^2 - 7y + 12.$
5. $p^2 - 8p + 12.$
6. $r^2 + 8r + 12.$
7. $15 + 2a - a^2.$
8. $b^2 + b - 12.$
9. $30 - r^2 + r.$
10. $c^2 - c - 72.$
11. $c^2 - 5c - 14.$
12. $x^2 - x - 110.$
13. $x^2 + 5ax + 6a^2.$
14. $x^2 - 6ax + 5a^2.$
15. $y^2 - 4by - 12b^2.$
16. $y^2 - 3ny - 28n^2.$
17. $z^2 - anz - 2a^2n^2.$
18. $-x^2 + 25x - 100.$
19. $x^4 + 19cx^2 + 90c^2.$
20. $x^6 + 12ax^3 + 20a^2.$
21. $x^{10} - 11b^2x^5 + 24b^4.$
22. $n^2x^2 - 11nxy + 30y^2.$

144. **To factor a trinomial of the form $ax^2 + bx + c$.**

EXERCISES

1. Factor $3x^2 + 11x - 4$.

Solution. — If this trinomial is the product of two binomial factors, they may be found by reversing the process of multiplication illustrated in exercise 32, page 73.

Since $3x^2$ is the product of the *first terms* of the binomial factors, the first terms, each containing x, are $3x$ and x.

Since -4 is the product of the last terms, § 78, they must have unlike signs, and the only possible *last terms* are 4 and -1, -4 and 1, or 2 and -2.

Hence, associating these pairs of factors of -4 with $3x$ and x in all possible ways, the possible binomial factors of $3x^2 + 11x - 4$ are:

$$\left.\begin{matrix}3x+4\\x-1\end{matrix}\right\},\quad \left.\begin{matrix}3x-1\\x+4\end{matrix}\right\},\quad \left.\begin{matrix}3x-4\\x+1\end{matrix}\right\},\quad \left.\begin{matrix}3x+1\\x-4\end{matrix}\right\},\quad \left.\begin{matrix}3x+2\\x-2\end{matrix}\right\},\quad \left.\begin{matrix}3x-2\\x+2\end{matrix}\right\}.$$

Of these we select *by trial* the pair that will give $+11x$ (the middle term of the given trinomial) for the algebraic sum of the 'cross-products,' that is, the second pair.

$$\therefore 3x^2 + 11x - 4 = (3x-1)(x+4).$$

By a reversal of the law of signs for multiplication and from the preceding solution it may be observed that:

1. *When the sign of the last term of the trinomial is +, the last terms of the factors must be both + or both −, and like the sign of the middle term of the trinomial.*

2. *When the sign of the last term of the trinomial is −, the sign of the last term of one factor must be +, and of the other −.*

Factor, and test each result:

2. $2y^2+3y+1$.
3. $5x^2+9x-2$.
4. $3x^2-7x-6$.
5. $4r^2+8r+3$.
6. $6x^2-7x+2$.
7. $2x^2-5x-12$.
8. $10t^2+t-3$.
9. $6n^2-13n+6$.
10. $2x^2+x-15$.
11. $5x^2+13x+6$.
12. $3x^2-17x+10$.
13. $6x^2-11x-35$.
14. $15x^2+17x-4$.
15. $15x^2-14x-8$.
16. $2x^2+3xy-2y^2$.
17. $3x^2-10xy+3y^2$.

When the coefficient of x^2 is a square, and when the square root of the coefficient of x^2 is exactly contained in the coefficient of x, the trinomial may be factored as follows:

18. Factor $9x^2+30x+16$.

Solution

$$\begin{aligned}&9x^2+30x+16\\&=(3x)^2+10(3x)+16\\&=(3x+2)(3x+8).\end{aligned}$$

Separate into factors, and test each result:

19. $9x^2-9x+2$.
20. $4x^2-4x-15$.
21. $9x^2-42x+40$.
22. $25x^2+15x+2$.
23. $16x^2+16x+3$.
24. $36x^2-36x+5$.
25. $49x^2-42x-55$.
26. $25x^2+25x-24$.
27. $16x^2-32x+15$.
28. $64x^2-32x-77$.
29. $100x^2+40x+3$.
30. $81x^2-108x+35$.

POLYNOMIALS

145. To factor a polynomial whose terms may be grouped to show a common polynomial factor.

EXERCISES

1. Factor $ax+ay+bx+by$.

Solution

$$\begin{aligned}ax+ay+bx+by&=(ax+ay)+(bx+by)\\&=a(x+y)+b(x+y)\\&=(a+b)(x+y).\end{aligned}$$

2. Factor $ax+by-ay-bx$.

Solution

$$\begin{aligned}ax+by-ay-bx&=ax-ay-bx+by\\ \text{§ 117, Prin. 2,}\quad&=(ax-ay)-(bx-by)\\&=a(x-y)-b(x-y)\\&=(a-b)(x-y).\end{aligned}$$

Remark. — The given polynomial must be arranged and grouped in such a way that after the monomial factor is removed from each group the polynomial factors in all the groups will be *alike in every respect.*

Factor, and test each result, especially for signs:

3. $am-an+mx-nx$.
4. $bc-bd+cx-dx$.
5. $pq-px-rq+rx$.
6. $ay-by-ab+b^2$.
7. $x^2-xy-5x+5y$.
8. $b^2-bc+ab-ac$.
9. $x^2+xy-ax-ay$.
10. $c^2-4c+ac-4a$.
11. $1-m+n-mn$.
12. $2x-y+4x^2-2xy$.
13. $2p+q+6p^2+3pq$.
14. $ar-rs-ab+bs$.
15. x^3+x^2+x+1.
16. y^3+y^2-3y-3.
17. $x^5+x^3+x^2y+y$.
18. $2-n^2-2n+n^3$.
19. $ax-x-a+x^2$.
20. $12a^2-8b-3a^3+2ab$.
21. $8ax+6ay-4bx-3by$.
22. $3x^3-15x+10y-2x^2y$.
23. $3r^2t-9rt^2+ar-3at$.
24. $ax-a-bx+b-cx+c$.

146. To factor by the factor theorem.

Zero multiplied by any number is equal to 0.

Conversely, if a product is equal to zero, at least one of the factors must be 0 or a number equal to 0.

If $5x=0$, since 5 is not equal to 0, x must equal 0.

If $5(x-3)=0$, since 5 is not equal to 0, x must have such a value as to make $x-3$ equal to 0; that is, $x=3$.

If $5(x-3)$, or $5x-15$, or any other polynomial in x reduces to 0 when $x=3$, $x-3$ is a factor of the polynomial.

Sometimes a polynomial in x reduces to 0 for more than one value of x. For example, x^2-5x+6 equals 0 when $x=3$ and also when $x=2$; or when $x-3=0$ and $x-2=0$. In this case both $x-3$ and $x-2$ are factors of the polynomial.

$$\therefore\ x^2-5x+6=(x-3)(x-2).$$

147. Factor Theorem. — *If a polynomial in x, having positive integral exponents, reduces to zero when r is substituted for x, the polynomial is exactly divisible by $x-r$.*

The letter r represents any number that may be substituted for x.

EXERCISES

148. 1. Factor x^3-x^2-4x+4.

SOLUTION. — When $x=1$, $x^3-x^2-4x+4=1-1-4+4=0$.

Therefore, $x-1$ is a factor of the given polynomial.

Dividing x^3-x^2-4x+4 by $x-1$, the quotient is found to be x^2-4.

By § 134, $x^2-4=(x+2)(x-2)$.

$$\therefore\ x^3-x^2-4x+4=(x-1)(x+2)(x-2).$$

SUGGESTIONS. — 1. Only factors of the absolute term of the polynomial need be substituted for x in seeking factors of the polynomial of the form $x-r$, for if $x-r$ is one factor, the absolute term of the polynomial is the product of r and the absolute term of the other factor.

2. In substituting the factors of the absolute term, try them in order beginning with the numerically smallest.

2. Factor x^3+x^2-9x-9.

SUGGESTION. — When $x=1$, $x^3+x^2-9x-9=-16$.

Therefore, $x-1$ is not a factor of the given polynomial.

When $x=-1$, $x^3+x^2-9x-9=0$.

Therefore, $x-(-1)$, or $x+1$, is a factor of the given polynomial.

Factor by the factor theorem:

3. $13x^2-5x-8$.
4. x^3-7x+6.
5. $x^3-9x^2+23x-15$.
6. $x^3-4x^2-7x+10$.
7. $x^3-6x^2-9x+14$.
8. $x^3-11x^2+31x-21$.
9. $x^3-10x^2+29x-20$.
10. $x^3-19x+30$.
11. $x^3-67x-126$.
12. $m^3+7m^2+2m-40$.
13. $x^4-25x^2+60x-36$.
14. $x^4+13x^2-54x+40$.
15. $x^4+22x^2+27x-50$.
16. $x^4-9x^3+21x^2+x-30$.

17. Factor $2x^3+x^2y-5xy^2+2y^3$.

SUGGESTION. — When $x=y$,

$$2x^3+x^2y-5xy^2+2y^3=2y^3+y^3-5y^3+2y^3=0.$$

Therefore, $x-y$ is a factor of $2x^3+x^2y-5xy^2+2y^3$.

Factor by the factor theorem:

18. $x^3-13xy^2+12y^3$.
19. $x^3-31xy^2-30y^3$.
20. $x^4-9x^2y^2+12xy^3-4y^4$.
21. $x^4-9x^2y^2-4xy^3+12y^4$.

MISCELLANEOUS EXERCISES

149. In the exercises under the preceding cases, except those under the factor theorem, the expressions given have been *completely* factored by *one* application of a *single* case, but frequently it is necessary to apply *two* or *more* cases in succession or *one* case *more than once* to factor the given expression *completely*.

Monomial factors should usually first be removed, as they often disguise a familiar type form.

1. Factor x^3+3x^2-10x.

SOLUTION

By § 133, $x^3+3x^2-10x=x(x^2+3x-10)$

By § 142, $\qquad\qquad\qquad =x(x+5)(x-2)$.

2. Factor $x^6 - y^6$.

Solution

Writing the expression as the difference of two squares, we have,

$$
\begin{aligned}
x^6 - y^6 &= (x^3)^2 - (y^3)^2 \\
\text{§ 134,} \quad &= (x^3 + y^3)(x^3 - y^3) \\
\text{§ 136,} \quad &= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2).
\end{aligned}
$$

3. Factor $x^8 - y^8$.

Solution

By successive applications of § 134,

$$
\begin{aligned}
x^8 - y^8 &= (x^4 + y^4)(x^4 - y^4) \\
&= (x^4 + y^4)(x^2 + y^2)(x^2 - y^2) \\
&= (x^4 + y^4)(x^2 + y^2)(x + y)(x - y).
\end{aligned}
$$

Factor completely, and test each result:

4. $x^3 - xy^2$.
5. $m - m^4$.
6. $a^4 - b^4$.
7. $x^3 - 1$.
8. $w^4 - 81$.
9. $3\,t^5 - 3\,t$.
10. $z^6 - 1$.
11. $5\,c^8 - 5$.
12. $4\,a - 4\,a^2 + a^3$.
13. $2\,x^2 + 5\,xy + 2\,y^2$.
14. $10\,x^2 - 20\,x + 10$.
15. $11\,a^2x - 55\,ax + 66\,x$.
16. $n^3 + 2\,n - 3\,n^2 - 6$.
17. $r^5 + 2\,r^2 + 5\,r^3 + 10$.
18. $18\,a^2b + 60\,ab^2 + 50\,b^3$.
19. $w^3 + uw^2 + vw^2 + uvw$.
20. $x^6 - y^3$.
21. $l^5 - 16\,l$.
22. $3\,r^4 - 3\,s^8$.
23. $4\,ax + 2\,ax^2 - 48\,a$.
24. $18\,m^2 - 3\,m - 36$.
25. $y + 10\,b^2y + 25\,b^4y$.
26. $27\,n + n^7$.
27. $64 - 2\,a^5$.
28. $m^7 + m^2n^5$.
29. $21\,a^2 - a - 10$.
30. $-a^2 - 4\,a + 45$.
31. $16\,x^2 + 20\,x - 66$.
32. $36\,x^2 - 48\,x - 20$.
33. $x^3 - 21\,xy^2 + 20\,y^3$.
34. $3\,a^2bx^2 - 3\,a^2bx - 6\,a^2b$.
35. $x^3 - 12\,x^2 + 41\,x - 30$.
36. $30 - m - 6\,m^2 + m^3$.
37. $3\,r^2s - 9\,rs^2 + br - 3\,bs$.
38. $15\,tl^2 - 9\,l^2n - 35\,tl + 21\,ln$.
39. $w^4 - w^3 - 7\,w^2 + w + 6$.
40. $m^4 - 15\,m^2 + 10\,m + 24$.

SPECIAL APPLICATIONS AND DEVICES

150. The method of factoring by grouping the terms of an expression in certain ways is very important. Polynomials may often be arranged in some one of the type forms already studied, and even many of these types themselves may be factored by grouping to show a common polynomial factor.

151. To factor a polynomial that may be grouped to form the difference of two squares.

Just as, § 134, $\quad a^2 - b^2 = (a + b)(a - b),$

so

$$
\begin{aligned}
a^2 - (b + c)^2 &= [a + (b + c)][a - (b + c)] \\
&= (a + b + c)(a - b - c).
\end{aligned}
$$

EXERCISES

152. Factor:

1. $(a + b)^2 - c^2$.
2. $r^2 - (s + t)^2$.
3. $x^2 - (y - z)^2$.
4. $(l + m)^2 - n^2$.
5. $(a - b)^2 - c^2$.
6. $1 - (v + w)^2$.
7. Factor $x^4 - (3\,x^2 - 2\,y)^2$.

Solution

$$
\begin{aligned}
x^4 - (3\,x^2 - 2\,y)^2 &= [x^2 + (3\,x^2 - 2\,y)][x^2 - (3\,x^2 - 2\,y)] \\
&= (x^2 + 3\,x^2 - 2\,y)(x^2 - 3\,x^2 + 2\,y) \\
&= (4\,x^2 - 2\,y)(2\,y - 2\,x^2) \\
&= 2\,(2\,x^2 - y)2(y - x^2) \\
&= 4\,(2\,x^2 - y)(y - x^2).
\end{aligned}
$$

Test. — When $x = 2$ and $y = 3$,
$x^4 - (3\,x^2 - 2\,y)^2 = 2^4 - (3 \cdot 2^2 - 2 \cdot 3)^2 = 16 - (12 - 6)^2 = 16 - 36 = -20$,
and $4(2\,x^2 - y)(y - x^2) = 4(2 \cdot 2^2 - 3)(3 - 2^2) = 4(8 - 3)(3 - 4) = -20$.

Factor, and test each result:

8. $4\,c^2 - (b + c)^2$.
9. $(2\,a + b)^2 - b^2$.
10. $9\,t^2 - (2\,t - 5)^2$.
11. $25\,a^2 - (b + c)^2$.
12. $49\,r^2 - (5\,r - 4\,s)^2$.
13. $36\,z^2 - (3\,z - 7\,y)^2$.
14. $(6\,w - 3\,k)^2 - 64\,k^2$.
15. $(3\,m + 8\,n)^2 - 16\,m^2$.

16. Factor a^2+4-c^2-4a.

SUGGESTION.—The given expression contains three square terms and a term that is not a square. The latter may be the middle term of a trinomial square. If so, it contains as factors the square roots of two of the square terms and these are the other terms of the trinomial square.

Then, arranging and grouping terms, we have

$$a^2+4-c^2-4a=(a^2-4a+4)-c^2=(a-2)^2-c^2.$$

NOTE.—It will be observed that the term that is *not* a *perfect monomial square* furnishes a *key* to the *grouping*.

Factor, and test each result:

17. $a^2-2ax+x^2-n^2$.
18. $b^2+2by+y^2-n^2$.
19. $1-4q+4q^2-a^2$.
20. $r^2-2rx+x^2-16t^2$.
21. $c^2-a^2-b^2-2ab$.
22. $b^2-x^2-y^2+2xy$.
23. $4c^2-x^2-y^2-2xy$.
24. $9a^2-6ab+b^2-4c^2$.

153. The principle by which the difference of two squares is factored may be extended to expressions that may be *written* as the difference of two squares by *adding* and *subtracting* the *same monomial perfect square.*

EXERCISES

154. 1. Factor $a^4+a^2b^2+b^4$.

SUGGESTION.—Since $a^4+a^2b^2+b^4$ lacks $+a^2b^2$ of being a perfect square, and since the value of the polynomial will not be changed by adding a^2b^2 and also subtracting a^2b^2, the polynomial may be written

$$a^4+2a^2b^2+b^4-a^2b^2, \text{ or } (a^2+b^2)^2-a^2b^2.$$

2. Factor x^4-21x^2+36.

SUGGESTION.—This expression has two square terms, but in order that it shall be a perfect trinomial square it must fulfill another condition, namely (§ 140), the other term must be numerically equal to twice the product of the square roots of the terms that are squares; that is, the middle term must be either $+12x^2$ or $-12x^2$.

Hence, the number to be added and subtracted is either $33x^2$ or $9x^2$, but the former will not give the difference of two squares, for $33x^2$ is not a perfect square; then, $9x^2$ is the number to be added and subtracted, giving

$$x^4-12x^2+36-9x^2, \text{ or } (x^2-6)^2-9x^2.$$

Separate into prime factors, and test each result:

3. $x^4+x^2y^2+y^4$.
4. $a^8+a^4b^4+b^8$.
5. $9x^4+20x^2y^2+16y^4$.
6. $4a^4+11a^2b^2+9b^4$.
7. $16a^4-17a^2x^2+x^4$.
8. x^4+x^2+1.
9. n^8+n^4+1.
10. $16x^4+4x^2y^2+y^4$.
11. $25a^4-14a^2b^4+b^8$.
12. $9a^4+26a^2b^2+25b^4$.

13. Factor a^4+4.

SUGGESTION. $a^4+4=a^4+4a^2+4-4a^2=(a^2+2)^2-4a^2$.

Factor completely, and test each result:

14. a^4+4b^4.
15. m^4+64.
16. x^8-16.
17. $4a^4+81$.
18. m^5+4mn^4.
19. $x^5y^2+4xy^2$.

155. The method of factoring by grouping to show a common polynomial factor applied to cases already solved by other methods.

The student has learned how to factor several type forms by special methods. He will now see how many of these forms may be factored by the method of § 145, which is of importance because its application is so general.

EXERCISES

156. *The forms* $a^2+2ab+b^2$ *and* $a^2-2ab+b^2$.

1. Factor $x^2+2xy+y^2$; also $9a^2m^2-6am+1$.

SOLUTION	SOLUTION
$x^2+2xy+y^2$	$9a^2m^2-6am+1$
$=x^2+xy+xy+y^2$	$=9a^2m^2-3am-3am+1$.
$=x(x+y)+y(x+y)$	$=3am(3am-1)-(3am-1)$
$=(x+y)(x+y)$.	$=(3am-1)(3am-1)$.

Factor by separating and grouping; test each result:

2. $l^2+14l+49$.
3. $r^2-18r+81$.
4. $4x^2+12xy+9y^2$.
5. $16a^2-24ab+9b^2$.

The form $a^2 - b^2$.

6. Factor $x^2 - y^2$; also $9r^2 - 4s^2$.

SOLUTION

$$\begin{aligned} & x^2 - y^2 \\ &= x^2 - xy + xy - y^2 \\ &= x(x-y) + y(x-y) \\ &= (x+y)(x-y). \end{aligned}$$

SOLUTION

$$\begin{aligned} & 9r^2 - 4s^2 \\ &= 9r^2 - 6rs + 6rs - 4s^2 \\ &= 3r(3r-2s) + 2s(3r-2s) \\ &= (3r+2s)(3r-2s). \end{aligned}$$

Factor by grouping; test each result:

7. $a^2 - z^2$.
8. $x^2 - 4$.
9. $n^2 - 1$.
10. $9x^2 - 25$.
11. $36z^2 - 25v^2$.
12. $49n^2 - 100l^2$.

The form $x^2 + px + q$.

13. Factor $x^2 + 8x + 15$; also $x^2 - 2x - 15$.

SOLUTION

$$\begin{aligned} & x^2 + 8x + 15 \\ &= x^2 + 5x + 3x + 15 \\ &= x(x+5) + 3(x+5) \\ &= (x+3)(x+5). \end{aligned}$$

SOLUTION

$$\begin{aligned} & x^2 - 2x - 15 \\ &= x^2 - 5x + 3x - 15 \\ &= x(x-5) + 3(x-5) \\ &= (x+3)(x-5). \end{aligned}$$

Factor by separating and grouping; test each result:

14. $x^2 + 12x + 20$.
15. $m^2 + 9m + 18$.
16. $y^2 + 8y - 20$.
17. $n^2 - 5n - 14$.

The form $ax^2 + bx + c$.

18. Factor $2x^2 + 11x + 12$; also $2x^2 + x - 15$.

SOLUTION

$$\begin{aligned} & 2x^2 + 11x + 12 \\ &= 2x^2 + 8x + 3x + 12 \\ &= 2x(x+4) + 3(x+4) \\ &= (2x+3)(x+4). \end{aligned}$$

SOLUTION

$$\begin{aligned} & 2x^2 + x - 15 \\ &= 2x^2 + 6x - 5x - 15 \\ &= 2x(x+3) - 5(x+3) \\ &= (2x-5)(x+3). \end{aligned}$$

Factor by separating and grouping; test each result:

19. $2x^2 + 3x + 1$.
20. $9y^2 + 21y + 10$.
21. $3m^2 + 5m - 22$.
22. $10r^2 - 3rs - 18s^2$.

Factor by separating and grouping:

23. $12n^2 + 31n + 9$.
24. $5z^2 + 48z - 77$.
25. $20x^2 + 13xy - 15y^2$.
26. $14a^2 - 23ab - 30b^2$.

REVIEW OF FACTORING

157. Summary of Cases. — In the previous pages the student has learned to factor expressions of the following types:

Monomial Factors

I. **Of monomials**; as a^2b^2c. (§ 129)

II. **Of expressions whose terms have a common factor**; as $nx + ny + nz$. (§ 133)

Binomials

III. **Difference of two squares**; as $a^2 - b^2$. (§§ 134, 151, 155)

IV. **Sum or difference of two cubes**; as $a^3 + b^3$ or $a^3 - b^3$. (§ 136)

V. **Sum or difference of same odd powers**; as $a^n + b^n$ or $a^n - b^n$ (when n is odd). (§ 137)

Trinomials

VI. **That are perfect squares**; as $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$. (§§ 138, 155)

VII. **Of the form** $x^2 + px + q$. (§§ 142, 155)

VIII. **Of the form** $ax^2 + bx + c$. (§§ 144, 155)

Polynomials

IX. **Whose terms may be grouped to show a common polynomial factor**; as $ax + ay + bx + by$. (§§ 145, 155)

X. **Having binomial factors** (Factor Theorem). (§ 146)

158. General Directions for Factoring Polynomials.—1. *Remove monomial factors if there are any.*

2. *Then endeavor to bring the polynomial under some one of the cases II–IX.*

3. *When other methods fail, try the factor theorem.*

4. *Resolve into prime factors.*

Each factor should be divided out of the given expression as soon as found in order to simplify the discovery of the remaining factors.

EXERCISES

159. Factor, and test each result:

1. y^4-1.
2. $1-x^8$.
3. $x^{10}-1$.
4. x^6-1.
5. $a-a^7$.
6. b^7+b.
7. p^4+4.
8. $1+x^{12}$.
9. $y-a^4y$.
10. x^2y-y^3.
11. $a^{13}-ab^{12}$.
12. $64-2y^5$.
13. $7n^7-7n$.
14. $4x^4-4x$.
15. $8-27a^3x^3$.
16. $32x-2x^5$.
17. $6b^4+24$.
18. a^5+27a^2.
19. b^2-196.
20. $450-2a^2$.
21. $7y^4-175$.
22. $x^2-xy-132y^2$.
23. $ax^2-3ax-4a$.
24. x^3+5x^2-6x.
25. $3x^2+30x+27$.
26. $128a^2-250a^5$.
27. $6x^2-19x+15$.
28. $x^{2n}+2x^ny^p+y^{2p}$.
29. $7x^2-77xy-84y^2$.
30. $y^2-25yx+136x^2$.
31. $9x^2-24xy+16y^2$.
32. $x^2-ax-72a^2$.
33. $n^2-an-90a^2$.
34. $a^2b^2+ab-56$.
35. $a^{2n}-2a^nb^n+b^{2n}$.
36. $25x^2+60xy+36y^2$.
37. $6ax^2+5axy-6ay^2$.
38. $169x^4-26ax^3+a^2x^2$.
39. $a^4c^4+a^2b^2c^2+b^4$.
40. $16x^4+4x^2y^2+y^4$.
41. b^4c-13b^2c+42c.

Factor, and test each result:

42. $9x^2+21x+10$.
43. $5x^2-26xy+5y^2$.
44. $y^2+16ay-36a^2$.
45. $8a^2-21ab-9b^2$.
46. $9x^2-15x-50$.
47. $30x^2-37x-77$.
48. $2x^3+28x^2+66x$.
49. $a^2+b^2-c^2-2ab$.
50. $ax^2+10ax-39a$.
51. $n^4+n^2a^2b^4+a^4b^8$.
52. $a^2z^4+a^2z^2+a^2$.
53. $a^{2m}-16a^m-17$.
54. $a^2x^2-4ax+3$.
55. $b^8+b^4y^2+y^4$.
56. $xy-3y+x-3y^2$.
57. $ax^2-axy-ax+ay$.
58. $9c^2-x^2-y^2+2xy$.
59. $x^3-a^2x-4b^2x-4abx$.
60. $bc^2-9a^2b-b^3-6ab^2$.
61. $ab^2-4a^3-12a^2c-9ac^2$.
62. $x^2-cx+2dx-2cd$.
63. $x^3y+4x^2y-31xy-70y$.
64. $x^2-3ax+4bx-12ab$.
65. $ax^3-9ax^2+26ax-24a$.
66. $12ax-8bx-9ay+6by$.
67. $25x^2-9y^2-24yz-16z^2$.
68. $2b^2t-3ab^2+2btx-3abx$.
69. $x^3y+14x^2y+43xy+30y$.
70. $x^3y-15x^2y+38xy-24y$.
71. $abx^3+3abx^2-abx-3ab$.
72. $3bmx+2bm-3anx-2an$.
73. $20ax^3-28ax^2+5a^2x-7a^2$.
74. $(a+b)^3-1$.
75. a^3-2a^2+1.
76. b^3-4b^2+8.
77. $3x^6+96x$.
78. $8x^4-6x^2-35$.
79. $x^5-x^2-x^4+x^3$.
80. $x^3-xy-x^2y+y^2$.
81. $12x^3+3x^2-8x-2$.
82. $2x^2+10x+ax+5a$.
83. $m^3+m^2-mn-mn^2$.
84. Factor $16+5x-11x^2$ by the factor theorem.
85. Factor $x^3-6bx^2+12b^2x-8b^3$ by the factor theorem.

EQUATIONS SOLVED BY FACTORING

160. Equations thus far solved have been such as involved, when in simplest form, only the *first power* of the unknown number.

Such equations are called **simple equations.**

161. A valuable application of factoring is found in the solution of equations that involve the unknown number in powers higher than the first.

An equation that in simplest form involves the second, but no higher, power of the unknown number is called a **quadratic equation.**

Thus, $x^2 = 4$ and $x^2 + 2x + 1 = 0$ are quadratic equations; but $x^2 + 3x = x^2 + 4$ is a simple equation, for in its simplest form, $3x = 4$, it has only the first power of x.

162. An equation that contains a higher power of the unknown number than the second is called a **higher equation.**

163. To solve quadratic equations by factoring.

EXERCISES

1. Find the values of x that satisfy $x^2+1=10$.

SOLUTION

$$x^2 + 1 = 10. \qquad (1)$$

Transposing so that all terms are in the first member and uniting terms,

$$x^2 - 9 = 0. \qquad (2)$$

Factoring the first member, § 134,

$$(x-3)(x+3) = 0. \qquad (3)$$

Since the product of the two factors is 0, one of them must equal 0; that is, the equation is satisfied for any value of x that will make either factor equal to 0.

If $x - 3 = 0$, $x = 3$; if $x + 3 = 0$, $x = -3$.

Hence, the values of x that satisfy (3) and therefore (1) are 3 or -3.

VERIFICATION. — When $x = 3$, (1) becomes $9 + 1 = 10$, or $10 = 10$.

When $x = -3$, (1) becomes $9 + 1 = 10$, or $10 = 10$.

Solve, and verify results:

2. $x^2+3=28$.

3. $x^2+1=50$.

4. $x^2-5=59$.

5. $x^2-7=29$.

6. $x^2+3=84$.

7. $x^2-24=120$.

8. $x^2+11=180$.

9. $x^2-11=110$.

10. Solve $x^2+4x=45$.

SOLUTION

$$x^2 + 4x = 45.$$

Transposing, $x^2 + 4x - 45 = 0.$

Factoring, § 142, $(x-5)(x+9) = 0.$

Hence, $x - 5 = 0$ or $x + 9 = 0$;

whence, $x = 5$ or -9.

Solve, and verify results:

11. $x^2-6x=40$.

12. $x^2-8x=48$.

13. $x^2-5x=-4$.

14. $x^2+4x+3=0$.

15. $r^2+6r+8=0$.

16. $y^2+42=13y$.

17. $t^2+63=16t$.

18. $v^2-60=11v$.

19. $x^2-7x=18$.

20. $x^2+10x=56$.

21. Solve $6x^2+5x-21=0$.

SOLUTION

$$6x^2 + 5x - 21 = 0.$$

Factoring, § 144, $(2x-3)(3x+7) = 0.$

Hence, $2x - 3 = 0$ or $3x + 7 = 0$;

whence, $x = \frac{3}{2}$ or $-\frac{7}{3}$.

Solve, and verify results:

22. $3x^2+2x-1=0$.

23. $5x^2+4x-1=0$.

24. $3y^2+y-10=0$.

25. $7x^2+6x-1=0$.

26. $2x^2+9x-18=0$.

27. $2v^2-9v-35=0$.

28. $6y^2-22y+20=0$.

29. $6z^2-11z-21=0$.

30. $4x^2-15x+14=0$.

31. $5x^2-48x-20=0$.

Solve, and verify results:

32. $x^2-21=4$.
33. $x^2-56=8$.
34. $x^2-9x=36$.
35. $x^2+11x=26$.
36. $x^2-12x=45$.
37. $y^2-15y=54$.
38. $y^2-21y=46$.
39. $3y^2-4y-4=0$.
40. $4y^2+9y-9=0$.
41. $32-x^2=28$.
42. $65-x^2=16$.
43. $x^2+x-132=0$.
44. $32=4w+w^2$.
45. $3s=88-s^2$.
46. $160=x^2-6x$.
47. $4y=y^2-192$.
48. $3x^2+13x-30=0$.
49. $4x^2+13x-12=0$.
50. $(2x+3)(2x-5)-(3x-1)(x-2)=1$.
51. $(2x-6)(3x-2)-(5x-9)(x-2)=4$.

Other methods of solving quadratics will be given in §§ 336–348.

164. To solve higher equations by factoring.

Any higher equation may be solved by the method just given for quadratic equations, whenever the expression obtained by transposing all of its terms to one member is factorable.

EXERCISES

165. 1. Solve the equation $x^3-2x^2-5x+6=0$.

SOLUTION

$$x^3-2x^2-5x+6=0.$$

Factoring, § 146, $(x-1)(x-3)(x+2)=0$.

Hence, $x-1=0$ or $x-3=0$ or $x+2=0$;

whence, $x=1$ or 3 or -2.

2. $x^3-15x^2+71x-105=0$.
3. $x^3+10x^2+11x-70=0$.
4. $x^3-12x+16=0$.
5. $x^3-19x-30=0$.
6. $x^4+x^3-21x^2-x+20=0$.
7. $x^4-7x^3+x^2+63x-90=0$.
8. $x^4-10x^3+35x^2-50x+24=0$.

Problems

166. 1. A sealing fleet carries 4000 men, and the number of men on each ship is 40 less than 8 times the number of ships. Find the number of ships and the number of men on each ship.

SOLUTION

Let x = the number of ships in the fleet.

Then, $8x-40$ = the number of men on each ship,

and $x(8x-40)$ = the total number of men with the fleet.

$\therefore x(8x-40)=4000$.

Expanding, dividing by 8, and transposing,

$$x^2-5x-500=0.$$

Factoring, $(x-25)(x+20)=0$.

Hence, $x-25=0$ or $x+20=0$;

whence, $x=25$ or -20,

and $8x-40=160$ or -200.

The second value of x and of $8x-40$ is evidently inadmissible, since neither the number of ships nor the number of men on a ship can be negative.

Hence, there are 25 ships in the fleet, and 160 men on a ship.

Solve the following problems and verify (§ 125) each solution:

2. The gold mined in a recent year would fill a square room, the height of which is 1 foot less than its length. If the area of one wall is 90 square feet, find the dimensions of the room.

3. Certain wooden paving blocks are twice as long as they are wide and the thickness of each is 4 inches. Find the length and width, if the volume of each block is 128 cubic inches.

4. A rectangular swimming tank on board a ship is 3 times as long as it is wide. If it were divided into 3 square tanks, the area of each would be 225 square feet. Find the dimensions of the tank.

5. A man bought as many tons of crude borax as it is worth dollars a ton, and crude borax is worth $\frac{1}{7}$ as much as refined borax. If the same amount of refined borax would be worth \$2800, find the value of crude borax a ton.

6. A farmer keeps his chickens in a rectangular lot that is 4 times as long as it is wide. If its area is 2500 square feet, find its length and width.

7. At a luncheon with Menelik of Abyssinia there was a pile of bread containing 448 cubic feet. Its height was twice its width and its length was 14 feet. Find its height and width.

8. A large rectangular freight station in Atlanta, Georgia, covers an area of 41,750 square feet. If the length is 16.7 times the width, what is the width of the station?

9. An automobilist paid $3.60 for gasoline. If the number of cents he paid per gallon was 2 less than the number of gallons he bought, find how many gallons he bought and the price per gallon.

10. St. Louis has the largest steam whistle in the world. The number of times it is blown each day is 3 more than the number of dollars it costs to blow it once. How many times is it blown a day, if the cost for 12 days is $48?

11. In the Panama Canal Zone a washerwoman washed as many dozen pieces as she received dollars a dozen for her labor. If she had washed 2 dozen more, she would have received $15. How much did she receive a dozen?

12. The number of pounds of duck feathers that a man bought was the same as the number of cents that he paid a pound for them. If he had bought 10 pounds more, they would have cost $20. How many pounds did he buy?

13. The area of one of the largest photographic prints ever made is 180 square feet. Its dimensions are 18 times those of the picture from which it was enlarged. Find the dimensions of the picture, if its length is 2 inches greater than its width.

14. The cages holding canaries imported into this country are arranged in rows in crates. The number of rows in a crate is 2 less than 5 times the number of cages in a row. If there are 231 cages in a crate, how many rows are there?

FRACTIONS

167. In algebra, an indicated division is called a **fraction**.

The fraction $\frac{a}{b}$ means $a \div b$ and is read 'a divided by b.'

168. The *dividend*, written above a line, is the **numerator**; the *divisor*, written below the line, is the **denominator**; the numerator and denominator are called the **terms** of the fraction.

An arithmetical fraction also indicates division, but the arithmetical notion is that a fraction is one or more of the *equal* parts of a unit; that is, in arithmetic, the terms of a fraction are *positive integers*, while in algebra they may be *any numbers whatever*.

169. The student will find no difficulty with algebraic fractions, if he will bear in mind that they are essentially the same as the fractions he has met in arithmetic. He will have occasion to change fractions to higher or lower terms; to write integral and mixed expressions in fractional form; to change fractions to integers or mixed numbers; to add, subtract, multiply, and divide with algebraic fractions just as he has learned to do with arithmetical fractions, except that *signs* must be considered in dealing with positive and negative numbers.

Signs in Fractions

170. There are *three* signs to be considered in connection with a fraction; namely, the sign of the numerator, the sign of the denominator, and the sign written before the dividing line, called the **sign of the fraction**.

In $-\frac{x}{3z}$ the sign of the fraction is $-$, while the signs of its terms are $+$.