REVIEW

48. 1. Tell how similar terms are added; subtracted. Tell what to do with dissimilar terms in addition; in subtraction.

2. Write a polynomial arranged according to the ascending powers of some letter; the descending powers.

3. State the law of exponents for multiplication; for division; the law of coefficients for each. $3^0 = ?$ $8^0 = ?$ $a^0 = ?$

4. What is an equation? Write one and point out the unknown numbers in it; the known numbers; its first member; its second member.

5. What is meant by 'solving an equation'? Give four methods by one or more of which equations may be solved. How may the value of an unknown number, obtained by solving an equation, be verified?

Solve and verify:

6. $3x = 21.$

7. $\frac{5}{6}x = 15.$

8. $7x - 3 + 4x = 21 - 2x + 2.$

9. $10 - 2x = 3x + 5 + 9 - 6x.$

10. Add $x + y + z$, $7x + 2z + 3y$, $4z + 5y$, and $9x + 3y + 2z$.

11. $11a + 5b + 2c - 4b + 2a - c + 4c - 9a + 5b - 3c + a = ?$

12. Subtract $5a^m + 7b^n + 18c$ from $7a^m + 25c + 8b^n + 8d$.

Expand:

13. $7p^3q^5r^7(pqr^2 + 4p^2qr + 2qr^3 + p^5 + 5p^3q^2r^4 + 3pq).$

14. $(x^4 + 7x^3y + 4x^2y^2 + 3xy^3 + 2y^4)(x^3 + 4x^2y + 2y^3).$

15. $(3z^4 + 4z^3w + 6z^2w^2 + 4zw^3 + 13w^4)(2z^2 + 4zw + 3w^2).$

16. $(a^5b + 3a^4b^2 + 6a^3b^3 + 5a^2b^4 + 11ab^5)(a^3 + 3a^2b + 4ab^2).$

Divide, and test each result:

17. $12l^6m^4n^2 + 18l^5m^2n^4 + 15l^4m^3n^3 + 3l^3mn^2$ by $3l^3mn^2$.

18. $35r^4 + 30r^3s + 69r^2s^2 + 12rs^3 + 22s^4$ by $5r^2 + 2s^2$.

19. $22x^2y^2 + 24xy^3 + 27x^3y^3 + 36x^2y^4 + 3x^3y$ by $3xy + 4y^2$.

20. $4a^5c + 8a^4bc + 11a^3b^2c + 24a^2b^3c + 24ab^4c + 7b^5c$ by $2a^2 + 3ab + b^2$.

POSITIVE AND NEGATIVE NUMBERS

49. The student of arithmetic knows the meaning of such an expression as $10 - 4$, but as yet an expression like $4 - 10$ has no meaning to him. It is the purpose of this chapter to extend the idea of number so that subtracting a larger number from a smaller one will have as much meaning as subtracting a smaller number from a larger one, to show that there is a practical demand for a new kind of number, and finally to show how operations involving this new kind of number are performed.

50. Suppose that at noon the temperature is 10° above 0 and that at 6 P.M. it has fallen 4°. The temperature is then $10° - 4°$, or 6° above 0, but if it has fallen 15° instead of 4°, it is then $10° - 15°$, and because the numbers on a thermometer extend below as well as above 0, we see that $10° - 15°$ means that the temperature is 5° below 0, 10° of the 15° of fall taking it to 0 and the other 5° of fall taking it to 5° below 0.

For convenience and brevity degrees 'above 0' are marked with the sign + and degrees 'below 0' with the sign −.

Such statements may be abbreviated algebraically, thus:

$$+10° - 4° = +6°,$$

and

$$+10° - 15° = -5°.$$

Similarly, if a ship now at 20° north latitude (latitude, $+20°$) sails south 30°, it will cross the equator (latitude, 0°) and be at 10° south latitude (latitude, $-10°$).

Again, a tourist in going from Lake Lucerne 1435 feet above sea level (altitude $+1435$ feet) to the Dead Sea 1295 feet below sea level (altitude -1295 feet) goes not only *to* 0 altitude (sea level), but *through* 0 altitude.

51. Such illustrations as those on the preceding page show a practical need of extending the number scale of arithmetic,

$$1, 2, 3, 4, 5, \cdots,^*$$

below 0, as on the thermometer.

The scale of **algebraic numbers,** then, including 0, is

$$\cdots, -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, \cdots,$$

the numbers in the scale increasing by 1 from left to right.

52. Numbers greater than 0, called **positive numbers,** are written either with or without the sign + prefixed.

Numbers less than 0, called **negative numbers,** *always* have the sign − prefixed.

53. By repeating the **positive unit,** +1, any positive integer may be obtained, and by repeating the **negative unit,** − 1, any negative integer may be obtained.

Fractions are measured by positive or negative *fractional* units.

54. It is seen, then, that while in arithmetic the signs + and − are used to indicate operations to be performed, they have an extended meaning and use in algebra, namely, to denote **opposition.** In this sense they are called **quality,** or **direction, signs.**

Thus, if gains are considered *positive*, indicated by +, *losses* are *negative*, indicated by − ; if *credits* are +, *debts* are − ; if distances *north* or *west* or *upstream* are +, distances *south* or *east* or *downstream* are −.

55. When it is necessary to distinguish between signs of operation and signs of quality the number with its sign of quality may be inclosed in parentheses.

Thus, the sum of 2 and − 3 may be written $2 + (-3)$.

56. Positive and negative numbers, whether integers or fractions, are called **algebraic numbers.**

Arithmetical numbers are positive numbers.

* The sign of continuation, ⋯, is read 'and so on' or 'and so on to.'

57. The value of a number without regard to its sign is called its **absolute value.**

Thus, the absolute value of both + 4 and − 4 is 4.

ADDITION AND SUBTRACTION

58. Addition and subtraction of positive and negative numbers may be performed by counting along the scale of algebraic numbers.

To illustrate, + 3 is *added* to − 2 by beginning at − 2 in the scale and counting 3 units in the *ascending*, or *positive*, direction, arriving at + 1; consequently, + 3 is subtracted from + 1 by beginning at + 1 and counting 3 units in the *descending*, or *negative*, direction, arriving at − 2.

59. The result of adding two or more algebraic numbers is called their **algebraic sum.**

This differs from their *arithmetical sum*, which is the sum of their absolute values.

Unless otherwise specified 'sum' in this book means 'algebraic sum'.

60. In addition, two numbers are given, and their algebraic sum is to be found. In subtraction, the algebraic sum and one of the numbers are given, and the other number is to be found.

Subtraction is thus the *inverse* of addition.

The **difference** is the algebraic number that *added to the subtrahend gives the minuend.*

Sum of Two or More Numbers

EXERCISES

61. Give algebraic sums:

1.	+5	−5	+5	+5	+5	−5	−5
	+5	−5	−5	−4	−9	+8	+2

Suggestions.—The sum of 5 positive units and 5 positive units is 10 positive units; of 5 negative units and 5 negative units, 10 negative units; of 5 negative units and 5 positive units, 0; of 4 negative units and 5 positive units, 1 positive unit; of − 9 and + 5, − 4; etc.

To add two algebraic numbers:

RULE. — *If they have* **like signs**, *add the absolute values and prefix the common sign; if they have* **unlike signs**, *find the difference of the absolute values and prefix the sign of the numerically greater.*

By successive applications of the above rule any number of numbers may be added.

Add:

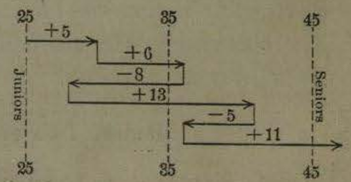
2.	3.	4.	5.	6.
8	$+8$	-8	$+4$	-5
2	$+2$	-2	-7	$+8$

7.	8.	9.	10.	11.
6	$+7$	-5	$+8$	-9
-3	-3	-3	-9	3
7	$+2$	-8	$+1$	2

12. $10+(-4)+(-6)+(-7)$.
13. $10-4-6-7$.
14. $-40+6+8+7+6$.
15. $-12+8+2+(-6)+2$.
16. $-12+8+2-6+2$.
17. $8-2+3+6-8+7-9$.

18. Julius Cæsar was born in the year -100 (100 B.C.), and was 56 years old when assassinated. In what year was he assassinated?

19. In a football game the ball was advanced 5 yards from the Juniors' 25-yard line toward the Seniors' goal, then 6 yards, then -8 yards (*i.e.* it went back 8 yards), and so on, as shown in the diagram. What was the position of the ball after 3 plays? after 4 plays? after 5 plays? after 6 plays?



20. Plot the following and find the last position of the ball: On 15-yard line; gained 4 yards; gained 5 yards; lost 2 yards; gained 30 yards; lost 6 yards; lost 2 yards; gained 12 yards.

21. How far from port is a vessel, if it sails 50 miles, -10 miles (driven back 10 miles), 40 miles, -30 miles, and 80 miles?

62. By doing the work in §§ 17–19, the student has become familiar with adding literal expressions in which the terms are all positive. When some of the terms are negative, the method is essentially the same, the only difference being in the matter of signs, which has just been explained.

EXERCISES

63. Add:

1.	2.	3.	4.
$2x$	a	$-a$	$-4c$
$3x$	$5a$	$4a$	$-3c$

5.	6.	7.	8.
$4v$	$-y$	$12mb$	$40x^2$
$-2v$	$4y$	$-2mb$	$-10x^2$
$-7v$	$-9y$	$-6mb$	$-60x^2$

9.	10.	11.
$6a-2b$	$3xy+2y^2$	$10x+3y+z$
$2a+3b$	$-2xy+6y^2$	$-x-y$
$-5a-4b$	$7xy-4y^2$	$2x+2y+z$

12. Simplify $11a^2b-7ab^2+2ac^2+10ab+ac^2-2a^2b+b^3+5ab^2-2b^3+2ab^2-8ab-6a^2b$.

PROCESS

$$\begin{array}{llllll} 11a^2b & -7ab^2 & +2ac^2 & +10ab & +b^3 \\ -2a^2b & +5ab^2 & +ac^2 & & -2b^3 \\ -6a^2b & +2ab^2 & & -8ab & \\ \hline 3a^2b & & +3ac^2 & +2ab & -b^3 \end{array}$$

RULE. — *Arrange the terms so that similar terms shall stand in the same column.*

Find the algebraic sum of each column, and write the results in succession with their proper signs.

13. Simplify $2a+2b+3c+4b-4a+6a-2c$.
14. Simplify $7w+4x-4y-x-2w+3y-3x+4w$.
15. Simplify $7l-6m+3n-8l+4m+11n+m-2l$.
16. Simplify $15r+6s-11t+r-9s+t-2s+5r-2t-10r$.

Simplify the following polynomials:

17. $7x-11y+4z-7z+11x-4y+7y-11z-4x+y-x-z.$

18. $a+3b+5c-6a+d+4b-2c-2b+5a-d+a-b.$

19. $4x^2-3xy+5y^2+10xy-17y^2-11x^2-5xy+12x^2-2xy.$

20. $2xy-5y^2+x^2y^2-7xy+3y^2-4x^2y^2+5xy+4y^2+x^2y^2.$

21. Add $2a-3b$, $2b-3c$, $5c-4a$, $10a-5b$, and $7b-3c$.

22. Add $x+y+z$, $x-y+z$, $y-z-x$, $z-x-y$, and $x-z$.

23. Add $4x^3-2x^2-7x+1$, x^3+3x^2+5x-6, $4x^2-8x^3+2-6x$, $2x^3-2x^2+8x+4$, and $2x^3-3x^2-2x+1$.

24. Add $5x-3y-2z$, $4y-2x+6z$, $3a-2x-4y$, $4b-2z-5y$, $a-5b$, $5y-6x$, $8x+2y-5a-2b$, and $6x-y-2z+4b$.

25. Add $.12x^3-4x^2+x+2$, $.4x^2-4x+.4-x^3$, $3\frac{1}{2}x-.6+3x^2+2x^3$, and $1-\frac{1}{2}x+1.2x^2+\frac{22}{25}x^3$.

26. Add $20x^{2m}-4x^my^n+36y^{2n}$, $4x^my^n-15y^{2n}-12x^{2m}$, $3y^{2n}+3x^{2m}$, $4x^my^n-11x^{2m}-16y^{2n}$, and $x^{2m}-y^{2n}$.

Difference of Two Numbers

EXERCISES

64. On account of the extension of the scale of numbers below zero (§ 51), subtraction is always possible in algebra.

When the subtrahend is positive, algebraic subtraction is like arithmetical subtraction, and consists in counting *backward* along the scale of numbers, as illustrated in § 58.

Subtract the lower number from the upper one:

1.	6	6	6	6	6	6	6
	3	4	5	6	7	8	9
2.	−3	−3	−3	−4	−5	−6	−7
	0	1	2	3	4	5	6

Observe that *subtracting a positive number is equivalent to adding a numerically equal negative number.*

When the subtrahend is negative, it is no longer possible to subtract as in arithmetic by counting backward.

3. Subtract -2 from 8.

PROCESS

$$\begin{array}{r} 8 \\ -2 \\ \hline 8+2=10 \end{array}$$

EXPLANATION. — If 0 were subtracted from 8, the result would be 8, the minuend itself.

The subtrahend, however, is not 0, but is a number 2 units below 0 in the scale of numbers. Hence, the difference is not 8, but is $8+2$, or the minuend plus the subtrahend with its sign changed.

Or, -2 is subtracted from 8 by beginning at 8 in the scale of numbers and counting 2 units in the direction *opposite* to that indicated by the sign of the subtrahend, arriving at 10.

REMARK. — Notice that any number is *added* by counting along the scale of numbers *in the direction indicated by its sign;* and any number is subtracted by counting *in the direction opposite to that indicated by its sign.*

Subtract the lower number from the upper one:

4.	4	4	4	4	5	7	9
	0	−1	−2	−3	−6	−7	−5
5.	−5	−5	−5	−5	−1	−4	−6
	0	−1	−2	−6	−3	−7	−5

Observe that:

PRINCIPLE. — *Subtracting any number (positive or negative) is equivalent to adding it with its sign changed.*

Subtract the lower number from the upper one:

6.	10	7.	12	8.	20	9.	16	10.	40
	−2		5		−6		−4		−8
11.	0	12.	−3	13.	−7	14.	10	15.	−5
	−2		−6		4		−5		10
16.	4	17.	4	18.	−4	19.	−9	20.	−7
	4		−4		4		3		8

21. Subtract 12 from -1.

22. Subtract -4 from 14.

23. From 0 subtract -3.

24. From -3 subtract 0.

25. From 0 subtract -7; from the result subtract -4; then add -2; add -3; add 7; subtract 11; and add -6.

26. Which is greater and how much, 3 or -5? -2 or -5? 6 or $8-3$? $-2+(-8)$ or $-2-(-8)$?

A weather map for January 16 gave the following minimum and maximum temperatures (Fahrenheit):

	CHICAGO	DULUTH	HELENA	MONTREAL	NEW ORLEANS	NEW YORK
Minimum	24°	-6°	-12°	-12°	64°	20°
Maximum	30°	2°	-4°	18°	76°	42°

27. The range of temperature in Chicago was 6°. Find the range of temperature in each of the other cities.

28. The freezing point is 32° F. How far below the freezing point did the temperature fall in Montreal?

29. How much colder was it in Duluth than in Chicago? in Montreal than in New York? in Helena than in New Orleans?

30. An elevator runs from a basement, -22 feet above the first floor, to the tenth story, 105 feet above the first floor. Express its total *rise* from the basement to the tenth floor; from the tenth floor to the basement.

65. From the work of this chapter, the student will have discovered that negative numbers give the definitions of addition, subtraction, sum, and difference a wider range of meaning than they had in arithmetic. In algebra addition does not always imply an increase, nor subtraction a decrease.

In §§ 20, 21, the student learned how to subtract one literal expression from another, all the terms being positive and the subtrahend being less than the minuend. This is arithmetical subtraction. He will now apply the *broader algebraic idea* of subtraction to literal expressions.

EXERCISES

66. 1. From $10x$ subtract $15x$.

PROCESS

$$\begin{array}{r} 10x \\ 15x \\ - \quad \\ \hline -5x \end{array}$$

EXPLANATION. — Since (§ 64, Prin.) subtracting any number is equivalent to adding it with its sign changed, $15x$ may be subtracted from $10x$ by changing the sign of $15x$ and adding $-15x$ to $10x$.

	2.	3.	4.	5.	6.
From	$5a$	$5x$	$9am$	$-8mn$	$3x^2y^2$
Take	$2a$	$-2x$	$21am$	$-4mn$	$-10x^2y^2$

7. From $8x-3y$ subtract $5x-7y$.

PROCESS

$$\begin{array}{r} 8x-3y \\ 5x-7y \\ - \quad + \\ \hline 3x+4y \end{array}$$

EXPLANATION. — Since (§ 64, Prin.) subtracting any number is equivalent to adding it with its sign changed, subtracting $5x$ from $8x$ is equivalent to adding $-5x$ to $8x$, and subtracting $-7y$ from $-3y$ is equivalent to adding $+7y$ to $-3y$.

RULE. — *Change the sign of each term of the subtrahend, and add the result to the minuend.*

After a little practice the student should make the change of sign mentally.

	8.	9.	10.	11.
From	$9a+7b$	$5r-10s$	$10x+2y$	$3m-3n$
Take	$2a+3b$	$7r+4s$	$6x-4y$	$2m-5n$

12. From $5x-3y+z$ take $2x-y+8z$.

13. From $3a^2b+b^3-a^3$ take $4a^2b-8a^3+2b^3$.

14. From $13a^2+5b^2-4c^2$ take $8a^2+9b^2+10c^2$.

15. From $15x-3y+2z$ subtract $3x+8y-9z$.

16. From a^2-ab-b^2 subtract $ab-2a^2-2b^2$.

17. From m^2-mn+n^2 subtract $2m^2-3mn+2n^2$.

18. From $4x^2+3xy+y^2$ subtract $2x^2-5xy+2y^2$.

19. From $3ab + a^2 + b^2$ subtract $a^2 + 4ab + b^2$.

20. From $6x^2 + 4xy - 3y^2$ subtract $4y^2 - 3xy + 6x^2$.

21. From the sum of $3a^2 - 2ab - b^2$ and $3ab - 2a^2$ subtract $a^2 - ab - b^2$.

22. From $3x - y + z$ subtract the sum of $x - 4y + z$ and $2x + 3y - 2z$.

23. From $a + b + c$ subtract the sum of $a - b - c$, $b - c - a$, and $c - a - b$.

24. Subtract the sum of $m^2n - 2mn^2$ and $2m^2n - m^3 - n^3 + 2mn^2$ from $m^3 - n^3$.

25. Subtract the sum of $2c - 9a - 3b$ and $3b - 5a - 5c$ from $b - 3c + a$.

26. From the sum of $3x^m + 4y^n + z^{m+n}$ and $2z^{m+n} + 2x^m - 3y^n$ subtract $4x^m - 2y^n + z^{m+n}$.

If $x = a^2 + b^2$, $y = 2ab$, $z = a^2 - b^2$, and $v = a^2 - 2ab + b^2$,

27. $x + y + z + v = ?$

28. $x - y - z + v = ?$

29. $x - y + z - v = ?$

30. $y - x - v + z = ?$

TRANSPOSITION IN EQUATIONS

67. In the solution of equations the student has used certain principles, stated in § 40 and § 42 and summed up in § 44.

They are usually stated in somewhat broader terms as in the following section and are so simple as to be self-evident. Such self-evident principles are called **axioms**.

68. AXIOMS. — 1. *If equals are added to equals, the sums are equal.*

2. *If equals are subtracted from equals, the remainders are equal.*

3. *If equals are multiplied by equals, the products are equal.*

4. *If equals are divided by equals, the quotients are equal.*

In the application of axiom 4, it is not allowable to divide by zero or any number equal to zero, because the result cannot be determined.

EXERCISES

69. 1. Solve $x - 6 = 4$ by adding 6 to both members (Ax. 1).

2. Solve the equation $x + 3 = 11$ by employing Ax. 2.

3. Solve $\frac{1}{3}x = 10$ by employing Ax. 3.

4. Solve $7x = 21$. Explain how Ax. 4 applies.

5. Solve $\frac{2}{3}x = 16$ in two steps, first finding the value of $\frac{1}{3}x$ by Ax. 4, then the value of x by Ax. 3.

Solve, and give the axiom applying to each step:

6. $2x = 6$.	13. $x + 2 = 10$.	20. $\frac{3}{2}m = 9$.
7. $5x = 5$.	14. $w - 5 = 11$.	21. $\frac{4}{3}n = 8$.
8. $4y = 8$.	15. $w + 1 = 12$.	22. $\frac{5}{2}x = 10$.
9. $3y = 9$.	16. $s - 7 = 10$.	23. $\frac{3}{5}x = 21$.
10. $\frac{1}{2}z = 5$.	17. $9 + s = 12$.	24. $\frac{5}{6}z = 20$.
11. $\frac{1}{3}z = 2$.	18. $5 + y = 15$.	25. $\frac{5}{8}z = 15$.
12. $\frac{1}{4}v = 3$.	19. $10 + x = 12$.	26. $\frac{7}{8}w = 49$.

70. 1. Adding 7 to both members of the equation

$$x - 7 = 3,$$

we obtain, by Ax. 1, $\quad x = 3 + 7.$

-7 has been removed from the first member, but reappears in the second member with the opposite sign.

2. Subtracting 5 from both members of the equation

$$x + 5 = 9,$$

we obtain, by Ax. 2, $\quad x = 9 - 5.$

When plus 5 is removed, or **transposed**, from the first member to the second, its sign is changed.

3. Explain the transposition of terms in each of the following:

$2x - 1 = 5;$	$3x + 2 = 11;$	$4x = 14 - 3x;$
$2x = 5 + 1.$	$3x = 11 - 2.$	$4x + 3x = 14.$

71. PRINCIPLE. — *Any term may be transposed from one member of an equation to the other, provided its sign is changed.*

EXERCISES

72. 1. Solve the equation $6 - 5x + 18 = 6 + 3x - 30$.

SOLUTION

By Ax. 2,	$\not{6} - 5x + 18 = \not{6} + 3x - 30.$
Transposing, § 71,	$-5x - 3x = -30 - 18.$
Uniting terms,	$-8x = -48.$
Changing signs,	$8x = 48.$
Dividing by 8, Ax. 4,	$x = 6.$

VERIFICATION. — Substituting 6 for x in the given equation,

$$6 - 5 \cdot 6 + 18 = 6 + 3 \cdot 6 - 30, \text{ or } -6 = -6.$$

Hence, 6 is the true value of x; that is, the value 6 substituted for x *satisfies* the equation.

SUGGESTIONS. — 1. By Ax. 2 the same number may be subtracted, or *canceled*, from both members.

2. By Ax. 2 the signs of all the terms of an equation may be changed, for each member may be subtracted from the corresponding member of the equation $0 = 0$.

Solve and verify:

2. $3 = 5 - x$.
3. $9 - 5x = -1$.
4. $10 + v = 18 - v$.
5. $2z + 2 = 32 - z$.
6. $7x + 2 = x + 14$.
7. $3p + 2 = p + 30$.
8. $6y - 2 = 3y + 7$.
9. $5m - 5 = 2m + 4$.
10. $8 + 7a = 5a + 20$.
11. $2 + 13h = 50 - 9$.
12. $50 - n = 20 + n$.
13. $3x - 23 = x - 17$.
14. $4x + 12 = 2x + 36$.
15. $2x + \frac{1}{2}x = 30 - \frac{1}{2}x$.
16. $3x - \frac{1}{4}x = 30 + \frac{1}{4}x$.
17. $5w - 10 = \frac{2}{3}w + 16$.

Simplify as much as possible before transposing terms, solve, and verify:

18. $10x + 30 - 4x - 9x + 33 + 12x = 90 + 12 - 4x$.
19. $16x + 12 - 75 + 2x - 12 - 70 = 8x - 50 - 25$.
20. $11s - 60 + 5s + 17 - 2s + 41 - 3s = 2s + 97$.
21. $10z - 35 - 12z + 16 + 32 = 4z - 35 + 10z + 32$.
22. $14n - 25 = 19 - 11n + 4 + 16 - 10n + n + 136 - 16n$.

Algebraic Representation

73. 1. How much does 8 exceed $3 + 2$? z exceed $10 + y$?

2. What number must be added to 5 so that the sum shall be 9? to m so that the sum shall be 4?

3. George rode a miles on his bicycle; then b miles on the cars; and walked 3 miles. How far did he travel?

4. A man bought a house for m dollars; spent n dollars for improvements; and then sold it for s dollars less than the entire cost. How much did he receive for it?

5. If 40 is separated into two parts, one of which is x, represent the other part.

6. A man made three purchases of a, b, and 2 dollars, respectively, and tendered a 20-dollar bill. Express the number of dollars in change due him.

7. Represent three times a number plus five times the double of the number.

8. What two integers are nearest to 8? to x, if x is an integer? to $a + b$, if $a + b$ is an integer?

9. What are the two even numbers nearest to 8? What are the two even numbers nearest to the even number $2n$?

10. Express the two odd numbers nearest to the odd number $2n + 1$; the two even numbers nearest to $2n + 1$.

11. There is a family of three children, each of whom is 2 years older than the next younger. When the youngest is x years old, what are the ages of the others? When the oldest is y years old, what are the ages of the others?

12. A boy who had 2 dollars spent 25 cents of his money. How much money had he left? If he had x dollars and spent y cents of his money, how much money had he left?

13. The number 376 may be written $300 + 70 + 6$. Write the number whose first digit is x, second digit y, and third digit z.

Problems

74. If $3x =$ a certain number and $x + 10 =$ the same number, then, $$3x = x + 10.$$

This illustrates another axiom to be added to the list that is given in § 68.

AXIOM 5. — *Numbers that are equal to the same number, or to equal numbers, are equal to each other.*

This axiom is useful in the solution of problems, for its application is always involved in writing the *equation of the problem.*

75. The student is advised to review the *general directions for solving problems* given on page 45.

1. The Borough of Manhattan contains 22,000 elevators. If 2000 more are for freight than for passengers, how many freight elevators are there?

2. The total height of a certain brick chimney in St. Louis is 172 feet. Its height above ground is 2 feet more than 16 times its depth below. How high is the part above ground?

3. There are 3141 of the Philippine Islands, of which the number that has been named is 195 more than the number that is nameless. Find the number of each.

4. The value of the toys made in Germany one year was $22,500,000, or $100 more than 4 times the amount purchased by the United States. Find the value of the latter's purchase.

5. The Canadian Falls in the Niagara River are 158 feet high. This is 8 feet more than $\frac{10}{11}$ of the height of the American Falls. Find the height of the American Falls.

6. The summer bird population of Illinois is estimated at 30,750,000 and the number of English sparrows is 19,750,000 less than the number of other birds. Find the number of sparrows.

7. The porch of a temple in India is 876 feet in perimeter, and $\frac{1}{3}$ of its length is 6 feet more than its width. Find its length and width.

8. With the machines of the present time a pin maker can turn out 1,500,000 pins a day, or 60,000 more than 300 times the daily output of a pin maker of early times. How many pins did the early pin maker turn out per day?

9. The cost of dressing the fur of a beaver is 2 cents more than 8 times that for a muskrat. For a muskrat the cost is 9 cents less than for a mink. If the cost of dressing all three furs is 71 cents, find the cost of dressing a beaver's fur.

10. The daily consumption of water per person in New York City is 22 gallons less than that in Boston. The daily consumption in Pittsburg is 250 gallons, or 30 gallons less than that in New York and Boston together. Find the daily consumption per person in Boston.

11. A carpenter, a plumber, and a mason together earn $12.70 a day. If the carpenter earns $1.70 less than the mason, and he and the plumber together earn $7.50, how much does each earn?

12. A letter sent from Indianapolis to Point Barrow, Alaska, travels 6800 miles. It goes 900 miles more by train than by steamer and 200 miles more by dog sleds than by train. How far does it travel by each?

13. In the first three years of excavation, 313,356 cubic yards more were taken from the Panama Canal than from the New York Barge Canal. The amount taken from the former was 6,364,484 cubic yards less than twice that from the latter. How much was excavated from each?

14. Of the wood used for pulp in New York State one year, 500,000 cords were supplied by the state. The amount imported was $\frac{6}{5}$ of that used by Maine. If New York used twice as much as Maine, how many cords were used by each?

15. At one time the coffee stored at the docks of Havre, France, was $\frac{1}{5}$ of the total yearly production of the world and $\frac{1}{4}$ that of Brazil. If Brazil produces $3\frac{1}{2}$ million bags more than all the rest of the world, find the amount stored at Havre.

MULTIPLICATION

76. Primarily *multiplication* is the process of taking one number as many times as there are units in another.

Thus, $3 \times 5 = 5 + 5 + 5 = 15.$

In this section and the next, the sign $\times$ is to be read 'times.'

Even in arithmetic multiplication is extended to cases that cannot by any stretch of language be brought under the original definition.

Thus, strictly, in $3\frac{2}{3} \times 4$, 4 cannot be taken $3\frac{2}{3}$ times any more than a revolver can be fired $3\frac{2}{3}$ times.

So in algebra there are still other cases, like -3×4, that do not come under the original definition.

What we are concerned with, however, is the method of finding the product (consistent with the laws of operation used in arithmetic) and the interpretation of the results obtained.

77. Sign of the product.

1. Just as in arithmetic 3 times 4 are 12, so 3 times 4 *positive* units are 12 *positive* units;

that is, $+3 \times +4 = +12.$ (1)

2. Also, 3 times 4 *negative* units are 12 *negative* units;

that is, $+3 \times -4 = -12.$ (2)

3. Just as in arithmetic $4 \times 3 = 3 \times 4$,

so $4 \times -3 = -3 \times 4;$

and since $4 \times -3 = -12,$

$-3 \times 4 = -12.$ (3)

4. Again, since $6 - 4 = 2,$

by Ax. 3, $-3(6-4) = -3 \times 2 = -6.$

Also, § 27, $-3(6-4) = -3 \times 6 - 3 \times -4 = -18 - 3 \times -4.$

Now, since both $-18 - 3 \times -4$ and -6 are equal to $-3(6-4)$,

by Ax. 5, $-18 - 3 \times -4 = -6.$

Transposing -18, § 71,

$-3 \times -4 = -6 + 18;$

that is, $-3 \times -4 = +12.$ (4)

5. The preceding conclusions may be written as follows:

From (1), $+a \times +b = +ab,$

from (2), $+a \times -b = -ab,$

from (3), $-a \times +b = -ab,$

and from (4), $-a \times -b = +ab.$

Hence, for multiplication:

78. Law of signs.—*The sign of the product of two factors is* + *when the factors have* **like signs**, *and* − *when they have* **unlike signs.**

EXERCISES

79. 1. Multiply each of the following by $+2$; then by -2:

$3,\ 5,\ -6,\ 10,\ -8,\ -9,\ -12,\ a,\ x,\ -b.$

2. Multiply	-8	9	6	4	-2
By	6	3	-5	-7	10

3. Multiply	a	$-b$	$-x$	$-y$	n
By	4	6	-8	-1	-12

80. When there are several factors, by the law of signs,

$$-a \times -b = +ab;$$

$$-a \times -b \times -c = +ab \times -c = -abc;$$

$$-a \times -b \times -c \times -d = -abc \times -d = +abcd;$$ etc. Hence,

The product of an even number of negative factors is positive; of an odd number of negative factors, negative.

Positive factors do not affect the sign of the product.

EXERCISES

81. Find the products indicated:

1. $(-1)(-1)(-1).$
2. $(-2)(-a)(-b)c.$
3. $(-1)(x)(y)(-3b).$
4. $(-3)(2)(-2a)b.$
5. $(-2a)(-3b)(-c)x.$
6. $(-1)(-2)(-3)(-4).$
7. $(-a)(-b)(+c)(-d).$
8. $(-x)(-y)(-1)z(-v).$
9. $(-x)(y)(z)(-v)(-w).$
10. $(-r)(-s)(-t)(x)(-3y).$

82. To multiply when the numbers are either positive or negative.

Having learned to multiply when the numbers are positive, §§ 22–29, and having just learned about the sign of the product when there are negative factors, the student is now prepared to multiply whether the terms of the factors are positive or negative.

EXERCISES

83. 1. Multiply $-4ax^2$ by $2a^3x^4$.

PROCESS	
$-4ax^2$	EXPLANATION. — Since the signs of the monomials are unlike, the sign of the product is $-$ (Law of Signs, § 78).
$2a^3x^4$	$4 \cdot 2 = 8$ (Law of Coefficients, § 24).
$-8a^4x^6$	$a \cdot a^3 = a^1 \cdot a^3 = a^{1+3} = a^4$ (Law of Exponents, § 23).
	$x^2 \cdot x^4 = x^{2+4} = x^6$ (Law of Exponents).
	Hence, the product is $-8a^4x^6$.

Multiply:

2. $10a^5$	3. $-5m^3n^2$	4. $-4abc$	5. $3a^2bc^3$
$5a^3$	$3mn$	$2a^2b$	$-7ab^2c$
6. x^2y^2	7. $5pq^2x^2$	8. $-8ab$	9. $-5a^2x^2$
xy^3	$-2rq^4x$	-1	$-2ax^3$
10. $-2x$	11. $-6a^2c^2x$	12. $-3ab$	13. $-2a^6x^2$
$2x^2$	$-4a^3bn$	$2ba$	$-4ax^4$
14. $-3n^3$	15. $4a^2b^3y^4$	16. -1	17. $-5m^3d^2$
$6b^3$	$3a^3b^2y$	-1	$-2m^{10}cd^3$
18. $2a^{m+1}$	19. $-2a^rb^{2z}$	20. $-x^ny^n$	21. $4x^{n-1}$
$3a^2$	$7a^{3r}b^{4z}$	xy	$-2x^{n+1}$
22. d^{s-n}	23. $8r^{x-y}$	24. $-x^{1-n}$	25. y^{n-m}
d^{2s+n}	$3r^ys^{a-2b}$	$-x^n$	y^{m-n+1}

26. $3x^2-2xy$ by $5xy^2$.

27. $3a^3-6a^2b$ by $-2b$.

28. $m^2n^3-3mn^4$ by $2mn$.

29. $p^2q^2-2pq^3$ by $-pq$.

30. $4a^2-5b^2c-c^3$ by abc^2.

31. $-2ac+4ax$ by $-5acx$.

Expand, and test each result:

32. $(a+b+c)(a+b+c)$.
33. $(a^2-ab+b^2)(a^2+ab+b^2)$.
34. $(a^3+3a^2b+3ab^2+b^3)(a+b)$.
35. $(a-b)(a+b)(a^2+b^2)(a^4+b^4)$.
36. $(x^4-x^3y+x^2y^2-xy^3+y^4)(x+y)$.
37. $(a^2+b^2+c^2+d^2)(a^2-b^2+c^2-d^2)$.
38. $(x^2-xy+y^2+x+y+1)(x+y+1)$.
39. $(a^3+3a^2b+3ab^2+b^3)(a^2+2ab+b^2)$.
40. $(a^2-ab-ac+b^2-bc+c^2)(a+b+c)$.

Arrange both multiplicand and multiplier according to the ascending or the descending powers of the letter involved; multiply, and test each result.

41. $x+x^3+1+x^2$ by $x-1$.
42. $x^3+10-7x-4x^2$ by $x-2$.
43. $14-9x-6x^2+x^3$ by $x+1$.
44. $a^3-30-11a+4a^2$ by $a-1$.
45. $4a^2-2a^3-8a+a^4-3$ by $2+a$.
46. $2m-3+2m^3-4m^2$ by $2m-3$.
47. $x+x^2-5$ by x^2-3-2x.
48. b^2+5b-4 by $-4+2b^2-3b$.
49. $4n^3+6-2n^4+16n-8n^2+n^5$ by $n+2$.
50. $1+x+4x^2+10x^3+46x^5+22x^4$ by $2x^2+1-3x$.
51. $5x^2+7-4x^3-6x+x^6+3x^4-2x^5$ by $2x+x^2+1$.

Multiply:

52. $ax^{2n}+ay^{2n}$ by $ax^{2n}-ay^{2n}$.
53. $ax^{n-1}+y^{n-1}$ by $3ax^{n-1}+2y^{n-1}$.
54. $x^{2n}+2x^ny^n+y^{2n}$ by $x^{2n}-2x^ny^n+y^{2n}$.
55. $a^{6n}+a^{4n}b^{2c}+a^{2n}b^{4c}+b^{6c}$ by $a^{2n}-b^{2c}$.
56. $m^{x+1}n^{x-1}+m^{x-1}n^{x+1}+1$ by $m^{x-1}n^{x+1}-m^{x+1}n^{x-1}+1$.

Special Cases in Multiplication

84. The square of the sum of two numbers.

1. Multiply $a + b$ by $a + b$; find the square of $x + y$.

$$\begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \\ + ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array} \qquad \begin{array}{r} x + y \\ x + y \\ \hline x^2 + xy \\ + xy + y^2 \\ \hline x^2 + 2xy + y^2 \end{array}$$

2. How is the first term of the product, that is, of the square of the sum of two numbers, obtained from the numbers? How is the second term obtained? the third term?

3. What signs have the terms of the result?

85. Principle. — *The square of the sum of two numbers is equal to the square of the first number, plus twice the product of the first and second, plus the square of the second.*

Since $5a^3 \times 5a^3 = 25a^6$, $3a^4b^5 \times 3a^4b^5 = 9a^8b^{10}$, etc., it is evident that in squaring a number the exponents of literal factors are doubled.

EXERCISES

86. Expand by inspection, and test each result:

1. $(p+q)(p+q)$.	11. $(5x+z)^2$.	21. $(a^3+b^3)^2$.
2. $(r+s)(r+s)$.	12. $(2a+x)^2$.	22. $(a^5+b^5)^2$.
3. $(a+x)(a+x)$.	13. $(ab+cd)^2$.	23. $(a^n+b^n)^2$.
4. $(x+4)(x+4)$.	14. $(5x+2y)^2$.	24. $(x^m+y^n)^2$.
5. $(a+6)(a+6)$.	15. $(7z+3c)^2$.	25. $(3a^2+5b^3)^2$.
6. $(y+7)(y+7)$.	16. $(3b+4x)^2$.	26. $(1+2a^3b)^2$.
7. $(z+1)(z+1)$.	17. $(2m+3n)^2$.	27. $(3xy^2+4x^2y)^2$.
8. $(c+9)(c+9)$.	18. $(3c+7d)^2$.	28. $(9a^{2m}+2b^{2n})^2$.
9 $(v+3)(v+3)$.	19. $(8s+5t)^2$.	29. $(4x^{2r}+y^{2t+1})^2$.
10. $(w+5)(w+5)$.	20. $(5w+3u)^2$.	30. $(x^{n-1}+y^{n-1})^2$.

87. The square of the difference of two numbers.

1. Multiply $a - b$ by $a - b$; find the square of $x - y$.

$$\begin{array}{r} a - b \\ a - b \\ \hline a^2 - ab \\ - ab + b^2 \\ \hline a^2 - 2ab + b^2 \end{array} \qquad \begin{array}{r} x - y \\ x - y \\ \hline x^2 - xy \\ - xy + y^2 \\ \hline x^2 - 2xy + y^2 \end{array}$$

2. How is the first term of the square of the difference of two numbers obtained from the numbers? How is the second term obtained? the third term?

3. What signs have the terms of the result?

88. Principle. — *The square of the difference of two numbers is equal to the square of the first number, minus twice the product of the first and second, plus the square of the second.*

EXERCISES

89. Expand by inspection, and test each result:

1. $(x-m)(x-m)$.	14. $(2a-x)^2$.	27. $(3x-2)^2$.
2. $(m-n)(m-n)$.	15. $(3m-n)^2$.	28. $(2x-5y)^2$.
3. $(x-6)(x-6)$.	16. $(4x-y)^2$.	29. $(5m-3n)^2$.
4. $(p-8)(p-8)$.	17. $(m-4n)^2$.	30. $(3p-5q)^2$.
5. $(q-7)(q-7)$.	18. $(p-3q)^2$.	31. $(a^n-b^n)^2$.
6. $(a-c)(a-c)$.	19. $(a-7b)^2$.	32. $(x^m-y^n)^2$.
7. $(a-x)(a-x)$.	20. $(4a-3)^2$.	33. $(a^2-2b^2)^2$.
8. $(x-1)(x-1)$.	21. $(5x-4)^2$.	34. $(y^3-6x)^2$.
9. $(b-5)(b-5)$.	22. $(ab-3)^2$.	35. $(ab-2c^2)^2$.
10. $(st-2)(st-2)$.	23. $(2a-3b)^2$.	36. $(4x^3-5y^2)^2$.
11. $(x-4)(x-4)$.	24. $(2z-7y)^2$.	37. $(x^{t+r}-y^{2r})^2$.
12. $(z-3)(z-3)$.	25. $(8x-5y)^2$.	38. $(mx^m-ny^n)^2$.
13. $(w-9)(w-9)$.	26. $(9w-2v)^2$.	39. $(x^{m-1}-y^{n-1})^2$.

90. The square of any polynomial.

1. Find the square of $a+b+c$.

$$\begin{array}{l} a+b+c \\ \underline{a+b+c} \\ a^2+ab+ac \\ \quad +ab \quad +b^2+bc \\ \underline{\quad\quad +ac \quad +bc+c^2} \\ a^2+2ab+2ac+b^2+2bc+c^2 \end{array}$$

That is, $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$.

2. Show by actual multiplication that

$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$$

3. Similarly, by squaring any polynomial by multiplication, it will be observed that:

91. Principle. — *The square of a polynomial is equal to the sum of the squares of the terms and twice the product of each term by each term, taken separately, that follows it.*

When some of the terms are negative, some of the double products will be negative, but the squares will always be positive. For example, since $(-b)^2=+b^2$, $(a-b+c)^2=a^2+(-b)^2+c^2+2a(-b)+2ac+2(-b)c = a^2+b^2+c^2-2ab+2ac-2bc$.

EXERCISES

92. Expand by inspection, and test each result:

1. $(x+y+z)^2$.
2. $(x+y-z)^2$.
3. $(x-y-z)^2$.
4. $(x-y+z)^2$.
5. $(x+y-3z)^2$.
6. $(x-y+3z)^2$.
7. $(a-2b+c)^2$.
8. $(2a-b-c)^2$.
9. $(rs+st-rt)^2$.
10. $(qb-pc-rd)^2$.
11. $(ax-by+cz)^2$.
12. $(xy-3c-ab)^2$.
13. $(3x-2y+4z)^2$.
14. $(2a-5b+3c)^2$.
15. $(2m-4n-r)^2$.
16. $(12-2x+3y)^2$.
17. $(a+m+b+n)^2$.
18. $(a-m+b-n)^2$.

93. The product of the sum and difference of two numbers.

1. Find the product of $a+b$ and $a-b$; of $x-y$ and $x+y$.

$$\begin{array}{l} a+b \\ \underline{a-b} \\ a^2+ab \\ \underline{\quad -ab-b^2} \\ a^2 \quad\quad -b^2 \end{array} \qquad\qquad \begin{array}{l} x-y \\ \underline{x+y} \\ x^2-xy \\ \underline{\quad +xy-y^2} \\ x^2 \quad\quad -y^2 \end{array}$$

2. How are the terms of the product of the sum and difference of two numbers obtained from the numbers?

3. What sign connects the terms?

94. Principle. — *The product of the sum and difference of two numbers is equal to the difference of their squares.*

EXERCISES

95. Expand by inspection, and test each result:

1. $(x+y)(x-y)$.
2. $(a+c)(a-c)$.
3. $(p+q)(p-q)$.
4. $(p+5)(p-5)$.
5. $(x+1)(x-1)$.
6. $(x^2+1)(x^2-1)$.
7. $(x^3+1)(x^3-1)$.
8. $(x^4-1)(x^4+1)$.
9. $(x^5-1)(x^5+1)$.
10. $(x^n+y^3)(x^n-y^3)$.
11. $(ab+5)(ab-5)$.
12. $(cd+d^2)(cd-d^2)$.
13. $(ab-c^2)(ab+c^2)$.
14. $(4y+z^2)(4y-z^2)$.
15. $(lx-5m)(lx+5m)$.
16. $(2x+3y)(2x-3y)$.
17. $(3m+4n)(3m-4n)$.
18. $(12+xy)(12-xy)$.
19. $(ab+cd)(ab-cd)$.
20. $(3m^2n-b)(3m^2n+b)$.
21. $(2x^3+5y^2)(2x^3-5y^2)$.
22. $(3x^6+2y^6)(3x^6-2y^6)$.
23. $(2a^2+2b^2)(2a^2-2b^2)$.
24. $(-5n-b)(-5n+b)$.
25. $(-x-2y)(-x+2y)$.
26. $(3x^m+7y^n)(3x^m-7y^n)$.
27. $(mx^a+2y^b)(mx^a-2y^b)$.
28. $(a^nb^m+a^mb^n)(a^nb^m-a^mb^n)$.
29. $(x^{m-1}+y^{n+1})(x^{m-1}-y^{n+1})$.
30. $(5a^3b^2+2x^z)(5a^3b^2-2x^z)$.